Chapter 5

On univariate and multivariate coefficient of variation

5.1 Introduction

Coefficient of variation (CV) is a measure of consistency of a set of observations on an rv. It gives the relative variability of the rv to its non-zero mean. The fact that the exponential distribution has CV equal to 1 has been exploited in reliability and survival analysis. Ahmed et al. (2008) introduce a new class of ageing distributions as the class of all dfs whose CV is at the most equal to 1 and study reliability properties of this class. Very few researchers have considered multivariate version of CV. Reyment (1960) appears to be the first to extend the concept of CV to the multivariate case. Zhang et al. (2010) compute a multivariate CV by using the Mahalanobis distance between the mean of a random vector and the origin. The sampling distribution of the sample multivariate CV introduced by Bennett has been studied in Bennett, B.M. (1977, 1978).

In this chapter, we first look at properties satisfied by compound distributions belonging to the ageing class introduced by Ahmed et al. (2008). We then introduce a possibly new definition of multivariate CV and compute CVs of several random vectors and discuss some consequences of the new definition, which needs further exploration.
5.2 CV of compound distributions

5.2.1 CV of geometric compound random variable

As in Ahmed et al. (2008), we define two classes of rvs, \( \mathcal{K} = \{X : CV(X) \leq 1\} \), and \( \mathcal{K}' = \{X : CV(X) \geq 1\} \). Also, let \( X_1, X_2, \ldots \) be iid lifetime rvs with common df \( F \), and finite mean and \( N \) denote a shifted geometric rv, independent of \( X_1, X_2, \ldots \) with pmf \( P(N = k) = pq^{k-1}, k = 1, 2, \ldots, 0 < p < 1, q = 1 - p \). Note that \( E(N) = \frac{1}{p} \), \( E(N^2) = \frac{2 - p}{p^2} \), \( V(N) = \frac{q}{p^2} \), so that the coefficient of variation of \( N \) is given by \( CV(N) = \sqrt{q} \leq 1 \), and hence \( N \in \mathcal{K} \). The geometric compound rv is defined as \( S_N = X_1 + X_2 + \cdots + X_N \). To find the CV of \( S_N \) we first find its expectation and variance. We have

\[
E(S_N) = E_N \left( E(X_1 + X_2 + \cdots + X_N | N) \right) = \sum_{k=1}^{\infty} pq^{k-1} E(X_1 + X_2 + \cdots + X_k),
\]

\[
= p \sum_{k=1}^{\infty} q^{k-1} k E(X_1) = p E(X_1) \sum_{k=1}^{\infty} k q^{k-1} = \frac{1}{p} E(X_1) = E(X_1) E(N), \text{ and}
\]

\[
E(S_N^2) = E_N \left( E((X_1 + X_2 + \cdots + X_N)^2 | N) \right) = \sum_{k=1}^{\infty} pq^{k-1} E((X_1 + X_2 + \cdots + X_k)^2),
\]

\[
= \sum_{k=1}^{\infty} pq^{k-1} \left( k E(X_1^2) + k(k - 1) E(X_1)^2 \right) = \frac{p E(X_1^2)}{p^2} + \frac{2q p E(X_1)^2}{p^2}
\]

\[
= \frac{p E(X_1^2) + 2q E(X_1)^2}{p^2} = E(N) E(X_1^2) + 2V(N) E(X_1)^2, \text{ so that}
\]

\[
V(S_N) = E(S_N^2) - E^2(S_N) = \frac{p E(X_1^2)}{p^2} + \frac{2q p E(X_1)^2}{p^2} - \frac{E^2(X_1)}{p^2},
\]

\[
= \frac{p E(X_1^2) + E^2(X_1)(2q - 1)}{p^2} = \frac{p E(X_1^2) + (1 - 2p) E^2(X_1)}{p^2},
\]

\[
= \frac{p E(X_1^2) + q E^2(X_1) - p E^2(X_1)}{p^2} = \frac{p[E(X_1^2) - E^2(X_1)] + q E^2(X_1)}{p^2},
\]

\[
= \frac{V(X_1)}{p} + \frac{q}{p^2} E^2(X_1) = E(N) V(X_1) + V(N) E^2(X_1).
\]
So, the CV of $S_N$ is given by

$$CV(S_N) = \left(\sqrt{\frac{V(X_1)}{p} + \frac{q}{p^2}E^2(X_1)}\right) \frac{p}{E(X_1)} = \sqrt{\frac{pV(X_1)}{E^2(X_1)} + q}$$

We have the following theorem.

