CHAPTER 2

MATHEMATICAL BACKGROUND
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2.1 Introduction

In this chapter we discuss the Mathematical background required for solving the diffusion problems in the subsequent chapters. The various methods applicable to solve the partial differential equations are Exact methods, Approximate methods, Numerical methods, Similarity methods etc. Among these similarity methods are discussed in this chapter.

A challenging problem faced by engineers and applied mathematicians working in a given field of study is to find solutions of the basic equations arising in that field while employing a minimum number of simplifying assumptions. In a large number of cases, the basic equations expressing physical laws are partial differential equations. Ideally one hopes to find exact solutions of these equations. In certain instances standard methods of solution are of value and solutions can be found. Even though there are number of problems in which solutions cannot be found by the usual classical methods. This is particularly true if the equations are encountered nonlinear. An excellent example of a nonlinear system of equations is the system of equations of motion for a viscous fluid through porous media. The solutions of certain sets of
partial differential equations can be found quite really in spite of the failure of the common classical methods. Notable among such solutions are those that have been obtained by employing transformations that reduce the system of partial differential equations to a system of ordinary differential equations. These solutions are generally designated as similarity solutions.

Unfortunately, texts and papers seldom give reason and motivations for selecting transformations that reduce partial differential equations to ordinary differential equations. Moreover, examinations of standard treaties on Applied Mathematics are usually of little help in providing information on similarity analysis. For example, Birkhoff [1] gives a rather extended treatment of similarity analysis. A number of authors; Hansen [2], Bluman, G.W. and Cole, J. D. [3] etc. have presented papers dealing with general similarity methods for solving certain classes of equations. In summary similarity solutions of partial differential equations are quite commonly found in technical literature [4],[5],[6].

The mathematical similarity in spite of having the property of simplification is evident in rare cases in natural sciences. Today, in the modern age, the use of similarity transforms is not restricted to fluid dynamics or heat transfer, but it much useful in studying physical, biological, medical and social sciences. These sciences use one of the
fundamental properties of similarity transforms, i.e. mathematical expression. It’s an important media to tackle the present day problem of science. The concept of mathematical similarity was first introduced by Hemholtz through the dimensional analysis approach. Later on importance of similarity parameters in physics and mechanics was demonstrated by Reynolds and Sedov [7] developed the concept of similarity in a more meaningful way. Birkhoff [1] showed that the reduction of one the number of independent variables in some system of partial differential equations. Hansen [2] introduced the concept of similarity through four different types of methods, namely Separation of variables method, Free parameter method, Group theory method and Dimensional analysis method. Lie has introduced the study of continuous group of transformations of ordinary differential equations based on the infinitesimal properties of the group. [4],[5],[6]

A similarity transforms is obtainable for boundary value problem if the governing differential equation and the associated boundary conditions are invariant under a group of transformations. However, if any of the equation and boundary conditions is not invariant under a group, then the problem becomes a non-similar. Lie’s showed that the order of the ordinary differential equations can be reduced by one constructively, if it is invariant under the one parameter.
The aim of the similarity methods is that, if the physical law is expressed as a differential equation then, we will transform the variables to solve it. Similarity methods are generally used to reduce the partial differential equations to the ordinary differential equations. Transformations may be employed to reduce a partial differential equation from an equation in \( n \) independent variables to an equation in \( (n - 1) \) independent variables. The techniques of similarity methods may provide the only simple way to obtain exact solutions to the certain equations, therefore the techniques for finding such solution are seldom included in the Mathematics of an engineer and Mathematician. Similarity methods are widely used in the field of fluid flow and heat transfer. Here we discuss the similarity methods for solving partial differential equation associated with diffusion. Similarity methods are applicable to linear and nonlinear partial differential equations and also system of partial differential equations. Sometimes the number of independent variable in a partial differential equation can be reduced by taking algebraic combination of the independent variables.

The idea of the similarity methods is to find new independent variables that are combinations of the old independent variables. The differential equations, when written in new variables, will not depend on all of the new variables. One technique for discovering the correct new variables is to choose temporary variables to be a parameter
to some power times the old variables. After writing the equations in terms of the temporary variables, the powers can be found by requiring homogeneity in the parameter. New variables are then constructed from the old variables in such a way that the parameter does not enter.

