Chapter 2

On reliability concepts for discrete lifetime random variables

2.1 Introduction

Discrete random variables (rvs) appear naturally while studying reliability of products and / or processes which work in cycles / periods wherein the random phenomenon occurs as counts. For example, rotations of a motor, shifts in a factory, etc. In this chapter, in the first section, we look at various definitions of reliability concepts given by several authors. The consequences of the definitions on several discrete distributions is reported in Section 2.2. Our study necessitates the need for unifying these definitions and clarifying contradictory and incomplete definitions. It is a practice in reliability to state and prove results for the discrete case separately. In the third and last section of this chapter, we list out a few definitions of stochastic orders among discrete rvs and prove some new results. The continuous counterparts of some of these results are reported in the next chapter.

Let X be an rv taking values 1, 2, 3, …, with respective probabilities $p_1, p_2, p_3, \ldots$.
Sometimes, it is convenient to include 0 as a value for the rv $X$. Unless stated otherwise, except in the last section of this chapter, we do not include 0. Some of the definitions have to be modified if $X$ takes the value 0.

## 2.2 Examples and discussion

In this section, we look at several examples of discrete rvs and classify them according to the definitions given in Section 1.2.3.

1. **Discrete uniform:**

   (a) **Probability mass function:** Pmf is $p_k = \frac{1}{n}$ for $k = 1, 2, 3, \ldots n$.

   (b) **Reliability function:** According to Xie (2002), Cyril and Olivier (2003), reliability function $R_k = P(X > k) = \sum_{j=k+1}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k} P(X = j) = 1 - \frac{k}{n}$. According to Kemp (2004), survival function $S_k = \sum_{j=k}^{\infty} P(X = j) = 1 - \frac{k}{n}$ with $S_0 = 1$.

   (c) **IFR / DFR:** According to Xie (2002), hazard rate is $r_k = \ln \left( \frac{R_{k-1}}{R_k} \right) = \ln \left( 1 + \frac{1}{n-k} \right)$. As $k$ increases $\ln \left( 1 + \frac{1}{n-k} \right)$ increases and therefore $X$ is IFR. According to Cyril and Olivier (2003), we have $\lambda_k = \frac{P(X=k)}{P(X \geq k)} = \frac{1}{n-k+1}$, and $\frac{p_{k+1}}{p_k} = \begin{cases} 1 & \text{if } 1 \leq k \leq n-1, \\ 0 & \text{if } k = n. \end{cases}$ So, $\frac{p_{k+1}}{p_k}$ decreases as $k$ increases so that $\lambda_k$ increases. Therefore $X$ is IFR. According to Kemp (2004), hazard rate $\lambda_k = \frac{P(X=k)}{P(X \geq k)} = \frac{1}{n-k+1}$. We have $\eta(k) = \frac{p_k - p_{k+1}}{p_k} = \begin{cases} 0 & \text{if } 1 \leq k \leq n-1, \\ 1 & \text{if } k = n; \end{cases}$ $\Delta \eta(k) = \eta(k+1) - \eta(k) = \begin{cases} 0 & \text{if } 1 \leq k \leq n-2, \\ 1 & \text{if } k = n-1, \\ \text{not defined} & \text{if } k = n. \end{cases}$ Therefore we cannot say anything about $X$ being IFR.
(d) **IFRA**: According to Xie (2002), 
\[ \frac{\ln R_k}{k} - \frac{\ln R_j}{j} = \ln \left( \frac{n-k}{n} \right) - \ln \left( \frac{n-j}{n} \right) \leq 0 \]
and hence discrete uniform is IFRA. According to Kemp (2004), 
\[(k+1)H_{k-1} = \frac{k}{n-k+1} - \sum_{j=1}^{k-1} \frac{1}{n-j+1} > 0. \]
Therefore, discrete uniform is IFRA.

(e) **NBU**: According to Xie (2002), we have 
\[ R_j R_k - R_{j+k} = \frac{kj}{n^2} \geq 0. \]
Therefore, discrete uniform is NBU. According to Kemp (2004), 
\[ S_{t+k} - S_t S_k = \frac{-n-kt-1+k+t}{n^2} < 0, \]
so that discrete uniform is NBU.

