Chapter 4

On stochastic orderings among equilibrium, residual lifetime and derived random variables

4.1 Introduction

This chapter looks at stochastic orderings between several random variables derived using the concepts of equilibrium rvs and residual lifetime at random time. In the second section, a result is proved concerning closure under a shock model. Given two lifetime rvs $X$ and $Y$, the probability that the residual lifetime $X_t$ is greater than the residual lifetime $Y_t$ is obtained in Zardash and Asadi (2010). In the third section of this chapter, we generalize this result to residual lifetimes at random times and find the probability that at random time $Z$, the residual lifetime $X_Z$ is greater than the residual lifetime $Y_Z$. In the last section, we prove converses of a couple of stochastic comparison properties proved in Yue and Cao (2000).
4.2 Stochastic orderings among some random variables

In Yue and Cao (2000), random variables are compared with exponential random variables to establish a few stochastic orders. Here we study such comparisons and establish other results among equilibrium, residual lifetime and derived random variables.

Throughout this section, we denote by $X$ a lifetime rv with finite mean. We define several random variables in the sequel and study stochastic orderings among these. First, we prove a result about HNBUE class of dfs.

**Theorem 4.2.1.** $X \leq_c Y$, where $Y$ is an exponential rv with the same mean as that of $X$, iff $X$ is HNBUE.

**Proof.** We have

\[
X \leq_c Y \iff \int_x^\infty F(u)du \leq \int_x^\infty G(u)du, \ G \text{ being the df of } Y;
\]
\[
\iff \int_x^\infty F(u)du \leq \int_x^\infty \exp(-\lambda u)du, \ \lambda = \frac{1}{\mu_X}, \ \mu_X, \ \text{the mean of } X;
\]
\[
\iff \int_x^\infty \mathcal{F}(u)du \leq \frac{\exp(-\lambda x)}{\lambda};
\]
\[
\iff \int_x^\infty \mathcal{F}(u)du \leq \mu_X \exp\left(-\frac{x}{\mu_X}\right);
\]
\[
\iff X \text{ is HNBUE.}
\]

Next we give definitions of some derived rvs to state the other results of this section.

**Some new definitions:**

\[
\]
(i) $X \sim F$, a lifetime rv with mean $\mu_X$;

(ii) $X_t \sim F_{X_t}(x) = F_t(x) = \frac{F(x+t)}{F(t)}$, $x \geq 0$, $t \geq 0$; the residual lifetime rv of $X$ at time $t$;

(iii) $X^e \sim F_{X^e}(x) = F_1(x) = \frac{1}{\mu_X} \int_x^\infty F(u)du$, $x \geq 0$; the equilibrium rv of $X$;

(iv) $X_Y \sim F_{X_Y}(x) = \int_0^\infty \frac{F(t+y)dG(y)}{F(t)}$, $x \geq 0$, the residual lifetime rv of $X$ at random time $Y \sim G$;

(v) $X^e_c \sim F_{X^e_c}(x) = \frac{1}{\mu_{X^e}} \int_x^\infty \frac{F(y+t)}{F(t)}dy$, $x \geq 0$, the equilibrium rv of $X^e$, with $\mu_{X^e} = \int_0^\infty \frac{F(t+x)}{F(t)}dx$, $t \geq 0$;

• (vi) $X^e_Y \sim F_{X^e_Y}(x) = \frac{1}{\mu_{X^e}} \int_x^\infty \frac{F(u+y)\int{y}G(y)du}{F(t)}$, $x \geq 0$, the equilibrium rv of $X_Y$;

(vii) $X^e_Y \sim F_{X^e_Y}(x) = \frac{\int_0^\infty (x+y)\int{y}G(y)du}{\int_0^\infty \int_0^y \frac{F(u+y)\int{y}G(y)du}{F(t)}}$, $x \geq 0$; the residual lifetime rv of $X$ at random time $Y^e$, where $G_Y^e(y) = \frac{1}{\mu_Y} \int_y^\infty G(u)du$.

\textbf{Theorem 4.2.2.}  

(i) $F_{X^e_Y}(x) = \frac{F_{X^e_Y}(x+t)}{F_{X^e_Y}(t)}$, $t \geq 0$, $x \geq 0$.

(ii) $X^e_Y \leq_{st} X^e_Y$ if $\mu_{X_Y} \geq 1$.

(iii) $X^e_t \leq_{st} X^e$ if $X^e$ is NBU.

(iv) $F_{X_Y}$ is NBUE if $X^e_t \leq_{st} X_Y$.

