CHAPTER - III

FRACTIONAL INTEGRATION AND GENERALISED HANKEL TRANSFORM.
3.1 **INTRODUCTION** - Operators of fractional integration are defined as,

\[
\int_{\mathbf{A}} f(x) = \frac{A}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt
\]

\[A > 0, \alpha > 0, \gamma > 0\]

Extension to $\alpha < 0$ is given by the relation,

\[
\int_{\mathbf{A}} f(x) = \frac{A}{\Gamma(\alpha)} \int_0^\infty (t-x)^{\alpha-1} f(t) \, dt
\]

The case $A = 1$ has been studied by Kober \[58\] and the general case $A > 0$ was considered by Erdélyi \[36\].

On the basis of the property of the $H$-function as a symmetric Fourier kernel as pointed out by Fox \[43\], a self-reciprocal transform can be defined in the form,

\[
g(t) = \int \left[ \Delta \left( a_1, b_1, c_1 \right), \Delta \left( 1-a_1-b_1, c_1 \right) \right] \left[ \Delta \left( a_p, b_p, c_p \right), \Delta \left( 1-a_p-b_p, c_p \right) \right] \right] f(t) \, dt
\]

provided the integral is absolutely convergent. $\Delta \left( a_p, b_p \right)$ indicates the set $(a_1, b_1), \ldots, (a_p, b_p)$.

We shall call (3.1.4) a generalised Hankel transform and denote it by $g(t) = U_{\omega} \left\{ f(t) \right\}$.

If $p = q = 0$, then using the relation,
\( H^{t,0}_{0,2} \left[ x \left| (a,1), (b,1) \right. \right] = x^{\frac{a+b}{2}} \cdot J_{a-b} \left( 2x^\frac{1}{2} \right) \),

(3.1.4) reduces to the ordinary Hankel transform,

\( g(t) = \sum_{c}^{\infty} J_{2\sqrt{st}} \cdot f(s) \ ds \), denoted by

\( g(t) = H_{2} \left\{ f(t) \right\} \).

The object of this chapter is to generalise (3.1.4) and to find out relations between fractional integrals and the new transform. Corresponding results about \( H_{2} \) follow as corollaries.

The following results will be used in our investigation:

\[
(3.1.6) \quad \gamma, \kappa < 1 \quad x^{\alpha} \mathbb{H}^{m,n}_{p+1,q+1} \left[ \Delta(a_p, e_p) \right]
= x^{\alpha} \mathbb{H}^{m,n}_{p+1,q+1} \left[ \Delta(b_q, f_q) \right]
\]

(3.1.7) \quad \gamma, \kappa < 1 \quad x^{-\beta} \mathbb{H}^{m,n}_{p+1,q+1} \left[ \Delta(a_p, e_p) \right]
= x^{-\beta} \mathbb{H}^{m,n}_{p+1,q+1} \left[ \Delta(b_q, f_q) \right].
\]
In the sequel we shall assume the absolute convergence of the integrals involved.

3.2 We present the generalisation in the form,

\[ G_{\nu,\alpha} f(t) = t^{-\frac{\alpha}{2}} \mathcal{H}_{\nu+\alpha} t^{-\frac{\alpha}{2}} \{ f(t) \} . \]

It may be noted that \( G_{\nu,\alpha} \) is not a self-reciprocal transform but may be inverted by a transform with an \( H \) - function in the kernel with parameters different from those in the original kernel.

If we define,

\[ S_{\nu,\alpha} \{ f(t) \} = t^{-\frac{\alpha}{2}} \mathcal{H}_{\nu+\alpha} t^{-\frac{\alpha}{2}} \{ f(t) \} , \]

\( S_{\nu,\alpha} \) is a special case of \( G_{\nu,\alpha} \).

3.3 Using the definitions (3.1.2) and (3.2.1) we have

\[
G_{\nu,\alpha} \left[ \mathcal{L}^{-1} K_{\nu}^{\alpha} \mathcal{L}^{-1} \{ f(t) \} \right] = t^{-\frac{\alpha}{2}} \mathcal{H}_{\nu+\alpha} t^{-\frac{\alpha}{2}} \{ f(t) \}
\]

Changing the order of integration and using (3.1.6) we have,

\[
\int_0^{\infty} \int_0^{\pi} \Delta \{(1-\alpha_\beta-\beta_\gamma, \beta_\gamma) \} \left[ \Delta \left( \frac{\alpha+\beta}{2}, \frac{\gamma+\beta}{2} \right) \right] \left( A(\frac{\beta+\gamma}{2}) \right)^{\alpha-1} \left( A(\frac{\alpha+\beta}{2}) \right)^{\beta-1} \left( A(\frac{\gamma+\beta}{2}) \right)^{\gamma-1} \{ f(x) \} \, dx \, d\beta.
\]
\[ \frac{\alpha}{1} \int_{A}^{t} G_{A, \alpha} \left\{ I_{A}^{-1} \frac{t^{\alpha+\frac{\alpha}{2}}}{\alpha} \frac{t^{1-A}}{\alpha} \left\{ f(t) \right\} \right\} d\Delta. \]