(f) **NBUE**: According to Kemp (2004), we have 
\[ \sum_{j=0}^{\infty} S_{t+j} - S_t \sum_{j=0}^{\infty} S_j = \sum_{j=0}^{\infty} \frac{n-t-j+1}{n} - \frac{n-t+1}{n} \sum_{j=0}^{\infty} \frac{n-j+1}{n} < 0, \]
so that discrete uniform is NBUE.

2. **Shifted geometric:**

   (a) **Pmf**: Pmf is 
   \[ p_k = p(1-p)^{k-1}, k = 1, 2, 3, \ldots, 0 < p < 1. \]

   (b) **Reliability function**: According to Xie (2002), Cyril and Olivier (2003), 
   reliability function 
   \[ R_k = P(X > k) = \sum_{j=k+1}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k} P(X = j) = 1 - \sum_{j=1}^{k} (1-p)^{j-1} = q^k. \]
   According to Kemp (2004), survival function 
   \[ R_{k-1} = S_k = \sum_{j=k}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k-1} P(X = j) = 1 - \sum_{j=1}^{k-1} p(1-p)^{j-1} = q^{k-1} \]
   and 
   \[ S_0 = 1. \]

   (c) **IFR/DFR**: According to Xie (2002), hazard rate is 
   \[ r_k = \ln \left( \frac{R_{k+1}}{R_k} \right) = -\ln(1-p), \]
   so that as \( k \) increases \( r_k \) remains constant and hence shifted geometric is both IFR and DFR. According to Cyril and Olivier (2003), \[ \lambda_k = \frac{P(X=k)}{P(X \geq k)} = p \]
   and \[ \frac{P_{k+1}}{P_k} = 1 - p, \]
   which is a constant and hence shifted geometric is both IFR and DFR. According to Kemp (2004), \[ \lambda_k = \frac{P(X=k)}{P(X \geq k)} = p, \]
   and \[ \eta(k) = \frac{p_k - p_{k+1}}{p_k} = p, \]
   \[ \Delta \eta(k) = \eta(k+1) - \eta(k) = p - p = 0, \]
   so that shifted geometric is both IFR and DFR.
(d) **IFRA/DFRA:** According to Xie (2002), we have $\ln R_k - \ln R_j = \ln ((1 - p)^k) - \ln ((1 - p)^j) = 0$. So, shifted geometric is both IFRA and DFRA.

But, according to Kemp (2004), $kH_k - (k+1)H_{k-1} = k^2 p - (k+1)(k-1)p = p > 0$, and hence shifted geometric is IFRA but not DFRA.

(e) **NBU/NWU:** According to Xie (2002), we have $R_j - R_k - R_{j+k} = (1 - p)^j - (1 - p)^k - (1 - p)^{k+j} = 0$ and hence shifted geometric is both NBU and NWU. But according to Kemp (2004), $S_{t+k} - S_t S_k = -p(1 - p)^{t+k-2} < 0$ and therefore it is NBU.

(f) **NBUE/NWUE:** According to Kemp (2004), we have $\sum_{j=0}^{\infty} S_{t+j} - S_t \sum_{j=0}^{\infty} S_j = \sum_{j=0}^{\infty} \frac{n-t-j+1}{n} - \sum_{j=0}^{\infty} \frac{n-j+1}{n} < 0$, so that shifted geometric is NBUE.

3. **Shifted Poisson:**

(a) **Pmf:** Pmf is $p_k = \frac{\exp^{-\lambda} \lambda^{k-1}}{(k-1)!}$, $k = 1, 2, 3, \ldots$, $\lambda > 0$.

(b) **Reliability function:** According to Xie (2002), Cyril and Olivier (2003), reliability function $R_k = P(X > k) = \sum_{j=k+1}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k} P(X = j) = 1 - \sum_{j=1}^{k} \frac{\exp^{-\lambda} \lambda^{j-1}}{(j-1)!}$. And according to Kemp (2004), survival function $R_{k-1} = S_k = \sum_{j=k}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k-1} P(X = j) = 1 - \sum_{j=1}^{k-1} \frac{\exp^{-\lambda} \lambda^{j-1}}{(j-1)!}$ with $S_0 = 1$.