**Theorem 5.2.1.** $X_1 \in \mathcal{K} \iff S_N \in \mathcal{K}.$

**Proof.** If $X_1 \in \mathcal{K},$ then $CV(X_1) \leq 1 \Rightarrow pCV^2(X_1) + q \leq 1 \Rightarrow CV(S_N) \leq 1 \Rightarrow S_N \in \mathcal{K}.$ Conversely, if $S_N \in \mathcal{K},$ then $\sqrt{pCV^2(X_1) + q} \leq 1 \Rightarrow pCV^2(X_1) + q \leq 1 \Rightarrow CV(X_1) \leq 1 \Rightarrow X_1 \in \mathcal{K}.$ \hfill \Box

**5.2.2 On CV of general compound random variable**

We assume that $P(N = k) = p_k, \ k = 1, 2, \ldots, 0 < p_k < 1; \sum_{k=1}^{\infty} p_k = 1;$ with probability generating function (pgf) $P(s) = E(s^N) = \sum_{k=1}^{\infty} p_k s^k, \ 0 \leq s \leq 1.$ We then have

$$E(S_N) = E_N\left(\left[\sum_{i=1}^{k} X_i\right] | N\right) = \sum_{k=1}^{\infty} p_k k E(X_1) = E(X_1) E(N), \ \text{and}$$

$$E(S_N^2) = E_N\left(\left[\left(\sum_{i=1}^{k} X_i\right)^2\right] | N\right) = \sum_{k=1}^{\infty} p_k E(\sum_{i=1}^{k} X_i^2),$$

$$= \sum_{k=1}^{\infty} p_k \left(\sum_{i=1}^{k} X_i^2 + \sum_{i \neq j}^{k} X_i X_j\right) = \sum_{k=1}^{\infty} k p_k E(X_1^2) + \sum_{k=1}^{\infty} k(k-1)p_k E^2(X_1),$$

$$= E(X_1^2) E(N) + E^2(X_1) \left( E(N^2) - E(N) \right), \ \text{so that}$$

$$V(S_N) = E(X_1^2) E(N) + E^2(X_1) \left( E(N^2) - E(N) \right) - \left( E(X_1) E(N) \right)^2,$$

$$= \left( E(X_1^2) - E^2(X_1) \right) E(N) + \left( E(N^2) - E^2(N) \right) E^2(X_1),$$

$$= V(X_1) E(N) + V(N) E^2(X_1).$$
We therefore have

\[
CV(S_N) = \sqrt{\frac{V(S_N)}{E^2(S_N)}} = \sqrt{\frac{V(X_1)E(N) + V(N)E^2(X_1)}{(E(X_1)E(N))^2}} = \sqrt{\frac{CV^2(X_1)}{E(N)}} + CV^2(N).
\]

We have the following theorem which relates the rvs \( N \) and \( S_N \) and the results do not depend on the df \( F \) of the summands \( X_1, X_2, \ldots \).

**Theorem 5.2.2.** (i) If \( N \in \mathcal{K}' \), then \( S_N \in \mathcal{K}' \). And, (ii) if \( S_N \in \mathcal{K} \), then \( N \in \mathcal{K} \).

