Chapter 4

Two Infinite Families of Arithmetic Identities for 5-Cores

4.1 Introduction

If \( a_5(n) \) denotes the number of partitions of \( n \) that are \( 5 \)-cores, then from (1.1.7) the generating function for \( a_5(n) \) is given by

\[
\sum_{n=0}^{\infty} a_5(n)q^n = \frac{(q^5; q^5)_\infty}{(q; q)_\infty}.
\] (4.1.1)

We note that, by (1.2.5), the formula (4.1.1) reduces to

\[
\sum_{n=0}^{\infty} a_5(n)q^n = \frac{f^5(-q^5)}{f(-q)}.
\] (4.1.2)

By using a modular equation of degree 5 recorded by Ramanujan in his second notebook [16, p. 280, Entry 13(iii)], Baruah and Berndt [4, Theorem 2.5] proved that

\[
a_5(4n + 3) = a_5(2n + 1) + 2a_5(n).
\] (4.1.3)

In fact, they first transcribed the said modular equation into the equivalent form

\[
\frac{f^5(-q^5)}{f(-q)} - 4q^3 \frac{f^5(-q^{20})}{f(-q^4)} = \frac{f^5(q^5)}{f(q)} + 2q \frac{f^5(-q^{10})}{f(-q^2)},
\]

which can be rewritten, with the aid of (4.1.2), as

\[
\frac{1}{2} \left( \sum_{n=0}^{\infty} a_5(n)q^n - \sum_{n=0}^{\infty} (-1)^n a_5(n)q^n \right) = q \sum_{n=0}^{\infty} a_5(n)q^{2n} + 2q^3 \sum_{n=0}^{\infty} a_5(n)q^{4n}.
\] (4.1.4)
Then by equating the coefficients of $q^{4n+3}$ on both sides of (4.1.4), they readily arrived at (4.1.3).

Now, by equating the coefficients of $q^{4n+1}$ on both sides of (4.1.4), we deduce that

$$a_5(4n + 1) = a_5(2n). \quad (4.1.5)$$

The above identity was missed by Baruah and Berndt [4].

In the next section, we give some preliminary results, which will be used in Section 4.3 to prove our main results on $a_5(n)$.

### 4.2 Preliminary results

In the following lemmas, we state some properties satisfied by Ramanujan’s theta functions, which will be used in the subsequent section.

**Lemma 4.2.1.** [16, p. 278] We have

$$\varphi(q^5)\varphi(-q) - \varphi(-q^5)\varphi(q) = -4qf(-q^4)f(-q^{20}). \quad (4.2.1)$$

**Theorem 4.2.2.** We have

$$\psi^3(q)\psi(q^5) - q\psi(q)\psi^3(q^5) = \sum_{n=0}^{\infty} a_5(n)q^n + q \sum_{n=0}^{\infty} a_5(n)q^{2n}. \quad (4.2.2)$$

**Proof.** From [16, p. 262, Entry 10(v)], we note that

$$\psi^2(q) - q\psi^2(q^5) = f(q, q^4)f(q^2, q^3). \quad (4.2.3)$$

Multiplying both sides by $\frac{\psi^3(q^5)}{\psi(q)}$ and employing (1.2.2), we find that

$$\psi(q)\psi^3(q^5) - q\frac{\psi^5(q^5)}{\psi(q)} = \frac{f^5(-q^{10})}{f(-q^2)}. \quad (4.2.4)$$

Again, squaring both sides of (4.2.3), and then multiplying by $\frac{\psi(q^5)}{\psi(q)}$, we obtain

$$\psi^3(q)\psi(q^5) + q^2\frac{\psi^5(q^5)}{\psi(q)} - 2q\psi(q)\psi^3(q^5) = \frac{f^5(-q^5)}{f(-q)}, \quad (4.2.5)$$
where we have repeatedly used (1.2.2).

Multiplying (4.2.4) by $q$, adding with (4.2.5), and then using (4.1.2), we arrive at (4.2.2).