(c) **IFR:** According to Cyril and Olivier (2003), hazard rate is $\lambda_k = \frac{P(X=k)}{P(X\geq k)} = \frac{\exp^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}}{\sum_{j=k}^{\infty} \exp^{-\lambda} \frac{\lambda^{j-1}}{(j-1)!}}$. We have $\frac{p_{k+1}}{p_k} = \frac{\lambda}{k}$ and as $k$ increases, $\frac{p_{k+1}}{p_k}$ decreases, so that $p_k$ is log-concave. Therefore, shifted Poisson is IFR. According to Kemp (2004), we have $\lambda_k = \frac{P(X=k)}{P(X \geq k)} = \frac{\exp^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}}{\sum_{j=k}^{\infty} \exp^{-\lambda} \frac{\lambda^{j-1}}{(j-1)!}}$. And, $\eta(k) = \frac{p_k - p_{k+1}}{p_k} = \frac{k-\lambda}{k}$ and $\Delta \eta(k) = \eta(k+1) - \eta(k) = \frac{\lambda}{k(k+1)} > 0$. Therefore, $\lambda_k$ is nondecreasing and shifted Poisson is IFR.
15

(d) **IFRA:** According to Xie (2002), we have
\[\ln R_k - \ln R_j = \frac{1}{k} \ln \sum_{i=k}^{\infty} \frac{\exp^{-\lambda i}}{(i-1)!} \]
\[-\frac{1}{j} \ln \left( \sum_{i=j}^{k-1} \frac{\exp^{-\lambda i}}{(i-1)!} + \sum_{i=k}^{\infty} \frac{\exp^{-\lambda i}}{(i-1)!} \right), \leq 0. \]
Therefore, shifted Poisson is IFRA.

4. Shifted binomial:

(a) **Pmf:** Pmf is
\[p_k = \left( \begin{array}{c} n \\ k-1 \end{array} \right) p^{k-1} q^{n-k+1}, k = 1, 2, 3, \ldots, n + 1, 0 < p < 1, q = 1 - p. \]

(b) **Reliability function:** According to Xie (2002) and Cyril and Olivier (2003), reliability function
\[R_k = P(X > k) = \sum_{j=k+1}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k} P(X = j) = 1 - \sum_{j=1}^{k} \left( \begin{array}{c} n \\ j-1 \end{array} \right) p^{j-1} q^{n-j+1}. \]
According to Kemp (2004), survival function
\[R_{k-1} = S_k = P(X \geq k) = \sum_{j=k}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k-1} P(X = j) = 1 - \sum_{j=1}^{k-1} \left( \begin{array}{c} n \\ j-1 \end{array} \right) p^{j-1} q^{n-j+1} \]
and \(S_0 = 1.\)

(c) **IFR:** According to Cyril and Olivier (2003), hazard rate is \(\lambda_k = \frac{P(X=k)}{P(X \geq k)}\)
and we have \(\frac{p_{k+1}}{p_k} = \left( \begin{array}{c} n-k+1 \\ k \end{array} \right) \frac{p}{q} \) which decreases as \(k\) increases. Therefore shifted binomial is IFR. According to Kemp (2004), \(\lambda_k = \frac{P(X=k)}{P(X \geq k)}\)
\[= \left( \begin{array}{c} n \\ k-1 \end{array} \right) p^{k-1} q^{n-k+1} \]
and \(\eta(k) = \frac{p_{k+1} - p_k}{p_k} = \frac{q^{k} - p^{k+1}}{q^k} \]
\(\Delta \eta(k) = \eta(k + 1) - \eta(k) = \frac{1}{q} \left[ \frac{p_{k+1}}{p_k} \right] > 0. \)
Therefore, \(\lambda_k\) is nondecreasing and hence shifted binomial is IFR.

