Chapter 1

Introduction

1.1 Introduction

A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of a natural number $n$ is a finite sequence of non-increasing positive integer parts $\lambda_i$ such that $n = \sum_{i=1}^{k} \lambda_i$. The number of partitions of $n$ is denoted by $p(n)$. For example, the partitions of 4 are (4), (3,1), (2,2), (2,1,1), (1,1,1,1), and hence, $p(4) = 5$. By convention, $p(0) = 1$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_{\infty}}, \tag{1.1.1}$$

where, here and throughout the thesis, we assume that $|q| < 1$ and use the standard notations

$$(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n),$$

and

$$(a_1, a_2, \ldots, a_k; q)_{\infty} := (a_1; q)_{\infty}(a_2; q)_{\infty} \cdots (a_k; q)_{\infty}.$$

In 1919, Ramanujan [52], [55, pp. 210-213] announced that he had found three simple congruences satisfied by $p(n)$, namely,

$$p(5n + 4) \equiv 0 \pmod{5}, \tag{1.1.2}$$
$$p(7n + 5) \equiv 0 \pmod{7}, \tag{1.1.3}$$
$$p(11n + 6) \equiv 0 \pmod{11}. \tag{1.1.4}$$
He gave proofs of (1.1.2) and (1.1.3) in [52] and later in a short one page note [53], [54, p. 230] announced that he had also found a proof of (1.1.4). In a posthumously published paper [54], [55, pp. 232–238], Hardy extracted different proofs of (1.1.2)–(1.1.4) from an unpublished manuscript of Ramanujan [56, pp. 133–177], [18]. In [52], Ramanujan also offered a more general conjecture which states that if $\delta = 5^a 7^b 11^c$ and $\lambda$ is an integer such that $24\lambda \equiv 1 \pmod{\delta}$ then

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta}. \quad (1.1.5)$$

Although Ramanujan gave a proof of this conjecture in his unpublished manuscript [56, pp. 133-177], [18] for arbitrary $a$ and $b = c = 0$, later on, his conjecture was needed to be corrected as

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta'}, \quad (1.1.6)$$

where, $\delta' = 5^a 7^b' 11^c$ with $b' = b$ if $b = 0$, 1, 2 and $b' = [(b + 2)/2]$ if $b > 2$. In 1938, G. N. Watson [59] published a proof of (1.1.6) for $a = c = 0$ and gave a more detailed version of Ramanujan’s proof of (1.1.6) in the case $b = c = 0$. It was not until 1967 that A. O. L. Atkin [2] proved (1.1.6) for arbitrary $c$ and $a = b = 0$.

A partition is very often represented with the help of a diagram called Ferrers–Young diagram. The Ferrers–Young diagram of the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of $n$ is formed by arranging $n$ nodes in $k$ rows so that the $i$th row has $\lambda_i$ nodes. For example, the Ferrers–Young diagram of the partition $\lambda = (3, 2, 1, 1)$ of 7 is

![Ferrers–Young diagram](image)

The conjugate of a partition $\lambda$, denoted $\lambda'$, is the partition whose Ferrers–Young diagram is the reflection along the main diagonal of the diagram of $\lambda$. Therefore, the conjugate of the partition $(3, 2, 1, 1)$ is the partition $(4, 2, 1)$. A partition $\lambda$ is *self-conjugate* if $\lambda = \lambda'$. For example, the partition $(4, 2, 1, 1)$ of 8 is self-conjugate.
The nodes in the Ferrers–Young diagram of a partition are labeled by row and column coordinates as one would label the entries of a matrix. Let \( \lambda'_j \) denote the number of nodes in column \( j \). The hook number \( H(i, j) \) of the \((i, j)\) node is defined as the number of nodes directly below and to the right of the node including the node itself. That is, \( H(i, j) = \lambda_i + \lambda'_j - j - i + 1 \). A partition \( \lambda \) is said to be a \( t \)-core if and only if it has no hook numbers that are multiples of \( t \).

**Example.** The Ferrers–Young diagram of the partition \( \lambda = (4, 2, 1) \) of 7 is

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The nodes \((1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2)\) and \((3, 1)\) have hook numbers 6, 4, 2, 1, 3, 1 and 1, respectively. Therefore, \( \lambda \) is a 5-core. Obviously, it is a \( t \)-core for \( t \geq 7 \).