There are four methods which are discussed in this chapter for deriving similarity solution:

(1) Free parameter method, (2) Separation of variables method, (3) Dimensional analysis method, (4) Group theory method: (a) Finite transformation and (b) Infinitesimal Transformation

Now-a-days some modification enhancement has taken place in the group theory method.

2.2 Free parameter method

One of the simplest and most straightforward methods of determining similarity solutions is a ‘Free parameter method’. It is assumed that the dependent variable occurring in a particular partial differential equation can be expressed as a product of two functions. One of the functions in this product is a function of all the independent variables except one. The other function is assumed to depend on a single parameter, say $\eta$; where $\eta$ is a variable obtain from a transformation of the variables involving the independent variable not occurring in the first function. As the form of $\eta$ is not specified, is called
a ‘Free parameter’ and hence designates this particular methods by Hansen [2]. This method is used for the problems with boundary conditions.

A form for the dependent variable was assumed. The expression was chosen in such a way that function of the single variable, could be introduced and boundary conditions could be transformed. Employing the transformation equation for the dependent variable, the partial differential equation was transformed into an equation involving the function of \( \eta \). The transformed equation was examined to determine what the conditions might be which result in the equation becoming an ordinary differential equation. It was obvious that a sufficient set of conditions was simply that the conditions of the various terms involving function and its derivatives be functions of \( \eta \). From this assumption an equation for \( u \) for any variable which is independent and not involved in the first function, in the product of the functions of the dependent variable involving a function of \( \eta \) is determined. The equation finally leads to an expression for \( \eta \). Using the equation for \( \eta \), the partial differential equation can be further simplified. Finally, the assumed constancy of one co-efficient reduced the equation to an ordinary differential equation.

There are some limitations of this method. The starting point, as mentioned earlier, is to examine the boundary conditions and then to attempt to construct a transformation function for the dependent
variable in such a way that these boundary conditions transfer to constant boundary values on the function of the free parameter.

In regards to analyses of the type of problems in fluid mechanics and heat transfer, several rather broad generalizations is based on the type of functions which appear in boundary conditions and in the transformation equations for the dependent variable. These functions have always been found to be either exponentials or power functions. While there may be exceptions to this. If we are faced with solving a problem in fluid mechanics or heat transfer which has boundary conditions containing powers or exponentials, an exact similarity solution will probably not be found. e.g. The wall temperature in heat transfer problem may be specified as varying sinusoidally with time. As the wall temperature is a boundary condition and as the temperature does not vary as a power or an exponential, probability of finding a similarity solution is extremely small. One major exception to this observation concerns solutions which are sums of simple similarity solutions.

Finally, the problem of analyzing equations are stemming from physical problems, having characteristic lengths. As long as characteristic lengths are manifest in boundary conditions specified at one or more points in the finite part of the plane, we can expect that a suitable form for the parameter $\eta$ cannot be determined.
2.3 Separation of variables method

The separation of variables method has been formulated by D.E. Abbott and S.J. Kline [8]. This method is quite a like the free parameter method. These two methods are alike in consideration of the role of the boundary conditions and the formulations of similarity transformations, at the outset of analysis. These types of problems are recognized and discussed in [8] and [9]. The first problem is the determination of similarity solutions for a partial differential equation, which has a complete set of boundary and initial values prescribed. Such a problem is called a ‘well-posed’ problem. Given a general form for the similarity transformations, either no specific similarity parameter exists or one similarity variable exists. The second problem is the determination of possible similarity solutions of a partial differential equation when some, but not all of the boundary conditions are given. In this type of problem, no similarity parameter, one parameter or many parameters may exist under an assumed general transformation. For example, two-dimensional boundary layers flow. There is a third type of problem in which we concerned only with reducing the number of independent variable of a given differential equation irrespective of any boundary condition.

This method is most often, applicable to the linear homogeneous partial differential equations. Using this method we get an
exact solution, generally in the form of an infinite series. We look for a solution to a partial differential equation by separating the solution into pieces, where each piece deals with a single dependent variable.

For linear homogeneous partial differential equation, to represent the solution as a sum of terms in which each term factors into a product of expressions, each expression dealing with a single independent variable.