(d) **IFRA:** According to Xie (2002),
\[\ln R_k - \ln R_j = \frac{1}{k} \ln \sum_{i=k+1}^{\infty} \left( \begin{array}{c} n \\ i-1 \end{array} \right) p^{i-1} q^{n-i+1} \]
\[-\frac{1}{j} \ln \left( \sum_{i=j+1}^{k} \left( \begin{array}{c} n \\ i-1 \end{array} \right) p^{i-1} q^{n-i+1} + \sum_{i=k+1}^{\infty} \left( \begin{array}{c} n \\ i-1 \end{array} \right) p^{i-1} q^{n-i+1} \right) \]
\[\leq 0. \]
Therefore, shifted binomial is IFRA.

5. Shifted negative binomial:
6. S distribution:

(a) **Pmf:** Pmf is 
\[
p_k = \binom{k + r - 2}{k - 1} p^r q^{k-1}
\]  
for \( k = 1, 2, 3, \ldots, r > 0, 0 < p < 1 \).

(b) **Reliability function:** According to Xie (2002) and Cyril and Olivier (2003), reliability function 
\[
R_k = P(X > k) = \sum_{j=k+1}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k} P(X = j) = 1 - \sum_{j=1}^{k} \left( \frac{j + r - 2}{j - 1} \right) p^r q^{j-1}.
\]  
According to Kemp (2004), survival function \( S_{k-1} = S_k = P(X \geq k) = \sum_{j=k}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k-1} P(X = j) = 1 - \sum_{j=1}^{k-1} \left( \frac{j + r - 2}{j - 1} \right) p^r q^{j-1} \) and \( S_0 = 1 \).

(c) **IFR:** According to Cyril and Olivier (2003), hazard rate is 
\[
\lambda_k = \frac{P(X=k)}{P(X \geq k)}
\]  
and we have \( \frac{p_{k+1}}{p_k} = \left( 1 + \frac{r-1}{k} \right) q \) which decreases as \( k \) increases. Therefore shifted negative binomial is IFR. According to Kemp (2004), we have 
\[
\lambda_k = \frac{P(X=k)}{P(X \geq k)} = \left( \frac{k + r - 2}{k - 1} \right) p^r q^{k-1}.
\]  
Also, \( \eta(k) = \frac{p_k - p_{k+1}}{p_k} = \frac{k+q-kq-r}{k} \) and \( \Delta \eta(k) = \eta(k+1) - \eta(k) = \frac{q(r-1)}{k(k+1)} > 0 \). Therefore, \( \lambda_k \) is nondecreasing and shifted negative binomial is IFR.

(d) **IFRA:** According to Xie (2002), 
\[
\frac{\ln R_k}{k} - \frac{\ln R_i}{j} = \frac{1}{k} \ln \sum_{i=k+1}^{\infty} \left( \frac{i + r - 2}{i - 1} \right) p^r q^{i-1} \]  
\[
- \frac{1}{j} \ln \left( \sum_{i=j}^{k} \left( \frac{i + r - 2}{i - 1} \right) p^r q^{i-1} + \sum_{i=k+1}^{\infty} \left( \frac{i + r - 2}{i - 1} \right) p^r q^{i-1} \right)
\]  
\[
\leq 0. \text{ Therefore shifted negative binomial is IFRA.}
\]
and hence S distribution is IFR .

(d) **IFRA:** According to Xie (2002), $F$ is IFRA if

$$-rac{\ln R_j}{j} \leq -\frac{\ln R_k}{k}$$

or

$$\ln R_k - \ln R_j = \ln \left( \frac{\prod_{i=1}^{k}(1-p+p\Pi^i)}{\prod_{i=1}^{j}(1-p+p\Pi^i)} \right)^{\frac{1}{j}} \leq 0.$$  

This is equivalent to

$$\left( \prod_{i=1}^{k}(1-p+p\Pi^i) \right)^{\frac{1}{j}} \leq \left( \prod_{i=1}^{j}(1-p+p\Pi^i) \right)^{\frac{1}{k}}.$$  

