Chapter 6

A note on limit laws for the maximum in the discrete case

6.1 Introduction

As a prelude to discuss entropy convergence for limit laws in the discrete case, this chapter gives preliminary results on partial maxima in the discrete case. Some results on maxima of discrete rvs are available in Anderson (1970, 1980), Anderson et al. (1997) and Nadarajah and Mitov (2004). This chapter looks at similar results under power normalization. At the time of submission of this thesis, entropy convergence for normalized partial maxima in the discrete case is being explored.

If $X_1, \ldots, X_n$ are iid discrete valued rvs on non-negative integers, then it is known (see Galambos, 1987) that the linearly normalized partial maxima $M_n = \max \{X_1, \ldots, X_n\}$ does not converge to any non-degenerate rv if

$$\lim_{n \to \infty} \frac{P(X = n)}{P(X \geq n + 1)} \neq 0,$$

(6.1.1)

which is the case for Poisson and geometric distributions.

However, if for each positive integer $n$, $R_{n,i}, \ i = 1, \ldots, n$ denote independent Poisson rvs with mean $\tau_n$ growing with $n$, Anderson et al. (1997) have obtained non-degenerate
limit laws for the normalized partial maxima \( \max_{1 \leq i \leq n} R_{n,i} \). The main tool used is Cramér’s theorem (see Petrov, 1995): For iid rvs \( X_i, i \geq 1 \), if \( E(X_i) = 0, \text{Var}(X_i) = \sigma^2 < \infty \), and \( S_n = \sum_{i=1}^{n} X_i \), then for \( x_n \) varying with \( n \) in such a way that as \( n \to \infty \), \( x_n = o(n^{1/3}) \),

\[
P\left( \frac{S_n}{\sigma \sqrt{n}} > x_n \right) = \exp \left( x_n^2 C \left( \frac{x_n}{\sqrt{n}} \right) \right) \left[ 1 + O \left( \frac{x_n}{n^{1/2}} \right) \right], \tag{6.1.2}
\]

where \( \eta \) is the standard normal df and \( C(\cdot) \) is the power series \( C(z) = c_1 z + c_2 z^2 + \ldots, c_j \) being a function of moments of order \( j + 2 \) and lower. Anderson et al. (1997) apply this result initially to centered unit Poisson variables \( X_i \), replacing \( n \) by \( \tau_n \), so that \( S_n \) in (6.1.2) follows the same centered Poisson distribution as \( R_{n,i} - \tau_n \). Thus

\[
P \left( \frac{R_{n,1} - \tau_n}{\sqrt{\tau_n}} > x_n \right) \sim (1 - \eta(x_n)) \exp \left( x_n^2 C \left( \frac{x_n}{\sqrt{\tau_n}} \right) \right), \tag{6.1.3}
\]

where \( x_n = o(\sqrt{\tau_n}), C \left( \frac{x_n}{\sqrt{\tau_n}} \right) = c_1 \frac{x_n}{\sqrt{\tau_n}} + \ldots + c_r \frac{x_n^{r+2}}{\tau_n^2}, \) and \( x_n^{r+3} = o(\tau_n^{-1/2}) \). And, it is shown that with \( u_n(x) = b^{(r)}(n) + a(n)x \),

\[
\lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} R_{n,i} \leq \sqrt{\tau_n} u_n(x) + \tau_n \right) = \Lambda(x), \tag{6.1.4}
\]

where \( \Lambda \) is the df of the Gumbel extreme value law, \( a(n) = (2 \log n)^{-1/2} \) and \( b^{(r)}(n) = b^{(0)}(n) + \epsilon_n \), with \( b^{(0)}(n) = \sqrt{2 \log n} - \frac{\log \log n + \log 4\pi}{2 \sqrt{2 \log n}} \), and

\[
\epsilon_n = b^{(0)}(n) \sum_{j=1}^{r} c_j \left( b^{(0)}(n) \right)^j \left( 1 + \sum_{j=1}^{r} (j+2)c_j \left( \frac{b^{(0)}(n)}{\sqrt{\tau_n}} \right)^j \right) + o \left( \frac{1}{b^{(r)}(n)} \right).
\]

Since \( R_{n,i} \)'s are independent, (6.1.4) is equivalent to

\[
\lim_{n \to \infty} nP \left( \frac{R_{n,1} - \tau_n}{\sqrt{\tau_n}} > u_n(x) \right) = \lim_{n \to \infty} n(1 - \eta(u_n(x))) \exp \left( (u_n(x))^2 C \left( \frac{u_n(x)}{\sqrt{\tau_n}} \right) \right) = e^{-x},
\]
and when $r = 0$

$$\lim_{n \to \infty} nP \left( \frac{R_{n,1} - \tau_n}{\sqrt{\tau_n}} > a(n)x + b^{(0)}(n) \right) = \lim_{n \to \infty} n(1 - \eta(a(n)x + b^{(0)}(n))) = e^{-x}.$$  