**Theorem 4.2.3.** We have

$$(-q; q^2)^3(-q^5; q^{10})_\infty - (q; q^2)^3(q^5; q^{10})_\infty = \frac{4q}{(q^{10}; q^{20})_\infty(q^2; q^4)_\infty^3} + 2q(q^{10}; q^{20})_\infty^2.$$  (4.2.6)

**Proof.** Setting $a = q$, $b = q^9$, $c = q^3$, $d = q^7$, and $n = q^2$ in (1.2.12), we have

$$f(q, q^9)f(q, q^7)f(q^5, q^5) - f(-q, -q^9)f(q^3, -q^7)f(-q^5, -q^5) = 2qf^2(q, q^8)f(q^4, q^6)\psi(q^{10}).$$  (4.2.7)

Applying (1.2.2) in (4.2.7), and then manipulating the $q$-products, we find that

$$(-q; q^2)_\infty(-q^3, -q^5, -q^7; q^{10})_\infty - (q; q^2)_\infty(q^3, q^5, q^7; q^{10})_\infty = 2q(-q^2; q^2)_\infty(-q^4, -q^6, -q^{10}; q^{10})_\infty.$$  (4.2.8)

Again, setting $a = q^{-1}$, $b = q^{11}$, $c = q^3$, $d = q^7$, and $n = q^2$ in (1.2.12), and then using the trivial identity $f(a, b) = af(a^{-1}, a^2b)$, we obtain

$$(-q; q^2)_\infty(-q, -q^5, -q^9, q^{10})_\infty + (q; q^2)_\infty(q, q^5, q^9, q^{10})_\infty = 2(-q^2; q^2)_\infty(-q^4, -q^6, -q^{10}; q^{10})_\infty.$$  (4.2.9)

Furthermore, setting $a = -q$, $b = -q^9$, $c = q^3$, and $d = q^7$, in (1.2.10), we find that

$$(q, q^9, -q^3, -q^7; q^{10})_\infty - (-q, -q^9, q^3, q^7; q^{10})_\infty = -2q(-q^{10}; q^{20})_\infty^2(q^2, q^6, q^{14}, q^{20})_\infty.$$  (4.2.10)

Multiplying (4.2.8), and (4.2.9) and then using (4.2.10), we arrive at (4.2.6) to finish the proof. □
Remark 4.2.4. The $q$-series identity (4.2.6) is equivalent to the modular equation [16, p. 281, Entry 13(vii)]

$$\left(\alpha^3\beta\right)^{1/8} + \{(1-\alpha)^3(1-\beta)\}^{1/8} = 1 - 2^{1/3} \left\{\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right\}^{1/24},$$

where $\beta$ has degree 5 over $\alpha$.

### 4.3 Main results on $a_5(n)$

At the beginning of this section, we prove (4.3.1) by showing the equivalence of their generating functions.

**Theorem 4.3.1.** If $a_5(n)$ denotes the number of 5-cores of $n$, then

$$5 \sum_{n=0}^{\infty} a_5(n)q^n = \sum_{n=0}^{\infty} a_5(5n+4)q^n. \quad (4.3.1)$$

Here we present two proofs of the above theorem.

**First Proof of Theorem 4.3.1.** Let $t(n)$ be defined by

$$\psi^3(q)\psi(q^5) - q\psi(q)\psi^3(q^5) = \sum_{n=0}^{\infty} t(n)q^n. \quad (4.3.2)$$

Then it is clear from (4.2.2) that

$$\sum_{n=0}^{\infty} t(2n)q^n = \sum_{n=0}^{\infty} a_5(2n)q^n \quad (4.3.3)$$

and

$$\sum_{n=0}^{\infty} t(2n+1)q^n = \sum_{n=0}^{\infty} a_5(2n+1)q^n + \sum_{n=0}^{\infty} a_5(n)q^n. \quad (4.3.4)$$

Now, employing (1.2.17) in (4.3.2), we have

$$\sum_{n=0}^{\infty} t(n)q^n = \psi(q^5) \left(f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25})\right)^3$$

$$- q\psi^3(q^5) \left(f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25})\right). \quad (4.3.5)$$
Extracting the terms involving $q^{5n+4}$ from both sides of (4.3.5), we find that
\begin{align*}
\sum_{n=0}^{\infty} t(5n + 4)q^n &= 6\psi(q)\psi(q^5)f(q^2, q^3)f(q, q^4) - (\psi^3(q)\psi(q^5) - q\psi(q)\psi^3(q^5)). \\
\end{align*}
(4.3.6)