Since there is no easy way to verify this for general $k$ and $j$, we looked at this for a few values for $k$ and $j$. When $k = 2, j = 1$, after simplification, we get

$$((1-p+p\Pi^2))^{\frac{1}{2}} \leq ((1-p+p\Pi^1))^{\frac{1}{1}},$$

which is true. For $k = 3, j = 2$, we have

$$\left( \prod_{i=1}^{k}(1-p+p\Pi^i) \right)^{\frac{1}{2}} \leq \left( \prod_{i=1}^{j}(1-p+p\Pi^i) \right)^{\frac{1}{j}},$$

$$\Rightarrow ((1-p+p\Pi^1)(1-p+p\Pi^2)(1-p+p\Pi^3))^{\frac{1}{3}} \leq ((1-p+p\Pi^1)(1-p+p\Pi^2))^{\frac{1}{2}},$$

$$\Rightarrow ((1-p+p\Pi^3))^{\frac{1}{3}} \leq ((1-p+p\Pi^2))^{\frac{2}{3}} - \frac{2}{3},$$

$$\Rightarrow ((1-p+p\Pi^3))^{\frac{1}{3}} \leq ((1-p+p\Pi^1)(1-p+p\Pi^2))^{\frac{1}{6}},$$

$$\Rightarrow ((1-p+p\Pi^3)^2) \leq ((1-p+p\Pi^1)(1-p+p\Pi^2)),$$

which is again true. When $k = 4, j = 1$ also the inequality is true. So, it appears that S distribution is IFRA, though, we are not able to prove it.

We could have also used a computer program to verify this.

(e) **NBU:** According to Xie (2002), $F$ is NBU if

$$R_j R_k \geq R_{j+k} \Leftrightarrow \prod_{i=1}^{j}(1-p+p\Pi^i) \prod_{i=1}^{k}(1-p+p\Pi^i) \geq \prod_{i=1}^{j+k}(1-p+p\Pi^i).$$

Again, it is not easy to verify this for $k$ and $j$, and when we verified for a few values of $k$ and $j$, we found that the inequality is satisfied and hence S distribution
appears to be NBU. For example, when \( k = 1, j = 1 \), we have

\[
\prod_{i=1}^{j}(1 - p + p\Pi^i) \prod_{i=1}^{k}(1 - p + p\Pi^i) \geq \prod_{i=1}^{j+k}(1 - p + p\Pi^i),
\]

\[
\Leftrightarrow (1 - p + p\Pi^1)(1 - p + p\Pi^1) \geq (1 - p + p\Pi^1)(1 - p + p\Pi^2),
\]

\[
\Leftrightarrow (1 - p + p\Pi^1) \geq (1 - p + p\Pi^2),
\]

which is true. When \( k = 1, j = 3 \),

\[
\prod_{i=1}^{j}(1 - p + p\Pi^i) \prod_{i=1}^{k}(1 - p + p\Pi^i) \geq \prod_{i=1}^{j+k}(1 - p + p\Pi^i),
\]

\[
\Leftrightarrow (1 - p + p\Pi^1)(1 - p + p\Pi^2)(1 - p + p\Pi^3)(1 - p + p\Pi^1)
\]

\[
\geq (1 - p + p\Pi^1)(1 - p + p\Pi^2)(1 - p + p\Pi^3)(1 - p + p\Pi^4),
\]

\[
\Leftrightarrow (1 - p + p\Pi^1) \geq (1 - p + p\Pi^4),
\]

which is true. When \( k = 5, j = 3 \) also the inequality is verified. We could have used a computer program to verify these.

### 2.2.1 Summary of examples

As is evident from the table below (where C. & O. ('03) refers to Cyril and Olivier (2003)), different definitions may lead to different conclusions and there is a need to unify and clarify these definitions.