Nadarajah and Mitov (2004) show that for the uniform, binomial, geometric, negative binomial and the generalized power series distributions, (6.1.1) is not satisfied by any sequences $a(n) > 0$ and $b(n)$ so that $\frac{M_n - b(n)}{a(n)}$ can have a non-degenerate limiting rv as $n \to \infty$. Assuming that the parameters in these distributions vary with $n$, they derived non-degenerate limit laws for normalized partial maxima as shown in the following results.

(a) For binomial distribution with parameter $N(n) \to \infty$, $p$ fixed, if $N(n)$ grows with $n$ according to $(\log n)^3 = o(N(n))$, then with $a(n) = \frac{1}{\sqrt{2\log n}}$, and $b(n) = \sqrt{2\log n} - \frac{\log \log n}{2\log n}$,

$$\lim_{n \to \infty} P \left( M_n \leq \sqrt{p(1-p)N(n)}a(n)x + pN(n) + \sqrt{p(1-p)N(n)}b(n) \right) = \Lambda(x), \ x \in \mathbb{R}.$$

(b) For discrete uniform distribution on integers $1, \ldots, N(n)$, if $N(n)$ grows with $n$ according to $n = o(N(n))$, then with $a(n) = \frac{\vartheta_1 N(n)}{n}$, $b(n) = N(n) - \frac{\vartheta_2 N(n)}{n}$, where $\vartheta_1 > 0$ and $\vartheta_2 \geq 0$,

$$\lim_{n \to \infty} P \left( M_n \leq a(n)x + b(n) \right) = e^{\vartheta_1 x - \vartheta_2}, \ x \in \mathbb{R}.$$

(c) For geometric distribution with parameter $p(n)$, if $\lim_{n \to \infty} p(n) = 0$, with $a(n) = \frac{\vartheta_1}{p(n)}$, $b(n) = \frac{\log \left( \frac{\vartheta_1}{p(n)} \right)}{p(n)}$, where $\vartheta_1 > 0$ and $\vartheta_2 \geq 0$,

$$\lim_{n \to \infty} P \left( M_n \leq a(n)x + b(n) \right) = e^{-\vartheta_2}e^{-\vartheta_1 x}, \ x \in \mathbb{R}.$$

(d) For negative binomial distribution with parameters $r \geq 2$ fixed and $p = p(n)$, if $\lim_{n \to \infty} p(n) = 0$, and $p(n) = o \left( \frac{1}{\log n} \right)$, then with $b(n) = \frac{(\log n + (r-1)(\log \log n) - (\log(r-1)!)p(n)}{p(n)}$, 

$$\lim_{n \to \infty} nP \left( \frac{R_{n,1} - \tau_n}{\sqrt{\tau_n}} > a(n)x + b^{(0)}(n) \right) = \lim_{n \to \infty} n(1 - \eta(a(n)x + b^{(0)}(n))) = e^{-x}.$$
\[ a(n) = \frac{\vartheta_1}{p(n)}, \text{ where } \vartheta_1 > 0, \]

\[ \lim_{n \to \infty} P(M_n \leq a(n)x + b(n)) = e^{-e^{-\vartheta_1 x}}, \quad x \in \mathbb{R}. \]

(e) For generalized power series distribution with parameter \(\vartheta_2\) fixed and \(p(n)\), if \(\lim_{n \to \infty} p(n) = 0\), and \(p(n) = o\left(\frac{1}{\log n}\right)\), then with \(a(n) = \frac{\vartheta_1}{p(n)}\), \(b(n) = \left(\frac{\log n + \vartheta_2 (\log \log n) - \log(p(\vartheta_2+1))}{p(n)}\right)\), where \(\vartheta_1 > 0\) and \(\vartheta_2 > 0\),

\[ \lim_{n \to \infty} P(M_n \leq a(n)x + b(n)) = e^{-e^{-\vartheta_1 x}}, \quad x \in \mathbb{R}. \]

**Remark 6.1.1.** Note that, in (b) above, if \(\vartheta_2 = 0\) and \(\vartheta_1 = 1\) then the limit law is \(\Psi_1\), in (c), if \(\vartheta_2 = \vartheta_1 = 1\) then the limit law is \(\Lambda\) and in (d) and (e), if \(\vartheta_1 = 1\), then the limit law is \(\Lambda\).