**Proof.** Since \( CV(S_N) = \sqrt{\frac{CV^2(X_1)}{E(N)}} + CV^2(N) \), if \( N \in \mathcal{K}' \), then \( CV(N) \geq 1 \) and hence \( CV(S_N) \geq 1 \) which means that \( S_N \in \mathcal{K}' \), proving (i). Similarly, if \( S_N \in \mathcal{K} \), then \( CV(S_N) \leq 1 \) and hence \( CV(N) \leq 1 \) so that \( N \in \mathcal{K} \), proving (ii).

We have the following situations, by definition of CV:

1. \( CV(N) \leq 1 \iff V(N) \leq E^2(N) \)
2. \( CV(X_1) \leq 1 \iff V(X_1) \leq E^2(X_1) \)
3. \( CV(S_N) \leq 1 \iff V(S_N) \leq E^2(S_N) \)

In view of the above observations we have the following lemma.

**Lemma 5.2.1.** If (1) holds, then (3) holds. And, if (3) holds, then (1) holds.

**Proof.** If (1) holds, we have \( V(S_N) - E^2(S_N) = E(N)V(X_1) + V(N)E^2(X_1) - E^2(N)E^2(X_1) \geq E(N)V(X_1) + E^2(N)E^2(X_1) - E^2(N)E^2(X_1) = E(N)V(X_1) \geq 0 \), so that (3) holds irrespective of what \( CV(X_1) \) is. Now, if (3) holds, then \( V(S_N) - E^2(S_N) = E(N)V(X_1) + V(N)E^2(X_1) - E^2(N)E^2(X_1) \) so that \( (V(N) - E^2(N))E^2(X_1) = V(S_N) - E^2(S_N) - E(N)V(X_1) \leq 0 \). Hence (1) holds.

We now look at a couple of examples. Note that in the case of the shifted geometric distribution, we have already seen \( CV(N) = \sqrt{q} \leq 1 \). In the case of shifted Poisson distribution, we have \( P(N = k) = e^{-\lambda} \frac{\lambda^k}{k!} \), \( k = 1, 2, \ldots \) and \( E(N) = \lambda + 1 \), \( V(N) = \lambda \).

So, \( CV(N) = \frac{\sqrt{\lambda}}{\lambda + 1} < 1 \), \( \lambda > 0 \).
5.3 On CV of equilibrium rv and its equilibrium rv

In this section, we compute the CV of the equilibrium rv generated by a lifetime rv and study some of its properties. We also look at the equilibrium rv of the equilibrium rv and study some of its properties. Let \( X_1 \) be the equilibrium rv generated by the lifetime rv \( X \) with df \( F \) and mean \( \mu(F) \) finite. The df of \( X_1 \) is given by \( F_{X_1}(x) = \frac{\int_0^x F(u) \, du}{\mu(F)} \), \( 0 \leq x \). We have

\[
E(X_1) = \mu(F_{X_1}) = \int_0^\infty F_{X_1}(u) \, du = \int_0^\infty \frac{1}{\mu(F)} \int_u^\infty F(v) \, dv \, du, \\
= \frac{1}{\mu(F)} \int_0^\infty \int_{v=u}^\infty F(v) \, dv \, du = \frac{1}{\mu(F)} \int_0^\infty \int_u^v F(v) \, dv \, du, \\
= \frac{1}{\mu(F)} \int_0^\infty \int_u^v v \, F(v) \, dv \, du = \frac{1}{\mu(F)} \int_0^\infty \int_{v=u}^\infty dF(u) \, dv, \\
= \frac{1}{\mu(F)} \int_0^\infty \int_{v=u}^\infty v \, dF(v) \, du = \frac{1}{\mu(F)} \int_0^\infty u^2 \, dF(u) = \frac{\mu_2(F)}{2\mu(F)}; \text{ and} \\
\mu_2(F_{X_1}) = \int_0^\infty u^2 \, dF_{X_1}(u) = \int_0^\infty \frac{u^2}{\mu(F)} \, du = \frac{1}{\mu(F)} \int_0^\infty u^2 \, dF(v), \\
= \frac{1}{\mu(F)} \int_0^\infty \frac{u^3}{3} \, dF(u) = \frac{\mu_3(F)}{3\mu(F)}; \text{ so that} \\
V(X_1) = \mu_2(F_{X_1}) - \mu^2(F_{X_1}) = \frac{\mu_3(F)}{3\mu(F)} - \left( \frac{\mu_2(F)}{2\mu(F)} \right)^2; \text{ and} \\
CV(F_{X_1}) = \sqrt{\frac{\mu_3(F)}{3\mu(F)} - \left( \frac{\mu_2(F)}{2\mu(F)} \right)^2} = \sqrt{\frac{4\mu(F)\mu_3(F)}{3\mu_2^2(F)}} - 1.
\]