If \( a_t(n) \) denotes the number of partitions of \( n \) that are \( t \)-cores, then the generating function for \( a_t(n) \) is given by [24, Equation (2.1)], [47, Proposition (3.3)]

\[
\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t; q^t)_\infty}{(q; q)_\infty},
\]

(1.1.7)

Now, if \( \text{asc}_t(n) \) denotes the number of self-conjugate \( t \)-cores of \( n \) then Garvan, Kim and Stanton [24, Eqs. (7.1a) and (7.1b)], and Olsson [48, Eq. (2.37)] found the generating function for \( \text{asc}_t(n) \) as

\[
\sum_{n=0}^{\infty} \text{asc}_t(n)q^n = (-q; q^2)_\infty(q^{2t}; q^{2t})^{t/2}, \quad \text{for } t \text{ even},
\]

(1.1.8)

and

\[
\sum_{n=0}^{\infty} \text{asc}_t(n)q^n = \frac{(-q; q^2)_\infty(q^{2t}; q^{2t})^{(t-1)/2}}{(-q^t; q^{2t})_\infty}, \quad \text{for } t \text{ odd}.
\]

(1.1.9)

Next, given a partition \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \) of \( n \) with distinct parts, the shifted Ferrers-Young diagram of \( \lambda \), \( S(\lambda) \), is the Ferrers-Young diagram of \( \lambda \) with each row shifted to the right by one node than the previous row. The doubled
distinct partition of \( \lambda \) is the partition of \( 2n \) obtained by adding \( \lambda_i \) nodes to the \( (i-1) \)st column of \( S(\lambda) \). For example, we consider the partition \((4, 2, 1)\) of 7 whose Ferrers-Young diagram is as follows:

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The shifted Ferrers-Young diagram of the above partition is given by the following diagram:

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Now adding 4, 2, and 1 nodes respectively to the null, first, and second columns of the above diagram we obtain the Ferrers-Young diagram

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which represents the doubled distinct partition \((5, 4, 4, 1)\) of 14 corresponding to the partition \((4, 2, 1)\) of 7.

The study of self-conjugate partitions and \(t\)-core partitions have played roles in variety of areas. The study of \(t\)-cores for \( t \) prime first arose in connection with Nakayama’s conjecture \([37, 57]\) in representation theory. At the turn of the last century, Young discovered that partitions of \( n \) label the irreducible characters of the symmetric group \( S_n \). At about the same time, Frobenius also found that the hook lengths on the diagonal of a self-conjugate partition determine the irrationalities that occur in the character table of the alternating groups \( A_n \). On account of these connections, \(t\)-core partitions find its place in the study of the representation theorists such as in \([44, 45, 57, 60]\). Hanusa and Jones \([28]\) observed that self-conjugate partitions and \(t\)-core partitions intersect in several important ways. Also,
R. Nath [46] found that self-conjugate $t$-core partitions are central to prove some representation theoretic conjectures in case of alternating groups.

Garvan, Kim and stanton [24, 26] found that $t$-cores are useful in establishing cranks, which are used to show a combinatorial proof of Ramanujan’s famous congruences for the partition function. Garvan [25] also proved some “Ramanujan-type” congruences for $a_p(n)$ for certain special small primes $p$. Hirschhorn and Sellers [32] proved multiplicative formulas for $a_4(n)$ and also conjectured similar multiplicative properties for $a_p(n)$ for other primes $p$.

The $t$-core conjecture has been the topic of a number of papers [23, 25, 27, 40, 41, 49, 50]. This conjecture asserted that every natural number has a $t$-core partition for every integer $t \geq 4$. Granville and Ono [27, 49, 50] have successfully completed the proof of this conjecture using the theory of modular forms and quadratic forms, and the proof has been simplified by Kiming [40].

Again, Baldwin, Depweg, Ford, Kunin and Sze [3] proved that every integer $n > 2$ has a self-conjugate $t$-core partition for $t > 7$, with the exception of $t = 9$, for which infinitely many integers do not have such a partition. We also refer to [31], [35], [32], [51], [14], [15], [4], [38], [9], [10] for further results and generalizations on $t$-cores.