In the next section, we prove some results similar to those given above under power normalization.

### 6.2 Limit laws in the discrete case under power normalization

**Theorem 6.2.1.** If \(X_1, X_2, \ldots, X_n\) are iid Poisson rvs with parameter \(\tau(n)\), growing with \(n\) according to \(\log n = o(\tau^{1/3}(n))\), then

\[ \lim_{n \to \infty} P\left(M_n \leq \sqrt{\tau(n)}(\delta(n) \mid x \mid^{\beta(n)} \text{ sign}(x)) + \tau(n)\right) = \Phi_1(x), \quad x \in \mathbb{R}, \]

where,

\[ \beta(n) = 1 \log\left(\frac{n^2}{\sqrt{4\pi \log n}}\right), \quad \delta(n) = \sqrt{2 \log n} - \frac{\log \log n + \log 4\pi}{2 \sqrt{2 \log n}}. \]

**Proof.** From Anderson et al. (1997) and Theorem 1 in Nadarajah and Mitov (2004), there
exist $a_n = \frac{1}{\sqrt{2 \log n}}$ and $b_n = 2 \log n - \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}}$ such that

$$\lim_{n \to \infty} P(M_n \leq \sqrt{\tau_n(a_n x + b_n)} + \tau_n) = \Lambda(x), \ x \in \mathbb{R}.$$ 

Arguing as in the proof of (a.(ii)) of Theorem 3.1 in Mohan and Ravi (1993) (Theorem 6.5.4) and setting $\delta(n) = b_n$, $\beta(n) = \frac{a_n}{b_n}$, $\nu_n(x) = \frac{x^{\beta(n)} - 1}{\beta(n)}$, $x > 0$ and $= -\frac{1}{\beta(n)}$, $x \leq 0$, and $\nu(x) = \log x$, $x > 0$ and $= -\infty$, $x \leq 0$, since $\Lambda(\nu(x)) = \Phi_1(x)$, we get

$$\lim_{n \to \infty} P(M_n \leq \sqrt{\tau(n)(\delta(n)x^{\beta(n)}) + \tau(n)}) = \lim_{n \to \infty} P(M_n \leq \sqrt{\tau(n)(a_n \nu_n(x) + b_n)} + \tau(n)),
= e^{-e^{-x}} = e^{-\frac{1}{\beta}} = \Phi(x), \ x > 0,$$

proving the theorem.

**Theorem 6.2.2.** Let $X_1, X_2, \ldots, X_n$ be iid binomial rv with df $F$ and parameters $N(n) \to \infty$ according to $(\log n)^3 = o(N(n))$, and $p$ fixed. Then there exist sequences

$$\beta(n) = \frac{1}{\log \left( \frac{n^2}{\sqrt{4\pi \log n}} \right)}, \ \delta(n) = \sqrt{2 \log n - \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}},$$

such that

$$\lim_{n \to \infty} P(M_n \leq \sqrt{p(1-p)N(n)(\delta(n) | x |^{\beta(n)} \text{ sign}(x)) + pN(n)}) = \Phi_1(x), \ x \in \mathbb{R}.$$

**Proof.** From Theorem 3 in Nadarajah and Mitov (2004) for the binomial distribution with parameter $N(n)$ and $p$ fixed there exist sequences $a_n = \frac{1}{\sqrt{2 \log n}}$ and $b_n = \sqrt{2 \log n - \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}}}$ such that

$$\lim_{n \to \infty} P(M_n \leq \sqrt{\tau_n(a_n x + b_n)} + \tau_n) = \Lambda(x), \ x \in \mathbb{R}.$$ 

Arguing as in the proof of (a.ii) of Theorem 3.1 in Mohan and Ravi (1993) (Theorem 6.5.4) and setting $\delta(n) = b_n$, $\beta(n) = \frac{a_n}{b_n}$, $\nu_n(x) = \frac{x^{\beta(n)} - 1}{\beta(n)}$, $x > 0$, and $= -\frac{1}{\beta(n)}$, $x \leq 0$, and

$$\nu(x) = \log x, x > 0 \text{ and } = -\infty, x \leq 0,$$

since $\Lambda(\nu(x)) = \Phi_1(x)$, we get

$$\lim_{n \to \infty} P(M_n \leq \sqrt{\tau(n)(\delta(n)x^{\beta(n)}) + \tau(n)}) = \lim_{n \to \infty} P(M_n \leq \sqrt{\tau(n)(a_n \nu_n(x) + b_n)} + \tau(n)),
= e^{-e^{-x}} = e^{-\frac{1}{\beta}} = \Phi(x), \ x > 0,$$