So, we have

\[
CV(F_1) \leq 1 \iff \frac{\mu(F)\mu_3(F)}{\mu_2^2(F)} \leq \frac{3}{2}.
\]
Similarly, if $X_2$ is the equilibrium rv of the equilibrium rv $X_1$, then

$$
\mu(F_{X_2}) = \frac{\mu_2(F_{X_1})}{2\mu(F_{X_1})} = \frac{\mu_3(F)}{3\mu(F)} \frac{\mu_4(F)}{\mu_2(F)} = \frac{\mu_3(F)}{3\mu_2(F)}, \quad \text{and}
$$

$$
\mu_2(F_{X_2}) = \frac{\mu_3(F_{X_1})}{3\mu(F_{X_1})} = \frac{\mu_4(F)}{4\mu(F)} \frac{2\mu(F)}{3\mu_2(F)} = \frac{\mu_4(F)}{6\mu_2(F)}, \quad \text{so that}
$$

$$
V(F_{X_2}) = \mu_2(F_{X_2}) - \mu^2(F_{X_2}) = \frac{\mu_4(F)}{6\mu_2(F)} - \left( \frac{\mu_3(F)}{3\mu_2(F)} \right)^2 \quad \text{and}
$$

$$
CV^2(F_{X_2}) = \frac{V(F_{X_2})}{\mu^2(F_{X_2})} = \left\{ \frac{\mu_4(F)}{6\mu_2(F)} - \left( \frac{\mu_3(F)}{3\mu_2(F)} \right)^2 \right\} \left( \frac{3\mu_2(F)}{\mu_3(F)} \right)^2,
$$

$$
= \frac{3\mu_4(F)}{2\mu_3^2(F)} - 1.
$$

Hence $CV(F_2) \leq 1$ iff $\frac{\mu_4(F)}{2\mu_3^2(F)} \leq \frac{4}{3}$ in which case $F_2 \in K$.

5.4 Bivariate CV

In this section, we give a definition of CV in the bivariate case which appears to be new. We then discuss a couple of consequences of the new definition and give a couple of examples. Detailed study of the new definition and its ramifications is proposed to be done in future. The bivariate definition is extended to the multivariate case in the next section. This concept of CV may be useful to compare consistency of data sets in the bivariate and multivariate cases.

Let the random vector $X = (X,Y)$ have the bivariate df $F$, with $E(X) \neq 0$, $E(Y) \neq 0$, $\rho$ equal to the correlation coefficient between $X$ and $Y$, so that

$$
E(X) = \left( E(X), E(Y) \right) \quad \text{and}
$$

$$
V(X) = E\left( (X - E(X))(X - E(X))' \right) = \begin{pmatrix} V(X) & Cov(X,Y) \\ Cov(Y,X) & V(Y) \end{pmatrix}.
$$
The CV of $X$ is defined as

$$CV^2(X) = \left( \frac{1}{|E(X)|} \right) \left( \frac{1}{|E(Y)|} \right) \left( \begin{array}{cc} V(X) & Cov(X,Y) \\ Cov(Y,X) & V(Y) \end{array} \right) \left( \frac{1}{|E(X)|} \right),$$