In 1999, Stanton [58] conjectured the monotonicity proposition that, if $n$ and $t$ are natural numbers such that $t \geq 4$ and $n \neq t + 1$, then $a_{t+1}(n) \geq a_t(n)$. This conjecture was proved for certain $t$ by Craven [22] and for large $n$ by Anderson [1]. More precisely, Craven [22] proved that if $n$ and $t$ are integers such that $t > 4$ and $n/2 < t < n - 1$, then $a_t(n) < a_{t+1}(n)$, and Anderson [1] found that if $t_1$ and $t_2$ are fixed integers satisfying $4 \leq t_1 < t_2$, then $a_{t_1}(n) < a_{t_2}(n)$, for sufficiently large $n$. Also, Kim and Rouse [39] use combination of techniques to find explicit bounds for $a_t(n)$ and as an application prove that for all $n \geq 0$, $n \neq t + 1$, $a_{t+1}(n) \geq a_t(n)$ provided $4 \leq t \leq 198$.

Although the monotonicity criterion is conjectured for $t$-core partitions in general, the set of self-conjugate $t$-cores are not found to satisfy a monotonicity criterion
for any \( n \geq 5 \).

However, C. R. H. Hanusa and R. Nath [29] conjectured that

\[
\text{asc}_{2t+2}(n) > \text{asc}_{2t}(n), \quad \text{for all } n \geq 20 \text{ and } 6 \leq 2t \leq 2 \lfloor n/4 \rfloor - 4,
\]

and

\[
\text{asc}_{2t+3}(n) > \text{asc}_{2t+1}(n), \quad \text{for all } n \geq 56 \text{ and } 9 \leq 2t + 1 \leq n - 17.
\]

They also provide the following partial answers to the conjectures:

\[
\text{asc}_{2t+2}(n) > \text{asc}_{2t}(n) \quad \text{when } n/4 < 2t \leq 2 \lfloor n/4 \rfloor - 4,
\]

and

\[
\text{asc}_{2t+3}(n) > \text{asc}_{2t+1}(n) \quad \text{for all } n \geq 48 \text{ and } n/3 < 2t + 1 \leq n - 17.
\]

In this thesis, we use various dissections of Ramanujan’s general theta function in obtaining infinite families of arithmetic identities for the partitions which are \( t \)-cores, self-conjugate \( t \)-cores, and doubled distinct \( t \)-cores for some small \( t \). In the following subsection, we state Ramanujan’s general theta function and a few of its important properties.

### 1.2 Ramanujan’s theta functions and some preliminary results

Ramanujan’s general theta function \( f(a, b) \) is defined by

\[
f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{1.2.1}
\]

Jacobi’s famous triple product identity [16, p. 35, Entry 19] takes the form

\[
f(a, b) = (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}. \tag{1.2.2}
\]
It is easy to verify that

\[ f(a, b) = f(b, a), \]
\[ f(1, a) = 2f(a, a^3), \]
\[ f(-1, a) = 0, \]

and, if \( n \) is an integer,

\[ f(a, b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}). \]

The three most important special cases of \( f(a, b) \) are

\[ \varphi(q) := f(q, q) = 1 + 2 \sum_{n=1}^{\infty} q^n = (-q; q^2)_{\infty}^2(q^2; q^2)_{\infty}, \quad (1.2.3) \]
\[ \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, \quad (1.2.4) \]

and

\[ f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} = (q; q)_{\infty}. \quad (1.2.5) \]

The product representation in (1.2.3)–(1.2.5) arise from (1.2.2).

In the following lemmas we state a few elementary results which will be used in the subsequent chapters of the thesis.

**Lemma 1.2.1.** [16, p. 40, Entries 25(i) and (ii)] We have

\[ \varphi(q) = \varphi(q^4) + 2q\psi(q^8). \quad (1.2.6) \]

**Lemma 1.2.2.** [16, p. 40, Entries 25(iv), (i) and (ii) and (v), (vi)] We have

\[ \psi^2(q) = \psi(q^2)(\varphi(q^4) + 2q\psi(q^8)) \quad (1.2.7) \]

and

\[ \varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \quad (1.2.8) \]
Lemma 1.2.3. [16, p. 45, Entries 29(i) and (ii)] If $ab = cd$, then

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc)$$

(1.2.9)

and

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af(b/c,ac^2d)f(b/d,acd^2)$$

(1.2.10)

and, adding (1.2.9) and (1.2.10), we have

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af(b/c,ac^2d)f(b/d,acd^2).$$

(1.2.11)