Employing (4.2.3) in (4.3.6), we obtain
\begin{align*}
\sum_{n=0}^{\infty} t(5n + 4)q^n &= 5(\psi^3(q)\psi(q^5) - q\psi(q)\psi^3(q^5)) = 5 \sum_{n=0}^{\infty} t(n)q^n. \\
\end{align*}
(4.3.7)

From (4.3.3) and (4.3.7), we find that
\begin{align*}
5 \sum_{n=0}^{\infty} a_5(2n)q^n = \sum_{n=0}^{\infty} a_5(10n + 4)q^n. \\
\end{align*}
(4.3.8)

To complete the proof, we need to show that (4.3.8) also holds when $2n$ is replaced by $2n + 1$. To this end, extracting the odd parts from both sides of (4.3.7), we have
\begin{align*}
\sum_{n=0}^{\infty} t(2(5n + 4) + 1)q^n &= 5 \sum_{n=0}^{\infty} t(2n + 1)q^n. \\
\end{align*}
(4.3.9)

From (4.3.9) and (4.3.4), we find that
\begin{align*}
5 \sum_{n=0}^{\infty} a_5(2n + 1)q^n + 5 \sum_{n=0}^{\infty} a_5(n)q^n &= \sum_{n=0}^{\infty} a_5(10n + 9)q^n + \sum_{n=0}^{\infty} a_5(5n + 4)q^n. \\
\end{align*}
(4.3.10)

Extracting the even parts from both sides of (4.3.10), and then using (4.3.8), we obtain
\begin{align*}
5 \sum_{n=0}^{\infty} a_5(4n + 1)q^n &= \sum_{n=0}^{\infty} a_5(5(4n + 1) + 4)q^n. \\
\end{align*}
(4.3.11)

Again, extracting the terms involving $q^{4n+1}$ from both sides of (4.3.10), and then using (4.3.11), we find that
\begin{align*}
5 \sum_{n=0}^{\infty} a_5(8n + 3)q^n &= \sum_{n=0}^{\infty} a_5(5(8n + 3) + 4)q^n. \\
\end{align*}
(4.3.12)

We continue the process, and find by mathematical induction that, for any integer $k \geq 2$,
\begin{align*}
5 \sum_{n=0}^{\infty} a_5(2^kn + 2^{k-1} - 1)q^n &= \sum_{n=0}^{\infty} a_5 \left(5(2^kn + 2^{k-1} - 1) + 4\right) q^n. \\
\end{align*}
(4.3.13)
Since any odd integer can always be written in the form \( 2^k n + 2^{k-1} - 1 \), \( n \geq 0 \), \( k \geq 2 \), we conclude from (4.3.13) that

\[
5 \sum_{n=0}^{\infty} a_5(2n + 1)q^n = \sum_{n=0}^{\infty} a_5(10n + 9)q^n. \tag{4.3.14}
\]

From (4.3.8) and (4.3.14), we arrive at (4.3.1) to finish the first proof of Theorem 4.3.1.

**Second proof of Theorem 4.3.1.** Let \( w(n) \) be defined by

\[
\sum_{n=0}^{\infty} w(n)q^n = \varphi(-q)\varphi^3(-q^5). \tag{4.3.15}
\]

Replacing \( q \) by \( -q \) in (1.2.15), using it in (4.3.15), and then extracting the terms involving \( q^{5n+r} \) for \( r = 0, 1, 2, 3, 4 \), respectively, from both sides of the resulting identity, we obtain

\[
\sum_{n=0}^{\infty} w(5n)q^n = \varphi^3(-q)\varphi(-q^5), \tag{4.3.16}
\]

\[
\sum_{n=0}^{\infty} w(5n + 1)q^n = -2\varphi^3(-q) f(-q^3, -q^7), \tag{4.3.17}
\]

\[
w(5n + 2) = 0, \tag{4.3.18}
\]

\[
w(5n + 3) = 0, \tag{4.3.19}
\]

and

\[
\sum_{n=0}^{\infty} w(5n + 4)q^n = 2\varphi^3(-q) f(-q, -q^9), \tag{4.3.20}
\]

respectively.