Suppose that \( L(u) = 0 \) is a linear partial differential equation, for \( u(X) \) that has the form, \( L(u) = \sum_i L_i[u] \), where \( L_i[u] \) are differential operators. We look for the solution of this partial differential equation in the form,

\[
u(X) = u(x_1, x_2, \ldots, x_n) = X_1(x_1)X_2(x_2) \ldots \ldots X_n(x_n)\]

where the functions \( (X_1, X_2, \ldots, X_n) \) are to be determined, by using the above form in the original and reasoning about which terms depend upon which variables we can often, reduce the original partial differential equation for each of the \( \{X_i\} \), in carrying this out, arbitrary constants will be introduced. After resulting ordinary differential equations are solved, the arbitrary constants can generally be found physical reasoning. Because superposition can be used in linear equations, any number of terms will also be a solution of the original equation. Also, if each of terms is multiplied by some constants, the resulting expression will also be a solution. Hence, the final solution will frequently be a sum of an
integral. This sum will have unknown constants in it due to the constants allowed in the superposition. These constants will be determined from the initial conditions and/or the boundary conditions.

The separation of variables method has certain advantages and disadvantages as compared to the free parameter method. One advantage is that it is a direct extension and the other is that in problems such as the heat conduction problem, we do not have to make any assumptions about the expression for the dependent variable beyond the fact that it is separable. Applying the free parameter method in this problem, we might be inclined to use a transformation of the form,

\[ u = \Phi(t)F(\eta) \]

the method would fail [8].

Comparing the free parameter method and the separation of variables method is to observe that, in applying the free parameter method, a form is chosen for the dependent variable which makes it possible for boundary conditions on that variable to be applied. However, nothing is done to insure that independent variable transformations are made in a manner which will result in constant values of the independent variables in the boundary conditions, being transformed into constant values for the similarity variables. Such is not the case in the separation of variables method. The form of the similarity variable is first chosen so that no problem regarding the constant values of the similarity variable
will occur. On the other hand, no steps are taken to insure that the dependent variable has an appropriate form consistent with the boundary conditions. We merely seek a separable expression for the variable. From all of these, we might state that the free parameter method concentrates on the dependent variable, while the separation of variable method concentrates on the independent variable.

2.4 Dimensional Analysis Method

The application of dimensional analysis to find separation transformations have been nicely developed by L. I. Sedov [7]. The principal methods are discussed at length in his book. Classical dimensional analysis stems from the famous Buckingham’s Pi theorem, which is stated as follows:

“If there are \( n \) variables in a problem and these variables contain \( m \) primary dimensions (for example \( M, L, T \)) then equation relating all the variables will have \( (n - m) \) non-dimensional groups.” i.e. Suppose that \( n \) quantities, describe some physical occurrence, and suppose further that each of these quantities are expressible in terms of certain fundamental dimensional quantities \( (M - mass, L - length, T - time) \), which are \( m \) in number, then the occurrence can be described in terms of \( (n - r) \) non-dimensional quantities (called Pi variables), where
$r$ is the rank of the $m \times n$ matrix formed from the dimensions of the $n$–quantities.

e.g. The pressure drop $\Delta p$ for an in-compressible laminar flow through a pipe is a function of the pipe length $l$, the fluid absolute viscosity $\mu$, radius of the pipe is $r$ and the average fluid velocity $U$:

$$\Delta p = f(l, \mu, r, U) \quad (2.5.1)$$

Let the fundamental dimensional quantities be mass $M$, length $L$ and time $T$. The dimensions of $\Delta p$ can be shown to be,

$$\Delta p \rightarrow ML^{-1}T^{-2}$$

The dimensions of each of the five quantities can be arranged in matrix form as follows:

$$
\begin{pmatrix}
\Delta p & l & \mu & r & U \\
M & 1 & 0 & 1 & 0 & 0 \\
L & -1 & 1 & -1 & 1 & 1 \\
T & -2 & 0 & -1 & 0 & -1 \\
\end{pmatrix}
$$

This matrix has the rank “3”.

The Pi theorem states that, whatever the physical law may be as described by equation-(2.5.1), that law can be expressed in terms of $5 - 3 = 2$ non-dimensional variables. Two independent dimensionless variables formed from $\Delta p, l, \mu, r$ and $U$ are:

$$\pi_1 = \frac{\Delta p l}{\mu U} \quad \text{and} \quad \pi_2 = \frac{1}{r}$$
Applying the Pi theorem, we would expect that,

\[ \pi_1 = f(\pi_2) \]

where \( f(\pi_2) \) is some function of \( \pi_2 \).