$$= \left( \frac{1}{|E(X)|} \right) \left( \frac{1}{|E(Y)|} \right) \left( \begin{array}{cc} V(X) + \frac{Cov(X,Y)}{|E(X)|} & \frac{Cov(X,Y)}{|E(Y)|} \\ \frac{Cov(Y,X)}{|E(X)|} & \frac{V(Y)}{|E(Y)|} \end{array} \right),$$

$$= \frac{V(X)}{E^2(X)} + \frac{Cov(X,Y)}{|E(X)|E(Y)} + \frac{Cov(X,Y)}{|E(Y)|E(Y)} + \frac{V(Y)}{E^2(Y)} = CV^2(X) + \frac{2}{|E(X)|E(Y)} Cov(X,Y) + CV^2(Y),$$

$$= CV^2(X) + 2\rho CV(X)CV(Y) + CV^2(Y).$$

Note that the above definition is equivalent to the following:

$$CV^2(X) = \left( CV(X) \right) R(X) \left( CV(X) \right)^t,$$

$$= \left( CV(X) \ CV(Y) \right) \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \left( CV(X) \ CV(Y) \right),$$

$$= \left( CV(X) \ CV(Y) \right) \begin{pmatrix} CV(X) + \rho CV(Y) \\ \rho CV(X) + CV(Y) \end{pmatrix},$$

$$= CV^2(X) + 2\rho CV(X)CV(Y) + CV^2(Y).$$

As a consequence of the new definition, we have the following three cases:

(i) $\rho = 0 \Rightarrow CV^2(X) = CV^2(X) + CV^2(Y),$

(ii) $\rho = 1 \Rightarrow CV^2(X) = (CV(X) + CV(Y))^2,$

(iii) $\rho = -1 \Rightarrow CV^2(X) = (CV(X) - CV(Y))^2.$

Since $-1 \leq \rho \leq 1$, we have $-2CV(X)CV(Y) \leq 2\rho CV(X)CV(Y) \leq 2CV(X)CV(Y).$

We now give a few illustrations of the new definition.
Examples illustrating the new definition

**Bivariate Normal Distribution**

If \( \mathbf{X} = (X, Y) \) has the bivariate normal distribution \( \text{BVN}(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) \), with \( \mu_1 \neq 1, \mu_2 \neq 0 \), then 
\[
CV^2(X) = \frac{\sigma_2^2}{\mu_1^2}, \quad CV^2(Y) = \frac{\sigma_1^2}{\mu_2^2},
\]
and hence 
\[
CV^2(X) = \frac{\sigma_1^2}{\mu_1^2} + \frac{\sigma_2^2}{\mu_2^2} + 2\rho \frac{\sigma_1 \sigma_2}{|\mu_1 \mu_2|}.
\]

**Gumbel Type I Distribution**

As in Balakrishnan and Lai (2009), the df of the bivariate Gumbel type I rv \( \mathbf{X} = (X, Y) \) is given by 
\[
H(x, y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+\theta xy)}, \quad x > 0, \ y > 0, \ 0 \leq \theta < 1.
\]
We have the univariate marginals as 
\[
H_1(x) = 1 - e^{-x}, \quad E(X) = 1, \quad V(X) = 1, \quad E(X^2) = 2,
\]
and 
\[
CV^2(X) = 1 = CV^2(Y).
\]
We then have 
\[
CV^2(X) = CV^2(X) + CV^2(Y) + 2\rho CV(X)CV(Y) = 1 + 1 + 2\left(1 + \int_0^\infty \frac{e^{-y}}{(1 + \theta y)} dy\right) = 2 - 2 + 2 \int_0^\infty \frac{e^{-y}}{(1 + \theta y)} dy = 2 \int_0^\infty \frac{e^{-y}}{(1 + \theta y)} dy = 2
\]
whenever \( \theta = 0 \).