(v) $X^e_Y \leq_{st} X$ if $X$ is NBU.
Proof. (i) We have, for $t \geq 0, x \geq 0$,

\[
F_{X^t}(x) = \frac{1}{\mu_{X^t}} \int_x^{\infty} F_{X^t}(y)dy, \quad \mu_{X^t} = \int_0^{\infty} F_{X^t}(y)dy
\]

\[
= \frac{1}{F(t)} \int_t^{\infty} F(y)dy = \frac{F(t)}{\mu_{X^t}} \int_t^{\infty} F(y)dy \int_x^{\infty} \frac{F(y+t)}{F(t)}dy
\]

\[
= \frac{F_{X^e}(x+t)}{F_{X^e}(t)}.
\]

(ii) For $t \geq 0$, we have

\[
F_{X^Y}(t) = \frac{1}{\mu_{X^Y}} \int_t^{\infty} F_{X^Y}(u)du,
\]

\[
\leq \frac{F_{X^Y}(t)}{\mu_{X^Y}} \text{ since } F_{X^Y}(.) \text{ is decreasing,}
\]

\[
\leq F_{X^Y}(t) \text{ if } \mu_{X^Y} \geq 1,
\]

\[
\Rightarrow X_{Y}^e \leq_{st} X_Y \text{ if } \mu_{X^Y} \geq 1.
\]

(iii) If $X^e$ is NBU, and $t \geq 0, x \geq 0$, we have

\[
F_{X^t}(x) = \frac{F_{X^e}(x+t)}{F_{X^e}(t)} \leq \frac{F_{X^e}(x)F_{X^e}(t)}{F_{X^e}(t)} = F_{X^e}(x),
\]

\[
\Rightarrow X_{t}^e \leq_{st} X^e.
\]

(iv) For $t \geq 0$, we have

\[
F_{X^Y} \text{ NBUE } \Rightarrow F_{X^Y}(t) \geq \frac{1}{\mu_{X^Y}} \int_t^{\infty} F_{X^Y}(u)du,
\]

\[
\Leftrightarrow \frac{\int_0^{\infty} F(t+y)dG(y)}{\int_0^{\infty} F(y)dG(y)} \geq \frac{\int_0^{\infty} F(u+y)dG(y)}{\int_0^{\infty} F(u)dG(y)},
\]

\[
\Rightarrow F_{X^Y}(t) \geq F_{X^Y}(t) \Rightarrow X_{Y}^e \leq_{st} X_Y.
\]
(v) If $F$ is NBU, for $t \geq 0$, we have

$$F_{X^*}(t) = \int_0^\infty \frac{F_{X^*}(t+y)dy}{G_{X^*}(y)} = \int_0^\infty \frac{F_{X^*}(t+y)dy}{G_{X^*}(y)} \frac{1}{\mu} \int_0^{\infty} \frac{1}{\mu} \int_0^{\infty} f(t)dt}{G_{X^*}(y)}dy,$$

$$= \int_0^\infty \frac{F_{X^*}(t+y)dy}{G_{X^*}(y)} \frac{1}{\mu} \int_0^{\infty} \frac{1}{\mu} \int_0^{\infty} f(t)dt}{G_{X^*}(y)}dy,$$

$$\leq F(t) \Rightarrow X^* \leq_{st} X.$$

\[\square\]

### 4.3 Closure of NBUE class under a shock model

We prove a result to confirm that the NBUE class is closed under geometric compounding. Suppose that $p$ denotes the pmf of a geometric rv $N$ with $P(N = k) = (1 - \rho)\rho^{k-1}, k = 1, 2, \ldots, 0 < \rho < 1$, and $X_1, X_2, \ldots$, are iid rvs, independent of $N$ and with df $F$. Let $C$ denote the df of $X_1 + \cdots + X_N$. Then $C(x) = \sum_{i=1}^\infty (1 - \rho)\rho^{k-1}F^*(x), x \in R$, where $F^k$ is the $k$-fold convolution of $F$ with itself.

We have the following result.