Lemma 1.2.4. [16, p. 47, Corollary] If $ab = cd$ then

$$f(a,b)f(c,d)f(an,b/n)f(cn,d/n) - f(-a,-b)f(-c,-d)f(-an,-b/n)f(-cn,-d/n) = 2af(c/a,ad)f(d/an,acn)f(n,ab/n)\psi(ab).$$

(1.2.12)

Lemma 1.2.5. [16, p. 48, Entry 31] If $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ for each integer $n$, then

$$f(U_1,V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right).$$

(1.2.13)

Lemma 1.2.6. [16, p. 49, Corollaries(i) and (ii)] We have

$$\varphi(q) = \varphi(q^9) + 2qf(q^3,q^{15}),$$

(1.2.14)

$$\varphi(q) = \varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4f(q^5,q^{45}),$$

(1.2.15)

$$\psi(q) = f(q^3,q^6) + q\psi(q^9),$$

(1.2.16)

and

$$\psi(q) = f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25}).$$

(1.2.17)

Lemma 1.2.7. [16, p. 39, Entry 24(iii) and p. 51, Example(v)] We have

$$f(q,q^5) = \psi(-q^3)\sqrt[3]{\frac{\varphi(q)}{\psi(-q)}}.$$
1.3 Work carried out in this thesis

The thesis consists of six chapters including this introductory chapter. In the following few paragraphs we briefly introduce the work done in our research.

By using the theory of modular forms, Granville and Ono [27] proved that
\[ a_3(n) = d_{1,3}(3n + 1) - d_{2,3}(3n + 1), \]
where \( d_{r,3}(n) \) is the number of divisors of \( n \) congruent to \( r \) (mod 3).

Again, Baruah and Berndt [4] used a modular equation of Ramanujan to prove that
\[ a_3(4n + 1) = a_3(n), \text{ for all } n \geq 0. \]
(1.3.2)

Though not explicitly written in [4], from the same modular equation it follows that
\[ a_3(4n + 3) = 0, \text{ for all } n \geq 0. \]
(1.3.3)

Hirschhorn and Sellers [35] used some elementary generating function manipulations to prove the result (1.3.1) and then employed that to derive an explicit formula for \( a_3(n) \) in terms of the factorization of \( 3n + 1 \). As corollaries, they derived several infinite families of arithmetic results involving \( a_3(n) \), including the generalizations of (1.3.2) and (1.3.3).

In Chapter 2, we use Ramanujan’s theta function identities to prove that \( u_1(12n + 4) = 6a_3(n) \), where \( u_1(n) \) denotes the number of representations of a nonnegative integer \( n \) in the form \( x^2 + 3y^2 \) with \( x, y \in \mathbb{Z} \). With the help of a classical result by L. Lorenz [43] in 1871, we then deduce (1.3.1). We also show that different proofs of the results by Hirschhorn and Sellers [35] can also be found without considering the factorization of \( 3n + 1 \).

Baruah and Berndt [4] also proved that if \( asc_3(n) \) denotes the number of self-conjugate 3-cores of \( n \) then \( asc_3(4n + 1) = asc_3(n) \), which is analogous to (1.3.2). We generalize this result and prove the following.

Let \( p \equiv 2 \) (mod 3) be prime and \( k \) be a positive even integer. If \( asc_3(n) \) denotes the number of 3-cores of \( n \) that are self-conjugates, then for any positive integer \( n \),
we have

\[ \text{asc}_3(n) = \text{asc}_3 \left( p^k n + \left( \frac{p^k - 1}{3} \right) \right). \]

Chapter 3 of this thesis deals with infinite families of arithmetic identities involving 4-Cores.

In [31, 32], Hirschhorn and Sellers used some elementary generating function manipulations to find certain congruences and the following infinite families of arithmetic relations involving 4-cores: for \( k \geq 1, \)

\[ 3^k a_4(3n) = a_4 \left( 3^{2k+1} n + \frac{5 \times 3^{2k} - 5}{8} \right), \tag{1.3.4} \]

\[ (2 \times 3^k - 1) a_4(3n + 1) = a_4 \left( 3^{2k+1} n + \frac{13 \times 3^{2k} - 5}{8} \right), \tag{1.3.5} \]

\[ \left( \frac{3^{k+1} - 1}{2} \right) a_4(9n + 2) = a_4 \left( 3^{2k+2} n + \frac{7 \times 3^{2k+1} - 5}{8} \right), \tag{1.3.6} \]

\[ \left( \frac{3^{k+1} - 1}{2} \right) a_4(9n + 8) = a_4 \left( 3^{2k+2} n + \frac{23 \times 3^{2k+1} - 5}{8} \right). \tag{1.3.7} \]