**Bivariate Lomax Distribution**

As in Balakrishnan and Lai (2009), the sf and pdf of the bivariate Lomax rv \( \mathbf{X} = (X, Y) \) are respectively given by 
\[
H(x, y) = (1 + ax + by + \theta xy)^c, \quad 0 \leq \theta \leq (c + 1)ab, \ a, b, c \geq 0,
\]
and 
\[
h(x, y) = \frac{c(a + \theta y + ax + by)}{(1 + ax + by + \theta xy)^{c+2}}.
\]
Then 
\[
E(X) = \frac{1}{a(c - 1)}, \quad E(Y) = \frac{1}{b(c - 1)},
\]
c > 1, and 
\[
V(X) = \frac{c}{(c - 1)^2}, \quad V(Y) = \frac{c}{(c - 1)^2(c - 2)}, \ c > 2.
\]
We also have 
\[
CV^2(X) = CV^2(Y) = \frac{c}{(c - 2)}, \quad CV^2(X) = \frac{2c}{(c - 2)(1 + \rho)}.
\]
5.5 Multivariate CV

Generalizing the definition in the previous section, if $X = (X_1, X_2, \ldots, X_d)$ with $\mu_i = E(X_i) \neq 0$, $\sigma_{ij} = \text{Cov}(X_i, X_j)$, $\rho_{ij} = \frac{\text{Cov}(X_i, X_j)}{\sqrt{\sigma_{ii}\sigma_{jj}}}$, we define

$$CV^2(X) = \left( \begin{array}{c|c|c|c|c} \frac{1}{|\mu_1|} & \frac{1}{|\mu_2|} & \cdots & \frac{1}{|\mu_d|} \\ \end{array} \right) \Sigma \left( \begin{array}{c|c|c|c|c} \frac{1}{|\mu_1|} \\ \hline \frac{1}{|\mu_2|} \\ \hline \vdots \\ \hline \frac{1}{|\mu_d|} \\ \end{array} \right)$$

$$= \left( \begin{array}{c|c|c|c|c} \frac{1}{|\mu_1|} & \frac{1}{|\mu_2|} & \cdots & \frac{1}{|\mu_d|} \\ \end{array} \right) \left( \begin{array}{cccc} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \cdots & \sigma_{dd} \\ \end{array} \right) \left( \begin{array}{c|c|c|c|c} \frac{1}{|\mu_1|} \\ \hline \frac{1}{|\mu_2|} \\ \hline \vdots \\ \hline \frac{1}{|\mu_d|} \\ \end{array} \right)$$

$$= \left( \sum_{j=1}^{d} \frac{\sigma_{j1}}{|\mu_j|} \right) \frac{\sigma_{j2}}{|\mu_j|} \cdots \left( \sum_{j=1}^{d} \frac{\sigma_{jd}}{|\mu_j|} \right) \left( \begin{array}{c|c|c|c|c} \frac{1}{|\mu_1|} \\ \hline \frac{1}{|\mu_2|} \\ \hline \vdots \\ \hline \frac{1}{|\mu_d|} \\ \end{array} \right)$$

$$= \sum_{k=1}^{d} \sum_{j=1}^{d} \frac{\sigma_{jk}}{|\mu_j \mu_k|} = \sum_{k=1}^{d} \frac{\sigma_{kk}}{|\mu_k|} + \sum_{k=1}^{d} \sum_{j \neq k=1}^{d} \frac{\sigma_{jk}}{|\mu_j \mu_k|}$$

$$= \sum_{k=1}^{d} CV(X_k)^2 + \sum_{k=1}^{d} \sum_{j \neq k=1}^{d} \rho_{jk} \sqrt{\frac{\sigma_{jj} \sigma_{kk}}{|\mu_j \mu_k|}}$$

$$= \sum_{k=1}^{d} CV(X_k)^2 + \sum_{k=1}^{d} \sum_{j \neq k=1}^{d} \rho_{jk} CV(X_j) CV(X_k).$$

