CHAPTER 1

Introduction and Fundamentals of Haar Wavelet Method

1.1 Background and Motivation

Integrals and derivatives are basic tools of calculus with numerous applications in science and engineering. Many researchers are involved in developing various numerical schemes for finding solutions of different problems which involves derivatives and integrals. Some integrals and derivatives cannot be found exactly, a few require special functions which themselves are a challenge for computation. Many systems in science and engineering are governed by differential, integral and integro-differential equations. The differential equations arise in many different contexts throughout mathematics and science one way or another, because when describing changes mathematically, the most accurate way uses differentials and derivatives (related, though not quite the same). The partial differential equations (PDEs) form the basis of many mathematical models of physical, chemical and biological phenomena, and more recently their use has spread into economics, financial forecasting, image processing and other fields. The vast majority of PDE models cannot be solved analytically. So, to investigate the predictions of PDE models of such phenomena it is often necessary to approximate their solutions numerically. In most cases, an approximate solution is represented by functional values at certain discrete points (grid points or mesh points). The integral and integro-differential equations provide an efficient tool for modelling a numerous phenomena and processes.

As there is no direct method to solve certain equations like Van-der-Pol equation, Fisher’s equation etc., hence for the solutions numerical methods can be used. For a better approximation of solution with lesser error, there are many numerical techniques available in the literature to solve differential and integral equations. Among them, the finite difference (FD), finite element (FE) and finite volume (FV) methods (Peir’O, 2005) fall under the category of low order methods, sometimes the methods of weighted residuals are
considered. Weighted residual methods (WRM) (Finlayson, 1973) assume that a solution can be approximated analytically or piecewise analytically. The method attempts to minimize the error, for instance, finite differences try to minimize the error specifically at the chosen grid points. Weighted residual methods represent a particular group of methods where an integral error is minimized in a certain way and thereby defining the specific method. The methods of weighted residuals are the approximate methods which determine the solution of the differential equation in the form of functions.

The word wavelet or ondelet has been started attracting the scientific community since 1980’s (Ruch and Fleet, 2009). Wavelet analysis assumed significance due to the successful applications in signal and image processing. The smooth orthonormal basis obtained by the translation and dilation of a single function in a hierarchical fashion proved very useful to develop compression algorithms for signals. From the viewpoint of approximation theory and harmonic analysis, the wavelet theory was important on several counts. The good features of wavelets have generated a huge interest and development of both wavelet theory and applications. Wavelet theory is relatively new and an emerging area in mathematical research. In recent years, wavelets have been applied in several areas including mathematics, physics, chemistry, biology, engineering, statistics and even time series analysis. In approximation theory, image processing and matrix theory, wavelets are recognized as a powerful tool. Different types of wavelets and approximating functions have been used in numerical solutions of initial and boundary value problems.

More recently wavelets have been applied to the numerical solution of differential, integral and integro-differential equations. Many methods have been studied both from the theoretical and the computational point of view. Wavelets have a few interesting applications. Many applications of wavelets have been developed over recent years and now the applications range from biomedical imaging and microarray analysis to data fusion.

Multi-resolution analysis (MRA) gives good understanding of the structure of wavelets. The general procedure in wavelet analysis is to transform a problem into its wavelet domain.
using an appropriate basis. The problem is solved in the wavelet domain and then transformed back to obtain the desired results. An important property of wavelet analysis is the capability to present information in a hierarchical manner. Wavelets are basis functions which are able to represent a signal in the time and frequency domain at the same time. They can be used to approximate an underlying trace or signal, similar to Fourier transforms. The advantages of wavelets are that they are well localized in frequency and time, so can handle a wider range of signals than Fourier analysis. Fourier transform analyzes the composition of a given function in terms of sinusoidal waves of different frequencies and amplitudes whereas wavelets analysis tells how a given function changes from one time period to the next. Wavelet analysis is also more flexible in sense that one can choose a specific wavelet to match the type of function being analyzed.

A wavelet is a wave-like oscillation with amplitude that starts out at zero, increases, and then decreases back to zero. Generally, wavelets are purposefully crafted to have specific properties that make them useful for signal processing. Wavelets can be combined, using a reverse, shift, multiply and sum technique called convolution, with portions of an unknown signal to extract information from the unknown signal. As a mathematical tool, wavelets can be used to extract information from many different kinds of data, including but certainly not limited to audio signals and images. A set of complementary wavelets will deconstruct data without gaps or overlap so that the deconstruction process is mathematically reversible. Thus, sets of complementary wavelets are useful in wavelet based compression/decompression algorithms where it is desirable to recover the original information with minimal loss. If we are interested in the low frequency part and hence discard the high frequency part, what remains is a smoother representation of the original signal with its low frequency components intact. Alternatively, if we are most interested in the high frequency part, we may be able to discard the low frequency part instead. This approach of decomposing a signal into two parts is common for all wavelets. Along this vein, the book by Strang and Nguyen (1996) describes a widely used application of wavelets.
Wavelet algorithms process data at different scales or resolutions. If we look at a signal at a large “window” we would notice gross features. Similarly, if we look at a signal with the small “window” we would notice small “window”. Wavelets permit the accurate representation of a variety of functions and operators. An analogy for how wavelets work is to think of a camera lens that allows taking broad landscape pictures as well as zooming in on microscopic detail that can’t normally be seen by the human eye. A usual starting point to explain how wavelets work is to start with the ideas of Fourier theory, which represent functions in terms of a series of sine and cosine functions (having infinite support).