Again, if \( h(-D) \) denotes the class number of primitive binary quadratic forms with discriminant \(-D\) and \( a_4(n) \) denotes the number of 4-cores of \( n \), then, for a square-free integer \( 8n + 5 \), Ono and Sze [51] proved that

\[ a_4(n) = \frac{1}{2} h(-32n - 20). \tag{1.3.8} \]

Employing (1.3.8) and the index formulae for class numbers, Ono and Sze [51] proved (1.3.4)–(1.3.7) and some general identities conjectured by Hirschhorn and Sellers [32].

In Section 3.2 of the thesis, we use Ramanujan’s theta function identities to prove that

\[ u(8n + 5) = 8a_4(n) = v(8n + 5) = \frac{1}{3} r_3(8n + 5), \]

where \( u(n) \) and \( v(n) \) denote the number of representations of a nonnegative integer \( n \) in the forms \( x^2 + 4y^2 + 4z^2 \) and \( x^2 + 2y^2 + 2z^2 \), respectively, with \( x, y, z \in \mathbb{Z} \) and
$r_3(n)$ denotes the number of representations of $n$ as a sum of three squares. With the help of this and a classical result of Gauss, we find a simple proof of (1.3.8).

We also find new proofs of (1.3.4)–(1.3.7) as well as the following analogous new infinite families of identities for $a_4(n)$.

For $k \geq 1$, we have

\[
7a_4(5n + 1) = a_4(125n + 40),
\]
\[
5a_4(5n + 2) = a_4(125n + 65),
\]
\[
5a_4(5n + 3) = a_4(125n + 90),
\]
\[
7a_4(5n + 4) = a_4(125n + 115),
\]
\[
\left(\frac{5^{k+1} - 1}{4}\right) a_4(25n) = a_4 \left( 5^{2k+2}n + \frac{5^{2k+1} - 5}{8} \right),
\]
\[
\left(\frac{5^{k+1} - 1}{4}\right) a_4(25n + 5) = a_4 \left( 5^{2k+2}n + \frac{9 \times 5^{2k+1} - 5}{8} \right),
\]
\[
\left(\frac{5^{k+1} - 1}{4}\right) a_4(25n + 10) = a_4 \left( 5^{2k+2}n + \frac{17 \times 5^{2k+1} - 5}{8} \right),
\]
\[
6a_4(25n + 15) = a_4(625n + 390) + 5a_4(n),
\]
\[
\left(\frac{5^{k+1} - 1}{4}\right) a_4(25n + 20) = a_4 \left( 5^{2k+2}n + \frac{33 \times 5^{2k+1} - 5}{8} \right),
\]

and

\[
\left(\frac{3^{k+1} - 1}{2}\right) a_4(9n + 5) = a_4 \left( 3^{2k+2}n + \frac{15 \times 3^{2k+1} - 5}{8} \right) + \frac{3^{k+1} - 1}{2} a_4(n).
\]

In Section 3.4 of the thesis, we also present several infinite families of new arithmetic identities for $r_3(n)$ and $t_3(n)$, and some new proofs for the infinite families of arithmetic identities earlier given by Hirschhorn and Sellers [33, 30].

Chapter 4 of our thesis is devoted to obtaining two infinite families of arithmetic identities for 5-Cores.

Garvan, Kim and Stanton [24] gave one analytic and another bijective proofs of

\[
a_5(5n + 4) = 5a_5(n).
\]
By using a modular equation of degree 5 recorded by Ramanujan in his second notebook [16, p. 280, Entry 13(iii)], Baruah and Berndt [4, Theorem 2.5] proved that
\[ a_5(4n + 3) = a_5(2n + 1) + 2a_5(n). \] (1.3.10)
From the same modular equation it follows that
\[ a_5(4n + 1) = a_5(2n), \] (1.3.11)
which was missed by Baruah and Berndt [4]. From the above two identities and with the help of mathematical induction, we also deduce the following two infinite families of arithmetic identities for 5-cores.