proving the theorem.
and $\nu(x) = \log x$, $x > 0$ and $= -\infty$, $x \leq 0$, since $\Lambda(\nu(x)) = \Phi_1(x)$, we get

$$
\lim_{n \to \infty} P(M_n \leq \sqrt{\tau(n)}(\tau(n)x^{\beta(n)}) + \tau(n)) = \lim_{n \to \infty} P(M_n \leq \sqrt{\tau(n)}(a_n\nu_n(x) + b_n) + \tau(n)) = e^{-e^{-\log x}} = e^{-\frac{1}{x}} = \Phi(x), \ x > 0,
$$

proving the theorem.

**Theorem 6.2.3.** Let $X_1, X_2, \ldots, X_n$ be iid uniform rvs with df $F$ and parameter $N(n)$ growing with $n$ according to $n = o(N(n))$. Then there exist sequences $\beta(n) = \frac{1}{n}$, $\delta(n) = N(n)$, such that

$$
\lim_{n \to \infty} P\left(M_n \leq \delta(n) \mid x \mid^{\beta(n)} \text{sign}(x)\right) = K_{2,1}(x), \ x \in \mathbb{R}.
$$

**Proof.** From Theorem 2 in Nadarajah and Mitov (2004) for the uniform distribution, if $\vartheta_1 = 1$ and $\vartheta_2 = 0$ then $F \in D_l(\Psi_1)$, with $0 < r(F)$ and $a_n = \frac{N(n)}{n}$, $b(n) = N(n)$. Arguing as in the proof of (e) of Theorem 3.1 in Mohan and Ravi (1993) (Theorem 6.5.4) and setting $\delta(n) = a_n$, $\beta(n) = \frac{a_n}{b_n}$, $\nu_n(x) = -\frac{1}{\beta(n)}$, $x \leq 0$, $= \frac{x^{\beta(n)} - 1}{\beta(n)}$, $0 < x \leq 1$, $= 0$, $x > 1$, and $\nu(x) = -\infty$, $x \leq 0$, $= \log x$, $0 < x \leq 1$, and $= 0$, $x > 1$, since $\Psi_1(\nu(x)) = K_{2,1}(x)$, we get

$$
\lim_{n \to \infty} P\left(M_n \leq \delta(n) x^{\beta(n)}\right) = \lim_{n \to \infty} P\left(M_n \leq a_n\nu_n(x) + b_n\right) = \Psi_1(\nu(x)) = K_{2,1}(x), \ 0 \leq x < 1,
$$

proving the theorem.

**Theorem 6.2.4.** Let $X_1, X_2, \ldots, X_n$ be iid geometric rvs with df $F$ and parameter $p(n) \to 0$ as $n \to \infty$. Then there exist sequences $\beta(n) = \frac{1}{\log n}$, $\delta(n) = \frac{\log n}{p(n)}$ such that

$$
\lim_{n \to \infty} P\left(M_n \leq \delta(n) \mid x \mid^{\beta(n)} \text{sign}(x)\right) = \Phi_1(x), \ x \in \mathbb{R}.
$$

**Proof.** From Theorem 4 in Nadarajah and Mitov (2004) for the geometric distribution, if
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\( \vartheta_1 = 1 \) and \( \vartheta_2 = 1 \) then \( F \in \mathcal{D}(\Lambda) \), with \( r(F) = \infty \) and \( a_n = \frac{1}{p(n)} \), \( b_n = \frac{\log n}{p(n)} \). Arguing as in the proof of (a.(ii)) of Theorem 3.1 in Mohan and Ravi (1993) (Theorem 6.5.4), and setting \( \delta(n) = b_n \), \( \beta(n) = \frac{a_n}{b_n} \), \( \nu_n(x) = \frac{x^{\beta(n)} - 1}{\beta(n)} \), \( x > 0 \), \( = -\frac{1}{\beta(n)} \), \( x \leq 0 \), and \( \nu(x) = \log x \), \( x > 0 \), \( = -\infty \), \( x \leq 0 \), since \( \Lambda(\nu(x)) = \Phi(x) \), we get

\[
\lim_{n \to \infty} P \left( M_n \leq \delta(n)x^{\beta(n)} \right) = \lim_{n \to \infty} P \left( M_n \leq a_n\nu_n(x) + b_n \right),
\]

proving the theorem.