From (3.1.1) and (3.1.6) we see that the right hand side of (3.3.1) is equal to \[ \frac{\alpha}{1} G_{A, \alpha} \left\{ f(t) \right\}. \]

Thus we have,

\[ \frac{\alpha}{1} \int_{A}^{t} G_{A, \alpha} \left\{ f(t) \right\} d\Delta. \]

Putting \( A = 1 \), we have,

\[ \frac{\alpha}{1} \int_{A}^{t} G_{A, \alpha} \left\{ f(t) \right\} d\Delta. \]

If further we put \( p = q = 0 \), then,

\[ \frac{\alpha}{1} \int_{A}^{t} G_{A, \alpha} \left\{ f(t) \right\} d\Delta. \]

Proceeding as in (3.3.1)
$$G_{\psi, \beta} \left[ t^{A-1} \int_0^t t^{1-A} \{ f(t) \} \right]$$

$$-p/2 \int_{-p/2}^p \left[ \frac{1}{t} \right] \left[ \Delta(\alpha, \beta), \Delta(\beta, \alpha) \right] \left[ \Delta(\delta^{+} - \delta^{-}, \epsilon^{+} - \epsilon^{-}) \right]$$

$$= t \int_0^t \left[ \frac{1}{t} \right] \left[ \Delta(b^+ f^+, f^-), \Delta(b^{-} f^{-}, f^{+}) \right]$$

$$\times \left[ \frac{1}{b} \right] \left[ \Delta \left( A - \frac{1}{2}, \alpha + \beta \right), \Delta \left( A - \frac{1}{2}, \beta + \alpha \right) \right]$$

Changing the order of integration and using (3.1.7) we have,

$$G_{\psi, \beta} \left[ t^{A-1} \int_0^t \frac{1}{t} t^{1-A} \{ f(t) \} \right]$$

$$= \int_0^t \left[ \frac{1}{t} \right] \left[ \Delta(\alpha, \beta), \Delta(\beta, \alpha) \right] \left[ \Delta(\delta^{+} - \delta^{-}, \epsilon^{+} - \epsilon^{-}) \right]$$

$$= t \int_0^t \left[ \frac{1}{t} \right] \left[ \Delta(b^+ f^+, f^-), \Delta(b^{-} f^{-}, f^{+}) \right]$$

$$\times \left[ \frac{1}{b} \right] \left[ \Delta \left( A - \frac{1}{2}, \alpha + \beta \right), \Delta \left( A - \frac{1}{2}, \beta + \alpha \right) \right]$$

From (3.1.2) and (3.1.7) the right hand side of (3.3.5) is also equal to $K_{\psi, \beta} G_{\psi, \beta} \{ f(t) \}$.

Thus we have,

$$G_{\psi, \beta} \left[ t^{A-1} \int_0^t \frac{1}{t} t^{1-A} \{ f(t) \} \right] = K_{\psi, \beta} G_{\psi, \beta} \{ f(t) \}$$

For $A = 1$, then

$$G_{\psi, \beta} \left[ \int_0^t \frac{1}{t} \{ f(t) \} \right] = K_{\psi, \beta} G_{\psi, \beta} \{ f(t) \}$$

$$= t \int_0^t \left[ \frac{1}{t} \right] \left[ \Delta(\alpha, \beta), \Delta(\beta, \alpha) \right] \left[ \Delta(\delta^{+} - \delta^{-}, \epsilon^{+} - \epsilon^{-}) \right]$$

$$\times \left[ \frac{1}{b} \right] \left[ \Delta \left( A - \frac{1}{2}, \alpha + \beta \right), \Delta \left( A - \frac{1}{2}, \beta + \alpha \right) \right]$$
If further we put \( p = q = 0 \), then

\[
(3.3.8) \quad S_{\nu+1, p} I_t \{f(t)\} = K_t I_t S_{\nu+1, p} \{f(t)\} = S_{\nu^+, \alpha+p} \{f(t)\}.
\]

From (3.3.4) and (3.3.8) we have after using the relations (3.1.3), the operator equivalences,

\[
(3.3.9) \quad K_t S_{\nu, \lambda} K_t = S_{\nu+1, \lambda} = I_t S_{\nu^+, \lambda} I_t.
\]

3.4 Let \( g(t) = \bigcup_{\nu} \{f(t)\} \). Then

\[
(3.4.1) \quad I_t \left\{ \sum_{\nu}^{3 \nu, -3 \nu} \int t^{-\nu} g(t) \right\} = \int_0^{\infty} \int_{\nu^2, 2 \nu+1}^\infty \left[ \Delta(a; f) \Delta(1-a; f) \right] d\lambda.
\]

Putting \( p = q = 0 \), we see that if \( g(t) = H_{\nu} \{f(t)\} \),

then \( I_t \left\{ t^{-\nu} g(t) \right\} = H_{\nu} \{ t^{\nu} \cdot f(t) \} \).