This brief explanation is not intended to be more than an illustration of the basic principles of dimensional analysis. The important fact is that dimensional analysis indicates how variables may be transformed.

The application of dimensional analysis for finding the similar solutions is valuable in special cases. Also, it is an extension of usually well-known techniques. But the methods covered in previous sections can be applied in a more straightforward manner and are more general.

### 2.5 Group theory method

One of the most mathematically sophisticated methods of determining similarity solutions of partial differential equations is based on concept derived from the theory of continuous transformation group [8]. Using this method, classes of transformations, this allows the number of independent variables in some systems of partial differential equations to be reduced by one. Suppose we have a single partial differential equation in two independent variables. We seek transformations which
reduce the number of independent variables by one, i.e. lead to an ordinary differential equation.

2.5.1 Finite Transformation

(1) Transformation group:
Let \( f^i(x^1, x^2, \ldots, x^n; a^1, a^2, \ldots, a^r); i = 1, 2, \ldots, n \) be a set of functions continuous in both the variables \( x^i \) and \( a^j \) will hereafter be called parameters of the functions. Given a specific set of parameters \( a^j \), values for the functions are found by assigning values to the variables \( x^i \). Now it may be possible to obtain the same functional values for assigned values of \( x^i \) by constructing a new function that has parameters \( a^1, a^2, \ldots, a^{r-1} \), where the \( a' \)'s are functions of the parameters \( a^j \) and

\[
f^i(x^1, x^2, \ldots, x^n; a^1, a^2, \ldots, a^r)
= F^i(x^1, x^2, \ldots, x^n; a^1, a^2, \ldots, a^{r-1})
\]
i.e. suppose that,

\[
f^1(x^1, x^2; a^1, a^2, a^3) = x^1 a^1 + (a^2 + a^3)e^{x^2}
\]
\[
f^2(x^1, x^2; a^1, a^2, a^3) = x^1 + a^1 \sin x^2
\]
Let \( a^1 = a^1; a^2 = a^2 + a^3 \)

We may write,

\[
f^1(x^1, x^2; a^1, a^2, a^3) = F^1(x^1, x^2; a^1, a^2) = a^1 x^1 + a^2 e^{x^2}
\]
\[
f^2(x^1, x^2; a^1, a^2, a^3) = F^1(x^1, x^2; a^1, a^2) = x^1 + a^1 \sin x^2
\]
If a reduction in the number of parameters of a function cannot be achieved, the parameters are called essential. More formally, the parameters $a^j$ are called ‘essential parameters’, if it is possible to find $r - 1$ functions of $a^j$.

\[ \alpha^1(a), \alpha^2(a), \ldots, \alpha^{r-1}(a), \] such that,

\[
f^i(x^1, x^2, \ldots, x^n; a^1, a^2, \ldots, a^r) = F^i(x^1, x^2, \ldots, x^n; \alpha^1, \alpha^2, \ldots, a^{r-1})
\]

If the $a^j$ are not essential parameters, it means that fewer parameters can be constructed from the $a^j$ to serve the same purpose in a function. We will interpret a set of functions as transformations which change a point with coordinates $(x^1, x^2, \ldots, x^n)$ into a new point. A complete set of transformations are taken as group elements. Successive transformations employing various sets of functions are considered to be the ‘operation’ between elements.

If we consider sets of functions that define a transformation as a possible group element, we will have to introduce the inverse of an element; i.e. the inverse of a set of functions defining the transformations. We know from the elementary calculus that if a single function $f(x)$ is a function of the variable $x$ and if $f^{-1}$ denotes the inverse of the function $f$, then $f^{-1}[f(x)] = x$. Thus, for the set of functions,

\[ f^i(x^1, x^2, \ldots, x^n; a^1, a^2, \ldots, a^r) \]
with essential parameters \(a^i\) have an inverse is that if the Jacobian of the \(f^i\) which represent to the \(x^i\) is not zero for a set of values \(x^i\), then in the neighbourhood of a point \((x^1, x^2, \ldots, x^m)\) an inverse transformation exists.