Note that the above definition is equivalent to the following:
\[ CV^2(X) = \left( CV(X_1) \ CV(X_2) \ \ldots \ CV(X_d) \right) R \left( \begin{array}{c} CV(X_1) \\ CV(X_2) \\ \ldots \\ CV(X_d) \end{array} \right) \]

\[ = \left( CV(X_1) \ CV(X_2) \ \ldots \ CV(X_d) \right) \left( \begin{array}{cccc} 1 & \rho_{12} & \ldots & \rho_{1d} \\ \rho_{21} & 1 & \ldots & \rho_{2d} \\ \rho_{d1} & \rho_{d2} & \ldots & 1 \end{array} \right) \left( \begin{array}{c} CV(X_1) \\ CV(X_2) \\ \ldots \\ CV(X_d) \end{array} \right) \]

\[ = \left( \sum_{j=1}^{d} \rho_{j1} CV(X_j) \ \sum_{j=1}^{d} \rho_{j2} CV(X_j) \ \ldots \ \sum_{j=1}^{d} \rho_{jd} CV(X_j) \right) \left( \begin{array}{c} CV(X_1) \\ CV(X_2) \\ \ldots \\ CV(X_d) \end{array} \right) \]

\[ = \sum_{k=1}^{d} \sum_{j=1}^{d} \rho_{jk} CV(X_j) CV(X_k) \]

\[ = \sum_{k=1}^{d} CV(X_k)^2 + \sum_{k=1}^{d} \sum_{j=1}^{d} \sum_{j \neq k}^{d} \rho_{jk} CV(X_j) CV(X_k). \]

**Remark 5.5.1.**

(i) Note that the definition given here does not require the variance covariance matrix \( \Sigma \) to be non-singular.

(ii) \( \rho_{jk} \equiv 0 \Rightarrow CV(X) = \sum_{k=1}^{d} CV(X_k)^2. \)

(ii) \( \rho_{jk} \equiv 1 \Rightarrow CV(X) = \left( \sum_{k=1}^{d} CV(X_k) \right)^2. \)

In the next section, we obtain CV of the multinomial distribution as an illustration. Other properties of the new definition will be explored in future.
CV of Multinomial distribution

Let \( X = (X_1, X_2, \ldots, X_{k-1}) \) denote a multinomial random vector with

\[
P(X_1 = x_1, X_2 = x_2, \ldots, X_{k-1} = x_{k-1}) = \frac{n!p_1^{x_1}p_2^{x_2}\cdots p_{k-1}^{x_{k-1}}}{x_1!x_2!\cdots(n-x_1, \ldots, x_{k-1})!},
\]
so that

\[
E(X_j) = np_j, \quad V(X_j) = np_j(1-p_j), \quad \text{Cov}(X_i, X_j) = -np_ip_j,
\]

\[
\rho_{ij} = -\left(\frac{p_ip_j}{(1-p_i)(1-p_j)}\right)^{\frac{1}{2}}, \quad CV^2(X_j) = \frac{np_j(1-p_j)}{n^2p_j^2} = \frac{1-p_j}{np_j} = \frac{1-p_j}{EX_j}, \quad \text{and hence}
\]

\[
CV^2(X) = \sum_{i=1}^{k-1} \left(\frac{1-p_j}{np_j}\right) + \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \frac{-(p_ip_j)^{\frac{1}{2}}}{(1-p_i)^{\frac{1}{2}}(1-p_j)^{\frac{1}{2}}} \frac{(1-p_i)^{\frac{1}{2}}(1-p_j)^{\frac{1}{2}}}{(np_i)^{\frac{1}{2}}(np_j)^{\frac{1}{2}}},
\]

\[
= \frac{1}{n} \sum_{i=1}^{k-1} \left(\frac{1}{p_j} - 1\right) - \frac{1}{n} (k-1)(k-2),
\]

\[
= \frac{1}{n} \left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{k-1}} - (k-1)(1+k-2)\right),
\]

\[
= \frac{1}{n} \left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{k-1}} - (k-1)^2\right),
\]

\[
= \frac{1}{EX_1} + \frac{1}{EX_2} + \cdots + \frac{1}{EX_{k-1}} - \frac{(k-1)^2}{n}.
\]