Now, employing (1.2.15) in (4.3.16), we have

\[
\sum_{n=0}^{\infty} w(5n)q^n = \varphi(-q^5) (\varphi(-q^{25}) - 2q f(-q^{15}, -q^{35}) + 2q^4 f(-q^5, -q^{45}))^3. \tag{4.3.21}
\]

Extracting the terms involving \( q^{5n} \) from both sides of (4.3.21), we find that

\[
\sum_{n=0}^{\infty} w(25n)q^n = \varphi(-q)\varphi^3(-q^5) - 24q\varphi(-q)\varphi(-q^5) f(-q^3, -q^7) f(-q, -q^9). \tag{4.3.22}
\]
Now, from [16, p. 262, Entry 10(iv)], we note that

\[ \varphi^2(q) - \varphi^2(q^5) = 4qf(q, q^9)f(q^3, q^7). \]  \hspace{1cm} (4.3.23)

Replacing \( q \) by \(-q\) in (4.3.23), we have

\[ \varphi^2(-q) - \varphi^2(-q^5) = -4qf(-q, -q^9)f(-q^3, -q^7). \]  \hspace{1cm} (4.3.24)

Employing (4.3.24) in (4.3.22),

\[ \sum_{n=0}^{\infty} w(25n)q^n = 6 \sum_{n=0}^{\infty} w(5n)q^n - 5 \sum_{n=0}^{\infty} w(n)q^n, \]  \hspace{1cm} (4.3.25)

which may also be written as

\[ \sum_{n=0}^{\infty} w(25n)q^n - \sum_{n=0}^{\infty} w(5n)q^n = 5 \left( \sum_{n=0}^{\infty} w(5n)q^n - \sum_{n=0}^{\infty} w(n)q^n \right). \]  \hspace{1cm} (4.3.26)

Next, multiplying both sides of (4.3.24) by \( \varphi^3(-q^5) \varphi(-q) \), and then employing (1.2.2), we find that

\[ (\varphi^2(-q) - \varphi^2(-q^5)) \varphi^3(-q^5) \varphi(-q) = -4q \frac{f^5(-q^5)}{f(-q)}. \]  \hspace{1cm} (4.3.27)

Furthermore, squaring both sides of (4.3.24), and then multiplying by \( \varphi(-q^5) \varphi(-q) \), we obtain

\[ (\varphi^2(-q) - \varphi^2(-q^5))^2 \varphi(-q^5) \varphi(-q) = 16q^2 \frac{f^5(-q^{10})}{f(-q^2)}, \]  \hspace{1cm} (4.3.28)

where we have repeatedly used (1.2.2). Adding (4.3.27) and (4.3.28), and then using (4.1.2), we arrive at

\[ \varphi^3(-q) \varphi(-q^5) - \varphi(-q) \varphi^3(-q^5) = 16q^2 \sum_{n=0}^{\infty} a_5(n)q^{2n} - 4q \sum_{n=0}^{\infty} a_5(n)q^n. \]  \hspace{1cm} (4.3.29)

Employing (4.3.15) and (4.3.16) in (4.3.29), we have

\[ \sum_{n=0}^{\infty} w(n)q^n - \sum_{n=0}^{\infty} w(5n)q^n = 4q \sum_{n=0}^{\infty} a_5(n)q^n - 16q^2 \sum_{n=0}^{\infty} a_5(n)q^{2n}. \]  \hspace{1cm} (4.3.30)

Extracting the even and odd terms from both sides of (4.3.30), we find that
\[ \sum_{n=0}^{\infty} w(2n)q^n - \sum_{n=0}^{\infty} w(10n)q^n = 4 \sum_{n=0}^{\infty} a_5(2n-1)q^n - 16 \sum_{n=0}^{\infty} a_5(n-1)q^n, \]  
(4.3.31)

\[ \sum_{n=0}^{\infty} w(2n+1)q^n - \sum_{n=0}^{\infty} w(10n+5)q^n = 4 \sum_{n=0}^{\infty} a_5(2n)q^n. \]  
(4.3.32)

Now, extracting the terms involving \( q^{2n+1} \) from both sides of (4.3.26), and then employing (4.3.32), we arrive at (4.3.8).

As in the case of the first proof, to complete the proof, we need to show that (4.3.8) also holds when \( 2n \) is replaced by \( 2n + 1 \).