It follows that, if we regard the function \(f^i(x, a)\) as transforming the variables \((x^1, x^2, \ldots, x^n)\) into the set of variables, 

\[(X^1, X^2, \ldots, X^n),\]

i.e. \(X^i = f^i(x^1, x^2, \ldots, x^n; a^1, a^2, \ldots, a^r)\) and if

\[
\text{det} \begin{vmatrix}
\frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f^n}{\partial x^1} & \cdots & \frac{\partial f^n}{\partial x^n}
\end{vmatrix} \neq 0
\]

in a neighbourhood of \((x^1, x^2, \ldots, x^n)\) then an inverse set of transformations is \(f^{*i}(X^1, X^2, \ldots, X^n; a^1, a^2, \ldots, a^r)\) exists, such that;

\[x^i = f^{*i}(X^1, X^2, \ldots, X^n; a^1, a^2, a^3, \ldots, a^r).\]

For further references, we say that a set of functions \(f^i\) is functionally independent if the Jacobian of the set of functions does not vanish. We consider the functions \(f^i\) as a set of transformations depending on the parameters \(a^1, a^2, a^3, \ldots, a^r\) and transforming a point \((x^1, x^2, \ldots, x^n)\) into \((X^1, X^2, \ldots, X^n)\).
For a particular set of values of the \( a^j \), say, \( a_1^1, a_1^2, \ldots, a_1^r \), we write the transformation \((x^1, x^2, \ldots, x^n)\) into \((X^1, X^2, \ldots, X^n)\) as;

\[ T_{a_1}x = X; \]

where \( x \) and \( X \) represent the general points. The functions \( f^{a_i} \) are inverse functions and we may also write transformation carrying \((X^1, X^2, \ldots, X^n)\) back into \((x^1, x^2, \ldots, x^n)\) as;

\[ T_{a_1}^{-1}X = x \]

Two different transformations are defined by different sets of values of the \( a^j \). Thus, if \( a_2^1, a_2^2, \ldots, a_2^s \) is a set of values distinct from \( a_1^1, a_1^2, \ldots, a_1^r \), we consider \( T_{a_1} \) and \( T_{a_2} \) be two different transformations. By the product of two transformations, \( T_{a_1} \) and \( T_{a_2} \), we mean application of the transformation successively, i.e. a point \( x \) is taken into a point \( X' \) as follows;

\[ X' = T_{a_2}X = T_{a_2}(T_{a_1}x) \equiv T_{a_2}T_{a_1}x \]

(2) Absolute invariance:

The concept of the absolute invariants is important in the study of the groups of transformations.

Let \( T_{a_i} \) be a continuous transformation group and let \( f(x) \) be a function of the variable \( x \).

Let \( X \) be defined by, \[ X = T_{a_i}x. \]
Now if \( f(X) = f(x) \), for every transformation \( T_{ai} \), the \( f(x) \) is called an absolute invariant of the group. Following the approach used by A.J.A.Morgan [10], one parameter continuous group of transformations defined by,

\[
T_a: \begin{cases} 
X^i = f^i(x^1, x^2, \ldots, x^m; a); (i = 1, 2, \ldots, m; m \geq 2) \\
Y^j = h^j(y^1, y^2, \ldots, y^m; a); (j = 1, 2, \ldots, n; n \geq 1)
\end{cases} \tag{2.5.1}
\]

The transformations defined by;

\( X^i = f^i(x^1, x^2, \ldots, x^m; a) \), are assumed to be defined a subgroup of the given group of transformations. The \( y^j \) will ultimately be designated as the dependent variables of a differential equations while the \( x^i \) will designated the independent variables. As such we assume that the \( y^j \) are differentiable functions of the \( x^i \) up to any required order. The set of transformations defined by, \( X^i = f^i(x^1, x^2, \ldots, x^m; a) \) form a subgroup (and hence a group), there exist \( m - 1 \) functionally independent absolute invariants of the subgroup:

\[
\eta_1(x^1, x^2, \ldots, x^m), \ldots, \eta_{m-1}(x^1, x^2, \ldots, x^m)
\]

Consider the group of transformations equation (2.4.1) as a whole, there are \( m + n - 1 \) functionally absolute invariants.