**Theorem 4.3.1.** If $p$ is geometric, then $F$ is NBUE implies that $C$ is NBUE.
Proof. We have

\[ F \text{ NBUE} \Rightarrow F(t) \geq \frac{1}{\mu_F} \int_t^\infty F(x)dx, \quad t \geq 0, \]

\[ \Rightarrow \mathcal{F}^{*k}(t) \geq \frac{1}{\mu_F} \int_t^\infty \mathcal{F}^{*k}(x)dx, \]

since NBUE class is closed under convolutions,

\[ \Rightarrow \sum_{k=1}^\infty (1 - \rho)\rho^k \mathcal{F}^{*k}(t) \geq \frac{1}{\mu_F} \int_t^\infty (1 - \rho) \sum_{k=1}^\infty \rho^k \mathcal{F}^{*k}(x)dx \]

\[ \Rightarrow \mathcal{C}(t) \geq \frac{1}{\mu_F} \int_t^\infty \mathcal{C}(x)dx, \]

\[ \Rightarrow (1 - \rho)\mathcal{C}(t) \geq \frac{(1 - \rho)}{\mu_C} \int_t^\infty \mathcal{C}(x)dx \]

\[ = \frac{\mu_F}{\mu_C} \int_t^\infty \mathcal{C}(x)dx, \]

since \( \mu_C = \mu_F \mu_N = \frac{\mu_F}{1 - \rho} \),

\[ \Rightarrow \mathcal{C}(t) \geq (1 - \rho)\mathcal{C}(t) \geq \frac{1}{\mu_C} \int_t^\infty \mathcal{C}(x)dx, \]

\[ \Rightarrow C \text{ is NBUE}. \]

\[ \square \]

4.4 On residual lifetime at random time

Zardash and Asadi (2010) have studied the probability that at time \( t \), the residual lifetime \( X_t \) of a lifetime rv \( X \) is greater than that of another independent rv \( Y \). We find a similar probability here when the residual lifetimes are considered not at a fixed time \( t \), but at a random time. More precisely, we find \( P(X_Z > Y_Z) \) for lifetime rvs \( X, Y, Z \). We assume that \( X, Y \) have dfs \( F, G \), respectively and that \( Z \) is independent of both \( X \) and \( Y \) with df \( H \) and pdf \( h \).

Theorem 4.4.1. If \( R(t) = P(X_t > Y_t), t \geq 0, \) then

\[ R = P(X_Z > Y_Z) = E_Z(R(Z)), \]
where the expectation is with respect to the rv $Z$.

Proof. We have

\[
R = P(X_Z > Y_Z) = \int_0^\infty P(X > z, Y > z) dP(Z \leq z),
\]

\[
= \int_0^\infty \int_z^\infty \frac{F(x) G(z)}{F(Z) G(Z)} dP(Z \leq z),
\]

\[
= \int_0^\infty \frac{\int_z^\infty F(x) dG(x)}{F(z) G(z)} h(z) dz,
\]

\[
= \int_0^\infty R(z) h(z) dz,
\]

\[
= E_Z(R(Z)),
\]

proving the theorem.

An example: If $X$ and $Y$ are exponentially distributed with parameters $\lambda$ and $\mu$, respectively, then

\[
R = \int_0^\infty R(z) h(z) dz,
\]

\[
= \int_0^\infty \int_z^\infty \frac{F(x) G(z)}{F(z) G(z)} h(z) dz,
\]

\[
= \int_0^\infty \frac{\mu e^{-(\lambda + \mu)z}}{e^{-(\lambda + \mu)z}} h(z) dz,
\]

\[
= \int_0^\infty \frac{\mu}{\mu + \lambda} h(z) dz,
\]

\[
= \frac{\mu}{\mu + \lambda}.
\]

Remark 4.4.2. 1. If $X =^d Y$, then $R(z) = \frac{1}{2}$ and so $R = \int_0^\infty \frac{1}{2} h(z) dz = \frac{1}{2}$.

2. If $R(z) \geq \frac{1}{2}$, then $R = \int_0^\infty R(z) h(z) dz \geq \frac{1}{2}$. 
3. If $R(z) \leq \frac{1}{2}$, then $R = \int_0^\infty R(z)h(z)dz \leq \frac{1}{2}$.

Now we prove a couple of results similar to the ones given in Zardash and Asadi (2010).

**Theorem 4.4.3.** If $X \leq_{st} Y$, then $R \leq 1 - \frac{\eta}{2}$ if $\frac{G(.)}{F(.)} \geq \eta > 0$.

**Proof.** Under the condition of the theorem, we have

\[
R = \int_0^\infty R(z)h(z)dz,
\]

\[
= \int_0^\infty \frac{\int_x^\infty F(x)dG(x)}{F(z)G(z)}h(z)dz,
\]

\[
= \int_0^\infty \left(1 - \frac{\int_x^\infty G(x)dF(x)}{F(z)G(z)}h(z)dz\right),
\]

\[
\leq \int_0^\infty \left(1 - \frac{F(z)}{2G(z)}\right)h(z)dz, \text{ by Zardash and Asadi (2010),}
\]

\[
= 1 - \int_0^\infty \frac{F(z)}{2G(z)}h(z)dz,
\]

\[
\leq 1 - \frac{\eta}{2},
\]

proving the theorem.