For any positive integers \( n \) and \( k \),
\[ a_5(2^{2k}n + 2^{2k-1} - 1) = \left( \sum_{r=1}^{k} 2^{2k-2r} \right) a_5(2n) = \frac{2^{2k} - 1}{3} a_5(2n) \] (1.3.12)
and
\[ a_5(2^{2k+1}n + 2^{2k} - 1) = \left( 1 + \sum_{r=1}^{k} 2^{(2k+1)-2r} \right) a_5(2n) = \frac{2^{2k+1} + 1}{3} a_5(2n). \] (1.3.13)
The following two infinite families of congruences for \( a_5(n) \) are apparent from (1.3.12) and (1.3.13).

For any positive integers \( n \) and \( k \),
\[ a_5(2^{2k}n + 2^{2k-1} - 1) \equiv 0 \pmod{\frac{2^{2k} - 1}{3}} \]
and
\[ a_5(2^{2k+1}n + 2^{2k} - 1) \equiv 0 \pmod{\frac{2^{2k+1} + 1}{3}}. \]

In the same chapter, in Section 4.3, we use Ramanujan’s theta function identities to find unified proofs of (1.3.9), (1.3.10) and (1.3.11).

Ramanujan found that, if \( p(n) \) is the number of partitions of \( n \), then
\[ \sum_{n=0}^{\infty} p(5n + 4)q^n = 5 \frac{(q^5; q^5)_{\infty}^5}{(q; q^6)_{\infty}}, \] (1.3.14)
which immediately implies one of Ramanujan’s famous partition congruences 

\[ p(5n+4) \equiv 0 \pmod{5} \]. Hardy [55, p. xxxv] says of (1.3.14): “It would be difficult to find more beautiful formulae than the ‘Rogers-Ramanujan’ identities, but here Ramanujan must take second place to Rogers; and, if I had to select one formula from all Ramanujan’s work, I would agree with Major MacMahon in selecting [(1.3.14)].” Hence, (1.3.14) is referred as “Ramanujan’s most beautiful identity”.

We find a new proof of (1.3.14) arising from the analytic version of (1.3.9). We refer to [36] for another elementary proof of “Ramanujan’s most beautiful identity”.

In Chapter 5, we deal with infinite families of arithmetic identities for self-conjugate 5-Cores and 7-Cores.

Let us recall that \( \text{asc}_5(n) \) denotes the number of 5-cores of \( n \) that are self-conjugates. Garvan, Kim and Stanton [24] gave bijective proofs of

\begin{align*}
\text{asc}_5(2n + 1) &= \text{asc}_5(n), \\
\text{asc}_5(5n + 4) &= \text{asc}_5(n), \\
\text{asc}_7(4n + 6) &= \text{asc}_7(n), \\
\text{asc}_7(n) &= 0, \quad \text{if } n + 2 = 4^k(8m + 1).
\end{align*}

Baruah and Berndt [4] proved (1.3.15). Recently, by applying some deep theorems developed by Cao [19], Baruah and Sarmah [7] proved that

\[ \text{asc}_7(8m - 1) = 0. \] (1.3.19)

Now, let \( r_2(n) \) and \( r_3(n) \) denote the number of representations of \( n \) as a sum of two squares and three squares, respectively. In our work, we use Ramanujan’s theta function identities to find relations between \( \text{asc}_5(n) \) and \( r_2(n) \), and between \( \text{asc}_7(n) \) and \( r_3(n) \). We then deduce (1.3.15)–(1.3.19). Interestingly, it turns out that (1.3.18) and (1.3.19) are equivalent [see Corollary 5.5.4]. We also find the following new infinite families of arithmetic properties of self-conjugate 5-cores and 7-cores.

For \( k \geq 1 \) and prime \( p \equiv 3 \pmod{4} \), we have

\[ \text{asc}_5(n) = \text{asc}_5\left(p^{2k}n + (p^{2k} - 1)\right), \]
\[(2 \times 3^k - 1)\text{asc}_7(3n + 2) = \text{asc}_7(3^{2k+1}n + 2(2 \times 3^{2k} - 1)),\]
\[3^k\text{asc}_7(3n) = \text{asc}_7(3^{2k+1}n + 2(3^{2k} - 1)),\]
\[\left(\frac{3^{k+1} - 1}{2}\right)\text{asc}_7(9n + 1) = \text{asc}_7(3^{2k+2}n + (3^{2k+1} - 2)),\]
\[\left(\frac{3^{k+1} - 1}{2}\right)\text{asc}_7(9n + 4) = \text{asc}_7(3^{2k+2}n + 2(3^{2k+1} - 1)),\]
\[\left(\frac{3^{k+1} - 1}{2}\right)\text{asc}_7(9n + 7) = \text{asc}_7(3^{2k+2}n + (3^{2k+2} - 2)) + \left(\frac{3^{k+1} - 3}{2}\right)\text{asc}_7(n - 1),\]