It is possible to choose the set of invariants

\[
g_1(x^1, x^2, \ldots, x^m; y^1, y^2, \ldots, y^n), \ldots, g_n(x^1, x^2, \ldots, x^m; y^1, y^2, \ldots, y^n)
\]

in such a way that the Jacobian,
This can be justified by the following procedure:

Select a one parameter transformation group. If $x$ and $y$ are the independent variables and if $u$ is the dependent variable, a reasonable first choice might be:

\[
\begin{align*}
X &= a^n x \\
Y &= a^m y \\
U &= a^p u
\end{align*}
\]

Find an absolute invariant which is a function of the independent variable alone, i.e.

\[
\eta = yx^s
\]

Now, we establish the relation between $n, m & s$, such that

\[
yx^s = YX^s.
\]

The absolute invariant $\eta$ will be the new independent variable.

Find a second absolute invariant $g$ involves the dependent variable $u$, i.e.

\[
g = ux^r.
\]

Find relations, such that,

\[
ux^r = UX^r.
\]

Set $g = F_1(\eta)$, then, $u = F_1(\eta)x^{-r}$ and $F_1(\eta)$ is the new dependent variable. Substituting the transformation for $u$ into the given equation and
employing the definition of $\eta$ should reduce the given partial differential equation to an ordinary differential equation.

If more dependent or independent variables are involved, the above procedure remains essentially the same, except that a group of independent variables $\eta_1, \eta_2, \ldots, \eta_{m-1}$ are sought from the original independent variables and are one less in number. The $\eta_i$ are absolute invariants. For each dependent variable, an absolute invariant $g_i$ is sought which involves the dependent variable. A good choice is $g_i = u_i h(x_1, x_2, \ldots, x_m)$, where $u_i$ is the dependent variable. The function $g_i$ is then equated to a function, $F_1(\eta_1, \eta_2, \ldots, \eta_{m-1})$.

If $g_i = u_i h(x_1, x_2, \ldots, x_m)$, then $u_i = \frac{F_1(\eta_1, \eta_2, \ldots, \eta_{m-1})}{h(x_1, x_2, \ldots, x_m)}$ is the dependent variable transformation substituting the various transformation into the original system of equations should lead to a new system with the number of the independent variables reduced by one.

There are two advantages of the group theory method. The first is that the method is rather simple to apply. We merely pick a transformation and proceed. There is no concern about boundary conditions, choices for various functions, etc. The second is that the number of independent variable is reduced by one. It is possible that a new system of partial differential equations without continuing to obtain an ordinary differential equations. The possible advantage of stopping
short of a system of ordinary differential equations is that may be possible
to solve a wider variety of problems in this manner. It would be very
interesting to explore this possibility in solving the diffusion equations.

It has been pointed out that one advantage of the group
typey method is to reduce a system of partial differential equations in $n$
independent variables to a system in $(n - 1)$ variables without continuing
to ordinary differential equations, the same type of results could be
achieved with a modification of the free parameter method.

2.5.2 Infinitesimal transformation

A group is said to be continuous if, between any two
operations of the group, a series of operations within the group can
always be found of which the effect of any operation in the series differs
from the effect of its previous operation only infinitesimally. The concept
of infinitesimal transformations comes as a natural consequence of the
definition of a continuous transformation group. An infinitesimal
transformation is one whose effects differ infinitesimally from the
identical transformation. Thus, any transformation of a finite continuous
transformation group which contains the identical transformation can be
obtained by infinite repetition of an infinitesimal transformation.

Let the identical transformation be

$$x_1 = \phi(x, y, a_0) = x; \quad y_1 = \psi(x, y, a_0) = y \quad (2.5.2.1)$$
where \( a_0 \) is a particular value of a general parameter \( a \). Then a transformation

\[
x_1 = \phi(x, y, a_0 + \delta \varepsilon); \quad y_1 = \psi(x, y, a_0 + \delta \varepsilon) \quad (2.5.2.2)
\]

where \( \delta \varepsilon \) is an infinitesimal quantity, defines an infinitesimal transformation in a broad sense.