**Theorem 4.4.4.** If $Y \leq_{st} X$, then $R \geq \frac{\epsilon}{2}$ if $\frac{F(.)}{G(.)} \geq \epsilon > 0$.

**Proof.** We have

\[
R = \int_0^\infty R(z)h(z)dz,
\]

\[
\geq \int_0^\infty \frac{G(z)}{2F(z)}h(z)dz, \text{ by Zardash and Asadi (2010),}
\]

\[
\geq \frac{\epsilon}{2},
\]

under the condition of the theorem and hence the proof.
4.5 Stochastic comparison properties

Stochastic comparison properties for the residual lifetime $X_Y$ of $X$ at random time $Y$ are obtained in Yue and Cao (2000). We reproduce the relevant results in Yue and Cao (2000) below for ease of reference. We establish the converses of these results in this section.

**Lemma 4.5.1** (Lemma 3.1, Yue and Cao, 2000). Let $X$ and $Y$ be two independent rvs. Let $\phi_1$ and $\phi_2$ be two bivariate functions. Denote $\Delta \phi_{21}(x,y) = \phi_2(x,y) - \phi_1(x,y)$. Then $X \leq_{rh} Y$ iff $E(\phi_1(X,Y)) \leq E(\phi_2(X,Y))$ for all $\phi_1$ and $\phi_2$ that satisfy the conditions: (i) for each $y$, $\Delta \phi_{21}(x,y)$, decreases in $x$ on $\{x \leq y\}$; and (ii) $\Delta \phi_{21}(x,y) \geq -\Delta \phi_{21}(y,x)$ whenever $x \leq y$ (see Theorem 1.B.28 in Shaked and Shanthikumar, 1994).

**Theorem 4.5.2** (Theorem 3.2, Yue and Cao, 2000). Suppose that $X, Y_1$, and $Y_2$ are independent non-negative rvs with dfs $F, G_1$ and $G_2$ respectively. If $Y_1 \leq_{rh} Y_2$ and $F$ is DFR (IFR) then $\underline{F_{Y_1}}(t) \leq (\geq) \underline{F_{Y_2}}(t)$.

**Theorem 4.5.3** (Theorem 3.3, Yue and Cao, 2000). Suppose that $X, Y_1$, and $Y_2$ are independent non-negative rvs with dfs $F, G_1$ and $G_2$ respectively. If $Y_1 \leq_{rh} Y_2$ and $F$ is IMRL (DMRL) then $E(X_{Y_1}) \leq (\geq) E(X_{Y_2})$.

**Theorem 4.5.4.** Suppose that $X, Y_1$, and $Y_2$ are independent non-negative rvs with dfs $F, G_1$ and $G_2$ respectively. If $F$ is DFR (IFR) and $X_{Y_1} \leq_{st} X_{Y_2}$ then $Y_1 \leq_{rh} Y_2$. 
Proof. The proof is on lines similar to that of Theorem 4.5.2. We have, for \( t \geq 0 \),

\[
X_{Y_1} \leq_{st} X_{Y_2} \Rightarrow F_{Y_1}(t) \leq F_{Y_2}(t),
\]

\[
\Rightarrow \frac{E(F(t + Y_1))}{E(F(Y_1))} \leq \frac{E(F(t + Y_2))}{E(F(Y_2))}, \text{ and}
\]

\[
F_{Y_1}(t) - F_{Y_2}(t) \leq 0 \Rightarrow \frac{E(F(t + Y_1))}{E(F(Y_1))} - \frac{E(F(t + Y_2))}{E(F(Y_2))} \leq 0,
\]

\[
\Rightarrow \frac{E(F(t + Y_1)F(Y_2)) - E(F(t + Y_2)F(Y_1))}{E(F(Y_1)F(Y_2))} \leq 0,
\]

\[
\Rightarrow E\phi_1(Y_1, Y_2) \leq E\phi_2(Y_1, Y_2),
\]