and
\[5\text{asc}_7(5n) = \text{asc}_7(175n + 48),\]
\[5\text{asc}_7(5n + 1) = \text{asc}_7(125n + 73),\]
\[7\text{asc}_7(5n + 2) = \text{asc}_7(125n + 98),\]
\[7\text{asc}_7(5n + 4) = \text{asc}_7(125n + 148),\]
\[\left(\frac{5^{k+1} - 1}{4}\right)\text{asc}_7(25n + 3) = \text{asc}_7(5^{2k+2}n + 5^{2k+1} - 2),\]
\[\left(\frac{5^{k+1} - 1}{4}\right)\text{asc}_7(25n + 8) = \text{asc}_7(5^{2k+2}n + 2 \times 5^{2k+1} - 2),\]
\[\left(\frac{5^{k+1} - 1}{4}\right)\text{asc}_7(25n + 13) = \text{asc}_7(5^{2k+2}n + 3 \times 5^{2k+1} - 2),\]
\[\left(\frac{5^{k+1} - 1}{4}\right)\text{asc}_7(25n + 18) = \text{asc}_7(5^{2k+2}n + 2(2 \times 5^{2k+1} - 1)),\]
\[6\text{asc}_7(25n + 23) = \text{asc}_7(625n + 623) + 5\text{asc}_7(n - 1).\]

In the final chapter we discuss identities for doubled distinct \(t\)-cores for \(t = 3, \ldots, 10\).

If \(\text{add}_t(n)\) denotes the number of doubled distinct partitions of \(n\) that are \(t\)-cores then the generating function for \(\text{add}_t(n)\) is given by Garvan, Kim and Stanton [24, Eq. (8.1a)] as
\[
\sum_{n=0}^{\infty} \text{add}_t(n)q^n = \frac{(-q^2; q^2)^\infty(q^{2t}; q^{2t})^{(t-2)/2}}{(-q; q^2)^\infty}, \quad \text{for \(t\) even},
\]
and
\[
\sum_{n=0}^{\infty} \text{add}_t(n)q^n = \frac{(-q^2; q^2)_{\infty} (q^{2t}; q^{2t})^{(t-1)/2}}{(-q^{2t}; q^{2t})_{\infty}}, \quad \text{for } t \text{ odd.}
\]

We note that \(\text{add}_t(n) = 0\) if \(n\) is odd.

Baruah and Sarmah [7] proved that
\[
\text{asc}_9(8n + 10) = \text{asc}_9(2n),
\]
and as 2 has no self-conjugate 9-core, there is an infinite sequence of positive integers having no self-conjugate 9-cores.

Among several results on \(\text{asc}_t(n)\) and \(\text{add}_t(n)\), Baruah and Sarmah [7] proved that
\[
\text{add}_3(n) = \text{asc}_3(4n),
\]
and
\[
\text{add}_5(n) = \text{asc}_5(2n).
\]

Now, let \(t_2(n)\) and \(t_3(n)\) denote the number of representations of \(n\) as a sum of two triangular numbers and three triangular numbers, respectively, and \(r_2(n)\) and \(r_3(n)\) denote the number of representations of \(n\) as a sum of two squares and three squares, respectively. We present simple alternative proofs of (1.3.20)–(1.3.22). Furthermore, we find several other relations involving \(t_2(n)\), \(t_3(n)\), \(r_2(n)\), \(r_3(n)\), \(\text{add}_t(n)\) and \(\text{asc}_t(n)\), for some small \(t\). For example, we deduce the following:

\[
r_2(24n + 5) = 8\text{asc}_4(3n) = 8\text{add}_6(4n),
\]
\[
r_3(16n + 14) = 48\text{add}_8(2n),
\]
\[
\text{add}_6(4n) = \text{asc}_4(3n),
\]
\[
\text{add}_4(3n) = \text{add}_3(n),
\]
\[
2\text{add}_5(2n) = \begin{cases} 
  t_2(5n + 1), & \text{if } n \equiv 0, 2, 3, 4 \pmod{5}; \\
  t_2(5n + 1) - t_2((n - 1)/5), & \text{if } n \equiv 1 \pmod{5}.
\end{cases}
\]
\[
6\text{add}_7(2n) = \begin{cases} 
  t_3(7n + 4), & \text{if } n \equiv 0, 1, 3, 4, 5, 6 \pmod{7}; \\
  t_3(7n + 4) - t_3((n - 2)/7), & \text{if } n \equiv 2 \pmod{7}.
\end{cases}
\]
As one of the corollaries, we find the following result.

If \( h(-D) \) denotes the class number of primitive binary quadratic forms with discriminant \(-D\) and \( \text{add}_8(n) \) denotes the number of doubled distinct 8-cores of \( n \), then, for a square-free integer \( 16n + 14 \), we have

\[
\text{add}_8(2n) = \frac{1}{4} h(-64n - 56).
\]

Finally, we present several infinite families of new arithmetic identities for \( \text{add}_3(n) \), \( \text{add}_4(n) \), \( \text{add}_5(n) \), \( \text{asc}_4(n) \), \( \text{add}_6(n) \), \( \text{add}_7(n) \), and \( \text{add}_8(n) \) along with a new arithmetic identity for \( \text{add}_{10}(n) \). For example, for any positive integer \( k \), we have the following infinite families of arithmetic identities for \( \text{add}_8(n) \).

\[
3^k \text{add}_8(6n) = \text{add}_8 \left( 2 \times 3^{2k+1} n + \frac{7(3^{2k} - 1)}{4} \right),
\]

\[
(2 \times 3^k - 1) \text{add}_8(6n + 4) = \text{add}_8 \left( 2 \times 3^{2k+1} n + \frac{23 \times 3^{2k} - 7}{4} \right),
\]

\[
\left( \frac{3^{k+1} - 1}{2} \right) \text{add}_8(18n + 2) = \text{add}_8 \left( 2 \times 3^{2k+2} n + \frac{5 \times 3^{2k+1} - 7}{4} \right),
\]

\[
\left( \frac{3^{k+1} - 1}{2} \right) \text{add}_8(18n + 8) = \text{add}_8 \left( 2 \times 3^{2k+2} n + \frac{13 \times 3^{2k+1} - 7}{4} \right),
\]

\[
\left( \frac{3^{k+1} - 1}{2} \right) \text{add}_8(18n + 14) = \text{add}_8 \left( 2 \times 3^{2k+2} n + \frac{21 \times 3^{2k+1} - 7}{4} \right) + \left( \frac{3^{k+1} - 3}{2} \right) \text{add}_8(2n),
\]

and

\[
5 \text{add}_8(10n) = \text{add}_8(250n + 42),
\]

\[
5 \text{add}_8(10n + 4) = \text{add}_8(250n + 142),
\]

\[
7 \text{add}_8(10n + 6) = \text{add}_8(250n + 192),
\]

\[
7 \text{add}_8(10n + 8) = \text{add}_8(250n + 242),
\]

\[
\left( \frac{5^{k+1} - 1}{4} \right) \text{add}_8(50n + 2) = \text{add}_8 \left( 2 \times 5^{2k+2} n + \frac{3 \times 5^{2k+1} - 7}{4} \right),
\]

\[
\left( \frac{5^{k+1} - 1}{4} \right) \text{add}_8(50n + 12) = \text{add}_8 \left( 2 \times 5^{2k+2} n + \frac{11 \times 5^{2k+1} - 7}{4} \right),
\]
\[
\left( \frac{5^{k+1} - 1}{4} \right) \text{add}_8(50n + 22) = \text{add}_8 \left( 2 \times 5^{2k+2} n + \frac{19 \times 5^{2k+1} - 7}{4} \right),
\]
\[
\left( \frac{5^{k+1} - 1}{4} \right) \text{add}_8(50n + 32) = \text{add}_8 \left( 2 \times 5^{2k+2} n + \frac{27 \times 5^{2k+1} - 7}{4} \right),
\]
\[
6\text{add}_8(50n + 42) = \text{add}_8(1250n + 1092) + 5\text{add}_8(2n).
\]