**Theorem 6.2.5.** Let \( X_1, X_2, \ldots, X_n \) be iid negative binomial rvs with df \( F \) and parameters \( r \geq 2 \) fixed and \( p(n) \to 0 \) as \( n \to \infty \) according to \( p(n) = o \left( \frac{1}{\log n} \right) \). Then there exist

sequences \( \beta(n) = (\log n + (r - 1)(\log \log n) - (\log (r - 1)!))^{-1}, \delta(n) = \frac{\beta(n)}{p(n)} \) such that

\[
\lim_{n \to \infty} P \left( M_n \leq \delta(n)x^{\beta(n)} \right| x^{\beta(n)} \text{ sign}(x) \right) = \Phi_1(x), \ x \in \mathbb{R}.
\]

**Proof.** From Theorem 5 in Nadarajah and Mitov (2004) for the geometric distribution, if \( \vartheta_1 = 1 \) then \( F \in \mathcal{D}(\Lambda) \), with \( r(F) = \infty \) and \( b_n = \frac{(\log n + (r - 1)(\log \log n) - (\log (r - 1)!)}{p(n)} \), \( a_n = \frac{1}{p(n)} \). Arguing as in the proof of (a.(ii)) of Theorem 3.1 in Mohan and Ravi (1993) (Theorem 6.5.4) and setting \( \delta(n) = b_n \), \( \beta(n) = \frac{a_n}{b_n} \), \( \nu_n(x) = \frac{x^{\beta(n)} - 1}{\beta(n)} \), \( x > 0 \), \( = -\frac{1}{\beta(n)} \), \( x \leq 0 \), and \( \nu(x) = \log x \), \( x > 0 \), \( = -\infty \), \( x \leq 0 \), since \( \Lambda(\nu(x)) = \Phi(x) \), we get

\[
\lim_{n \to \infty} P \left( M_n \leq \delta(n)x^{\beta(n)} \right) = \lim_{n \to \infty} P \left( M_n \leq a_n\nu_n(x) + b_n \right),
\]

proving the theorem.

**Theorem 6.2.6.** Let \( X_1, X_2, \ldots, X_n \) be iid generalized power series rvs with df \( F \) and parameters \( r \) fixed and \( p = p(n) \to 0 \) as \( n \to \infty \), according to \( p(n) = o \left( \frac{1}{\log n} \right) \). Then there
exist sequences $\beta(n) = ((\log n) + r(\log \log n) - (\log \Gamma(r + 1)))^{-1}, \delta(n) = \frac{\beta(n)}{p(n)}$ such that

$$\lim_{n \to \infty} P\left(M_n \leq \delta(n) \mid x \mid^{\beta(n)} \operatorname{sign}(x)\right) = \Phi_1(x), \ x \in \mathbb{R}.$$ 

Proof. From Theorem 6 in Nadarajah and Mitov (2004) for the generalized power series distribution, if $\vartheta_1 = 1$ then $F \in D_l(\Lambda)$, with $r(F) = \infty$ and $a_n = \frac{1}{p(n)}$, $b(n) = (\log n) + r(\log \log n) - (\log \Gamma(r + 1)) p(n)$.

Arguing as in the proof of (a.(ii)) of Theorem 3.1 in Mohan and Ravi (1993) (Theorem 6.5.4) and setting $\delta(n) = b_n$, $\beta(n) = \frac{a_n}{b_n}$, $\nu_n(x) = x^{\beta(n)} - 1, \ x > 0, = -\frac{1}{\beta(n)}, \ x \leq 0, \ \nu(x) = \log x, \ x > 0, = -\infty, \ x \leq 0$, since $\Lambda(\nu(x)) = \Phi(x)$, we get

$$\lim_{n \to \infty} P\left(M_n \leq \delta(n)x^{\beta(n)}\right) = \lim_{n \to \infty} P\left(M_n \leq a_n\nu_n(x) + b_n\right),$$

$$= \Lambda(\nu(x)) = \Phi_1(x), \ x > 0,$$

proving the theorem.

At the time of submission the entropy and relative entropy convergences in the above results are being explored. We next give a couple of graphs about tail behaviour of Poisson and binomial in the above results.
6.3 Graphs of tails

Figure 6.1: (A) Tail of Fréchet df with $\alpha = 1$ (dashed line) and tail of power normalized partial maxima of Poisson with rate $\tau_n = (\log n)^{7/2}$ and $n = 10^{13}$ (line) (Theorem 6.2.1). (B) Tail of Fréchet df with $\alpha = 1$ (dashed line) and tail of power normalized partial maxima of binomial for $p = 0.5$ and $N = 10^{13}$, (line) (Theorem 6.2.2).