To this end, extracting the terms involving \( q^{2n} \) from both sides of (4.3.26) and then using (4.3.31), we have

\[ 5 \sum_{n=0}^{\infty} a_5(2n-1)q^n - \sum_{n=0}^{\infty} a_5(10n-1)q^n \]
\[ = 4 \left( 5 \sum_{n=0}^{\infty} a_5(n-1)q^n - \sum_{n=0}^{\infty} a_5(5n-1)q^n \right). \]  
(4.3.33)

Replacing \( n \) by \( (2n+1) \) in (4.3.33), and then employing (4.3.8), we find that

\[ 5 \sum_{n=0}^{\infty} a_5(4n+1)q^n = \sum_{n=0}^{\infty} a_5(20n+9)q^n = \sum_{n=0}^{\infty} a_5(5(4n+1)+4)q^n. \]  
(4.3.34)

Again, replacing \( n \) by \( 4n + 2 \) in (4.3.33), and employing (4.3.34), we arrive at (4.3.12). The remaining part of the proof is similar to that of the first proof. \( \Box \)

We now deduce Ramanujan’s “Most Beautiful Identity.”

Corollary 4.3.2. The following identity

\[ \sum_{n=0}^{\infty} p(5n+4)q^n = 5 \frac{(q^5; q^5)_\infty}{(q; q)_\infty^6} \]  
(4.3.35)

holds.

Proof. Since

\[ \frac{1}{f(-q)} = \sum_{n=0}^{\infty} p(n)q^n, \]
by (4.1.2), we have
\[\sum_{n=0}^{\infty} a_5(n)q^n = f^5(-q^5) \sum_{n=0}^{\infty} p(n)q^n.\]  
(4.3.36)

Extracting the terms involving \( q^{5n+4} \) from both sides of (4.3.36), we find that
\[\sum_{n=0}^{\infty} a_5(5n+4)q^n = f^5(-q) \sum_{n=0}^{\infty} p(5n+4)q^n.\]  
(4.3.37)

Employing (4.3.1) in (4.3.37), we obtain
\[\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \frac{1}{f^5(-q)} \sum_{n=0}^{\infty} a_5(n)q^n,\]
from which, with the aid of (4.1.2) again, (4.3.35) follows readily. \qed

In the next theorem, we present various arithmetic properties of \( a_5(n) \).

**Theorem 4.3.3.** If \( a_5(n) \) denotes the number of 5-cores of \( n \), then

\[5a_5(4n + 3) - a_5(8n + 7) = 8a_5(n) + 2a_5(2n + 1),\]  
(4.3.38)

\[5a_5(4n + 1) - a_5(8n + 3) = 2a_5(2n),\]  
(4.3.39)

\[a_5(2n) = 4a_5(4n + 1) - a_5(8n + 3),\]  
(4.3.40)

\[a_5(4n + 1) - a_5(16n + 7) + 4a_5(8n + 3) = 8a_5(2n),\]  
(4.3.41)

\[a_5(4n + 3) - a_5(16n + 15) + 4a_5(8n + 7) = 8a_5(2n + 1) + 4a_5(n).\]  
(4.3.42)

**Proof.** Let \( v_1(n) \) be defined by
\[\sum_{n=0}^{\infty} v_1(n)q^n = 4q\varphi(q)\varphi(q^5)f(q, q^9)f(q^3, q^7).\]  
(4.3.43)

Applying (1.2.2) and (1.2.3) in (4.3.43) and also using (1.2.5), we obtain
\[\sum_{n=0}^{\infty} v_1(n)q^n = 4q(-q; q^2)_{\infty}^3(-q^5; q^{10})_{\infty} f^3(-q^{10})f(-q^2).\]  
(4.3.44)

Replacing \( q \) by \(-q\) in (4.3.44),
\[\sum_{n=0}^{\infty} (-1)^n v_1(n)q^n = -4q(q^2)_{\infty}^3(q^5; q^{10})_{\infty} f^3(-q^{10})f(-q^2).\]  
(4.3.45)
Adding (4.3.44) and (4.3.45), we have
\[ \sum_{n=0}^{\infty} v_1(n)q^n + \sum_{n=0}^{\infty} (-1)^n v_1(n)q^n = 4q f(-q^2)f^3(-q^{10}) ((-q; q^2)^3_{\infty}(-q^5; q^{10})_{\infty} - (q; q^2)^3_{\infty}(q^5; q^{10})_{\infty}). \] (4.3.46)