Expanding (2.5.2.2) in Taylor series, we get,

\[
x_1 = \phi(x, y, a_0) + \frac{\delta \varepsilon}{1!} \left( \frac{\partial \phi}{\partial a} \right)_{a_0} + \frac{(\delta \varepsilon)^2}{2!} \left( \frac{\partial^2 \phi}{\partial a^2} \right)_{a_0} + \cdots \quad (2.5.2.3)
\]

and

\[
y_1 = \psi(x, y, a_0) + \frac{\delta \varepsilon}{1!} \left( \frac{\partial \psi}{\partial a} \right)_{a_0} + \frac{(\delta \varepsilon)^2}{2!} \left( \frac{\partial^2 \psi}{\partial a^2} \right)_{a_0} + \cdots \quad (2.5.2.4)
\]

Since \( \delta \varepsilon \) is infinitesimal, higher-order terms of \( \delta \varepsilon \) can be neglected, Eq. (2.5.2.3) and (2.5.2.4) then become;

\[
\begin{align*}
x_1 &= x + \xi(x, y) \delta \varepsilon, \\
y_1 &= y + \eta(x, y) \delta \varepsilon
\end{align*}
\]

(2.5.2.5)

where

\[
\xi(x, y) = \left( \frac{\partial \phi}{\partial a} \right)_{a_0} \quad (2.5.2.6)
\]

and

\[
\eta(x, y) = \left( \frac{\partial \psi}{\partial a} \right)_{a_0} \quad (2.5.2.7)
\]

The employment of the infinitesimal transformation, equation (2.5.2.5), in conjunction with the function \( f(x, y) \) will be to transform \( f(x, y) \) into \( f(x_1, y_1) \) which, upon expanding in Taylor series, becomes:
\[ f(x_1, y_1) = f(x + \xi \delta \epsilon, y + \eta \delta \epsilon) = f(x, y) + \frac{\delta \epsilon}{1!} Uf + \frac{(\delta \epsilon)^2}{2!} U^2 f + \ldots, \]  
(2.5.2.8)

where \( Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} \)  
(2.5.2.9)

is called the group representation and \( U^n f \) means repeating the operator \( U \) for \( n \) times. The function \( f(x, y) \) is said to be an invariant function under the infinitesimal transformations, (2.5.2.5), if \( f(x, y) = f(x_1, y_1) \).

It can be shown from (2.5.2.9) that the necessary and sufficient condition that the function \( f(x, y) \) be invariant under the infinitesimal group of transformations represented by \( Uf \) is \( Uf = 0 \),

\[ \text{i.e.} \quad \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} = 0 \]  
(2.5.2.10)

To determine the invariant function, \( f \), it is necessary to solve the equation (2.5.2.10) by the method of Lagrange given in the theories of linear differential equations. Thus, we solve the related differential equation

\[ \frac{dx}{\xi} = \frac{dy}{\eta} \]  
(2.5.2.11)

Suppose the solution is \( \phi(x, y) = \text{constant} \), this function is invariant function of the infinitesimal transformation represented by \( Uf \). Since equation (2.5.2.11) has only one independent solution depending on a single arbitrary constant, a one parameter group in two variables has one and only one independent invariant. In the case of \( n \) variables all the
theories correspondence to two variables can be generalized by the following the same pattern, if the function of \( n \) variables \( f(x_1, x_2, \ldots, x_n) \) is invariant under the infinitesimal transformation

\[
x_i' = x_i + \xi_i(x_1, x_2, \ldots, x_n) \delta \varepsilon
\]

where \( i = 1, 2, \ldots, n \)

then a necessary and sufficient condition is again \( Uf = 0 \) which, its expanded form, is

\[
\xi_1(x_1, \ldots, x_n) \frac{\partial f}{\partial x_1} + \cdots + \xi_n(x_1, \ldots, x_n) \frac{\partial f}{\partial x_n} = 0
\]

(2.5.2.13)

Following the same reasoning as in two-dimensional case, the invariant functions can be obtained by integrating the following equations:

\[
\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \cdots = \frac{dx_n}{\xi_n}
\]

(2.5.2.14)

Since there are \( (n - 1) \) independent solutions to the equation (2.5.2.14) a one parameter group in \( n \) variables has \( (n - 1) \) independent invariants.

The invariant functions are therefore

\[
\varphi_1(x_1, x_2, \ldots, x_n) = \text{constant} = C_m,
\]

(2.5.2.15)

where \( m = 1, 2, \ldots, (n - 1) \)

and are the solutions of the system of equations given by (2.5.2.14).

On the other hand, there are two evident disadvantages to employing the group theory methods. The first is that boundary conditions are not taken into account in any way until the entire analysis is completed. The second is the uncertainty in choosing a proper
transformation group, this does not mean that another which would prove to be adequate, does not exist.

2.6 References