where \( \phi_1(x, y) = F(t + x)F(y) \), and \( \phi_2(x, y) = F(x)F(t + y) \). Since \( F \) is DFR, we now show that \( \phi_1, \phi_2 \) satisfy the conditions (i) and (ii) of Lemma 4.5.1. Condition (ii) is trivially true since \( \Delta \phi_{21}(x, y) = -\Delta \phi_{21}(y, x) \) for all \( x, y \geq 0 \). For \( y \geq 0 \), and \( x_1, x_2 \), such that \( 0 \leq x_1 \leq x_2 \leq y \), we have

\[
\Delta \phi_{21}(x_1, y) = F(x_1)F(t + y) - F(t + x_1)F(y),
\]

\[
\Delta \phi_{21}(x_1, y) = F(x_1)F(y) (F(t | y) - F(t | x_1)).
\]

Since \( F \) is DFR, \( F(t | x) \) is increasing in \( x \geq 0 \) and hence \( F(t | x_2) \leq F(t | y) \), and hence \( \Delta \phi_{21}(x_1, y) \leq F(x_1)F(y) (F(t | y) - F(t | x_2)) \) \( \geq F(x_2) F(y) (F(t | y) - F(t | x_2)) = \Delta \phi_{21}(x_2, y) \). So, condition (i) is also satisfied, completing the proof for IFR. The proof for DFR is similar and is omitted. \( \square \)

**Theorem 4.5.5.** Suppose that \( X, Y_1, \) and \( Y_2 \) are independent non-negative rvs with dfs \( F, G_1 \) and \( G_2 \) respectively. If \( F \) is IMRL (DMRL) and \( E(X_{Y_1}) \leq (\geq) E(X_{Y_2}) \) then \( Y_1 \leq_{rh} Y_2 \).

Proof. The proof is on lines similar to that of Theorem 4.5.3. By the definition of
we have, for $i = 1, 2$,

\[
E(X_i) = \int_0^\infty \overline{F}(t)dt,
\]

\[
= \int_0^\infty \int_0^\infty \int_0^\infty \overline{F}(t + y)dG_i(y)dt,
\]

\[
= \int_0^\infty \int_y^\infty \overline{F}(u) dudG_i(y),
\]

\[
= \int_0^\infty m(Y_i) \overline{F}(y)dG_i(y) \int_0^\infty \overline{F}(y)dG_i(y),
\]

\[
= \frac{E(m(Y_i)\overline{F}(Y_i))}{E(F(Y_i))},
\]

where $m(y) = E(X_Y) = \int_0^\infty \frac{\overline{F}(u)du}{\overline{F}(y)}$ is the MRL at time $Y$. Now,

\[
E(X_{Y_1}) \leq E(X_{Y_2})
\]

\[
\Leftrightarrow \frac{E(m(Y_1)\overline{F}(Y_1))}{E(F(Y_1))} \leq \frac{E(m(Y_2)\overline{F}(Y_2))}{E(F(Y_2))},
\]

\[
\Leftrightarrow \frac{E(m(Y_1)\overline{F}(Y_1))E(F(Y_2)) - E(m(Y_2)\overline{F}(Y_2))E(F(Y_1))}{E(F(Y_1))E(F(Y_2))} \leq 0,
\]

\[
\Leftrightarrow E\phi_1(Y_1, Y_2) \leq E\phi_2(Y_1, Y_2),
\]

where $\phi_1(x, y) = m(x)\overline{F}(x)\overline{F}(y)$, $\phi_2(x, y) = m(y)\overline{F}(y)\overline{F}(x)$. From Lemma 4.5.1, this implies that $Y_1 \leq rh Y_2$ if $\phi_1, \phi_2$ satisfy conditions (i) and (ii). Since $\Delta \phi_21(y, x) = -\Delta \phi_21(y, x)$, for all $x, y \geq 0$, condition (ii) is trivially true. For, $x_1, x_2$ such that $0 \leq x_1 \leq x_2 \leq y$, we have $\Delta \phi_21(x_1, y) = (m(y) - m(x_1)) \overline{F}(y) \overline{F}(x_1)$. Since $F$ is IMRL, $m(x_1) \leq m(x_2) \leq m(y)$. Therefore, $\Delta \phi_21(x_1, y) = (m(y) - m(x_1)) \overline{F}(y) \overline{F}(x_1) \geq (m(y) - m(x_2)) \overline{F}(y) \overline{F}(x_2) = \Delta \phi_21(x_2, y)$, and hence condition (ii) is satisfied, proving the theorem for IMRL. The proof for DMRL is similar and is omitted. \qed