Employing (4.2.6) in (4.3.46), we find that
\[ \sum_{n=0}^{\infty} v_1(2n)q^n = 8q f(-q)f^3(-q^5)_{\infty} + \frac{4q f(-q)f^3(-q^5)(q; q^2)^3_{\infty}}{(q^5; q^{10})_{\infty}^2}. \] (4.3.47)

Manipulating the \( q \)-products and recalling the product representation of \( \varphi(q) \) from (1.2.3), we rewrite the above in the form
\[ \sum_{n=0}^{\infty} v_1(2n)q^n = 8q f^2(-q^2)f^2(-q^{10})\frac{\varphi(-q^5)}{\varphi(-q)} + 4q(q; q^2)^3_{\infty}(q^5; q^{10}) f(-q^2)f^3(-q^{10}), \]

which, with the aid of (4.3.45), implies
\[ \sum_{n=0}^{\infty} v_1(2n)q^n + \sum_{n=0}^{\infty} (-1)^n v_1(n)q^n = 8q f^2(-q^2)f^2(-q^{10})\frac{\varphi(-q^5)}{\varphi(-q)}. \] (4.3.48)

Replacing \( q \) by \(-q\) in (4.3.48), we have
\[ \sum_{n=0}^{\infty} (-1)^n v_1(2n)q^n + \sum_{n=0}^{\infty} v_1(n)q^n = -8q f^2(-q^2)f^2(-q^{10})\frac{\varphi(q^5)}{\varphi(q)}. \] (4.3.49)

Adding (4.3.48) and (4.3.49), and then using (4.2.1) and the trivial identity \( \varphi(q)\varphi(-q) = \varphi^2(-q^2) \), we find that
\[ \sum_{n=0}^{\infty} v_1(2n)q^n + \sum_{n=0}^{\infty} (-1)^n v_1(2n)q^n + \sum_{n=0}^{\infty} v_1(n)q^n + \sum_{n=0}^{\infty} (-1)^n v_1(n)q^n \]
\[ = 32q^2 f^2(-q^2)f^2(-q^{10})\frac{f(-q^4)f(-q^{20})}{\varphi^2(-q^2)}. \] (4.3.50)

Extracting the terms involving \( q^{2n} \) from both sides of (4.3.50) and then replacing \( q^2 \) by \( q \), we deduce that
\[ \sum_{n=0}^{\infty} v_1(4n)q^n + \sum_{n=0}^{\infty} v_1(2n)q^n = 16q f^2(-q^2)f^2(-q^{10})\frac{\varphi(-q^5)}{\varphi(-q)}, \]
which, by (4.3.48), reduces to
\[
\sum_{n=0}^{\infty} v_1(4n)q^n = \sum_{n=0}^{\infty} v_1(2n)q^n + 2 \sum_{n=0}^{\infty} (-1)^n v_1(n)q^n. \tag{4.3.51}
\]

Now, employing (4.3.23) in (4.3.43), we have
\[
\sum_{n=0}^{\infty} v_1(n)q^n = \phi^3(q)\phi(q^5) - \phi(q)\phi^3(q^5). \tag{4.3.52}
\]

From (4.3.52) and (4.3.29), we obtain
\[
\sum_{n=0}^{\infty} v_1(n)q^n = 16q^2 \sum_{n=0}^{\infty} a_5(n)q^{2n} + 4q \sum_{n=0}^{\infty} (-1)^n a_5(n)q^n. \tag{4.3.53}
\]

Using (4.3.53) in (4.3.51), we deduce that
\[
5 \sum_{n=0}^{\infty} a_5(2n-1)q^n - \sum_{n=0}^{\infty} a_5(4n-1)q^n = 4 \sum_{n=0}^{\infty} a_5(n-1)q^n + 8q^2 \sum_{n=0}^{\infty} a_5(n)q^{2n} - 2q \sum_{n=0}^{\infty} a_5(n)q^n. \tag{4.3.54}
\]

Equating the coefficients of even and odd terms, respectively, from both sides of (4.3.54), we arrive at (4.3.38) and (4.3.39), respectively.