<table>
<thead>
<tr>
<th>Discrete rv</th>
<th>Xie ('02)</th>
<th>C. &amp; O. ('03)</th>
<th>Kemp ('04)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>IFR, IFRA, NBU</td>
<td>IFR</td>
<td>IFRA, NBU, NBUE</td>
</tr>
<tr>
<td>Shifted geom.</td>
<td>IFR &amp; DFR, IFRA &amp;</td>
<td>IFR &amp; DFR</td>
<td>IFR &amp; DFR,</td>
</tr>
<tr>
<td>Shifted geom.</td>
<td>DFRA, NBU &amp; NWU</td>
<td>-</td>
<td>IFRA, NBU, NBUE</td>
</tr>
<tr>
<td>Shifted Poisson</td>
<td>IFRA</td>
<td>IFR</td>
<td>IFR</td>
</tr>
<tr>
<td>Shifted bin.</td>
<td>IFRA</td>
<td>IFR</td>
<td>IFR</td>
</tr>
<tr>
<td>Shifted neg. bin.</td>
<td>IFRA</td>
<td>IFR</td>
<td>IFR</td>
</tr>
<tr>
<td>S-distribution</td>
<td>IFR, IFRA (?), NBU (?)</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
2.3 Stochastic orderings and some new results

In this section we prove some new results.

2.3.1 Some new results

Let $X_Y = X - Y | X > Y$ with survival function $P(X_Y > k) = P(X - Y > k | X > y) = P(X > k + y | X > y) = P(X > k) P(Y = l) / P(Y = l)$, $k \geq 0$.

**Theorem 2.3.1.** For discrete rvs $X,Y$ taking non-negative integer values, $X$ is NBU implies that (i) $X_Y \leq_{st} X$, (ii) $X_Y \leq_{e} X$, (iii) $X_Y \leq_p X$, and (iv) $X$ is NBUL.

**Proof.** Let $X,Y$ be discrete rvs taking non-negative integer values, and $X$ be NBU.

\[
\begin{align*}
(i)\ X \text{ is NBU} & \Rightarrow P(X > k + l) \leq P(X > k)P(X > l), k, l \geq 0, \\
& \Rightarrow \sum_{l=0}^{\infty} P(X > k + l, Y = l) \leq \\
& \sum_{l=0}^{\infty} P(X > k) P(X > l, Y = l), k \geq 0, \\
& \Rightarrow \sum_{l=0}^{\infty} P(X > k + l | Y = l) P(Y = l) \leq \\
& P(X > k) \sum_{l=0}^{\infty} P(X > l | Y = l) P(Y = l), k \geq 0, \\
& \Rightarrow \sum_{l=0}^{\infty} P(X > k + l | Y = l) P(Y = l) / \sum_{l=0}^{\infty} P(X > l | Y = l) P(Y = l) \leq P(X > k), k \geq 0, \\
& \Rightarrow P(X_Y > k) \leq P(X > k), k \geq 0, \\
& \Rightarrow X_Y \leq_{st} X.
\end{align*}
\]
(ii) Arguing as above, we have

\[ X \text{ is } \text{NBU} \Rightarrow P(X_Y > k) \leq P(X > k), k \geq 0, \]

\[ \Rightarrow \sum_{k=i}^{\infty} P(X_Y > k) \leq \sum_{k=i}^{\infty} P(X > k), i \geq 0, \]

\[ \Rightarrow E(X_Y) \leq E(X), \]

\[ \Rightarrow X_Y \leq_c X. \]

(iii) Arguing as in the proof of (i) above, we have

\[ X \text{ is } \text{NBU} \Rightarrow P(X_Y > k) \leq P(X > k), k \geq 0, \]

\[ \Rightarrow \sum_{k=0}^{\infty} P(X_Y > k) z^k \leq \sum_{k=0}^{\infty} P(X > k) z^k, 0 < z \leq 1, \]

\[ \Rightarrow X_Y \leq_p X. \]

(iv) We have

\[ X \text{ is } \text{NBU} \Rightarrow P(X > k + l) \leq P(X > k)P(X > l), k, l \geq 0, \]

\[ \Rightarrow \sum_{k=0}^{\infty} P(X > k + l) z^k \leq \]

\[ P(X > l) \sum_{k=0}^{\infty} P(X > k) z^k, 0 < z \leq 1, l \geq 0, \]

\[ \Rightarrow X \text{ is NBUL}. \]

\[ \blacksquare \]

This chapter forms the basis for the article Ravi and Prathibha (2012b), which has been accepted for publication in myScience, the science journal of the University of Mysore.