Next, employing (1.2.6) in (4.3.52), and then extracting the terms involving \(q^{4n}\), we find that
\[
\sum_{n=0}^{\infty} v_1(4n)q^n = \sum_{n=0}^{\infty} v_1(n)q^n + 16q^2 \left( \psi^3(q^2)\psi(q^{10}) - q^2\psi(q^2)\psi^3(q^{10}) \right). \tag{4.3.55}
\]

Extracting the terms involving \(q^{2n}\) and \(q^{2n+1}\), respectively, from both sides of (4.3.55), we obtain
\[
\sum_{n=0}^{\infty} v_1(8n)q^n - \sum_{n=0}^{\infty} v_1(2n)q^n = 16q \left( \psi^3(q)\psi(q^5) - q\psi(q)\psi^3(q^5) \right) \tag{4.3.56}
\]

and
\[
\sum_{n=0}^{\infty} v_1(2n+1)q^n = \sum_{n=0}^{\infty} v_1(8n+4)q^n, \tag{4.3.57}
\]
respectively. From (4.3.57) and (4.3.53), we easily deduce (4.3.40).
Again, employing (4.2.2) and (4.3.53) in (4.3.56), we find that
\[
\left( \sum_{n=0}^{\infty} a_5(2n-1)q^n - 4 \sum_{n=0}^{\infty} a_5(n-1)q^n \right) - \left( \sum_{n=0}^{\infty} a_5(8n-1)q^n - 4 \sum_{n=0}^{\infty} a_5(4n-1)q^n \right) \\
= 4q \left( \sum_{n=0}^{\infty} a_5(n)q^n + q \sum_{n=0}^{\infty} a_5(n)q^{2n} \right). \tag{4.3.58}
\]
Equating the coefficients of $q^{2n+1}$ and $q^{2n+2}$, respectively, from both sides of (4.3.58), we arrive at (4.3.41) and (4.3.42), respectively, to finish the proof.

Finally, we are in a position to prove (4.1.3) and (4.1.5).

**Theorem 4.3.4.** Identities (4.1.3) and (4.1.5) hold.

*Proof.* Employing (4.3.40) in (4.3.39), we find that
\[
a_5(8n + 3) = 3a_5(4n + 1). \tag{4.3.59}
\]
Employing (4.3.59) in (4.3.40) we readily arrive at (4.1.3).

Next, replacing $n$ by $2n + 1$ in (4.3.38), we have
\[
a_5(16n + 15) = 5a_5(8n + 7) - 8a_5(2n + 1) - 2a_5(4n + 3) \tag{4.3.60}
\]
Employing (4.3.60) in (4.3.42), we obtain
\[
3a_5(4n + 3) - a_5(8n + 7) = 4a_5(n). \tag{4.3.61}
\]
Using (4.3.38) in (4.3.61), we easily deduce (4.1.5) to complete the proof.

With the aid of (4.1.3), (4.1.5), and mathematical induction, we easily prove the following two infinite families of arithmetic identities for $a_5(n)$.

**Theorem 4.3.5.** Let $a_5(n)$ denote the number of 5-cores of $n$. Then, for any positive integers $n$ and $k$, we have
\[
a_5(2^{2k}n + 2^{2k-1} - 1) = \left( \sum_{r=1}^{k} 2^{2k-2r} \right) a_5(2n) = \frac{2^{2k} - 1}{3} a_5(2n) \tag{4.3.62}
\]
and
\[
a_5(2^{2k+1}+n + 2^{2k} - 1) = \left( 1 + \sum_{r=1}^{k} 2^{(2k+1)-2r} \right) a_5(2n) = \frac{2^{2k+1} + 1}{3} a_5(2n). \tag{4.3.63}
\]
From (4.3.62) and (4.3.63), we readily arrive at the following two infinite families of congruences for $a_5(n)$.

**Corollary 4.3.6.** For any positive integers $n$ and $k$, we have

$$a_5(2^{2k}n + 2^{2k-1} - 1) \equiv 0 \left( \mod \frac{2^{2k} - 1}{3} \right)$$

and

$$a_5(2^{2k+1}n + 2^{2k} - 1) \equiv 0 \left( \mod \frac{2^{2k+1} + 1}{3} \right).$$