A disadvantage of wavelet is that the transformation obtained only has representation of the data at a discrete number of resolution levels, each resolution level having a representation approximately twice the frequency of the previous level. The idea of wavelets can be summarized as a family of functions constructed from transformation and dilation of a single function called mother wavelet. There are many kinds of wavelets with different properties, some examples are the Haar wavelets (Haar, 1910), Daubechies wavelets and Meyer’s wavelets (Daubechies, 1998).

The wavelet algorithms for solving differential equations are based on the Galerkin or collocation technique (see book by Politis, 1996 for detail). One possibility for such a type of problems is to make use of the Haar wavelet family. Haar wavelets (which are in fact Daubechies of order one) consists of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. A drawback of the Haar wavelets is their discontinuity. Since the derivatives do not exist at the breaking points it is not possible to apply the Haar wavelets for solving differential and integral directly. There are some possibilities for getting out of this situation.

This motivates the study and application of numerical methods for approximating integrals and derivatives. Therefore numerical approximation by Haar wavelets is one of the convenient ways to find approximate solutions of the differential, integral and integro-differential equations.
1.2 Some Basic Definitions

The concept of wavelet was introduced by J. Morlet, a French geophysicist working in oil company (Guan, 2004), as an alternative for short term Fourier transform. In fact, the first orthonormal wavelet basis was discovered by Alfred Haar in 1909 and P. Franklin in 1927 constructing orthonormal basis of $L^2(R): \{ \psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k) \}, \quad j,k \in \mathbb{Z}.$

**Wavelet:** A wavelet system $\{ \psi_{j,k} = 2^{j/2} \psi(2^j t - k) \}, \quad j,k \in \mathbb{Z}$ is a complete orthonormal set of $L^2(R).$ The functions $\psi_{j,k}$ are called wavelets. The real or complex valued continuous oscillatory function $\psi \in L^2(R)$ with zero mean is a mother wavelet if it has the desirable properties:

1. **Smoothness:** $\psi(t)$ is $n$ times differentiable and that their derivatives are continuous.

2. **Localization:** $\psi(t)$ is well localized both in time and frequency domains, i.e. $\psi(t)$ and its derivatives must decay very rapidly. For frequency localization $\hat{\psi}(\omega)$ must decay sufficiently fast as $|\omega| \rightarrow \infty$ and that $\hat{\psi}(\omega)$ becomes flat in the neighborhood of $\omega = 0.$ The flatness is associated with number of vanishing moments of $\psi(t),$ i.e.

$$\int_{-\infty}^{\infty} t^k \psi(t) dt = 0 \quad \text{or equivalently} \quad \frac{d^k}{d\omega^k} \hat{\psi}(\omega) = 0 \quad \text{for} \quad k = 0, 1, \ldots, n$$

in the sense that larger the number of vanishing moments more is the flatness when $\omega$ is small.

3. **The Admissibility Condition**

$$C_{\psi} = \int_{-\infty}^{\infty} \frac{\left| \hat{\psi}(\omega) \right|}{\omega} d\omega, \quad 0 < C_{\psi} < \infty$$

suggests that $|\hat{\psi}(\omega)|^2$ decays at least as $|\omega|^{-1}$ or $|t|^{-1}$ for $\epsilon > 0.$
Wavelet Series: A function \( \psi(t) \in L^2(R) \) is said to be orthonormal wavelet if the family

\[
\psi_{j,k} = 2^{j/2} \psi(2^j t - k), \quad \| \psi_{j,k} \|_2 = \| \psi \|_2
\]
satisfies the conditions

\[
\langle \psi_{j,k}, \psi_{l,m} \rangle = O_{j,l}, \ldots, \delta_{k,m} : j, k, l, m \in Z
\]

Wavelet series expansion of \( f \in L^2(R) \) is defined by

\[
f(t) = \sum_{j,k=-\infty}^{\infty} a_{j,k} \psi_{j,k}(t)
\]

where the wavelet coefficient are defined as

\[
a_{j,k} = \langle f, \psi_{l,m} \rangle = \int_{-\infty}^{\infty} f(t) \psi_{j,k}(t) dt
\]

Wavelet Transform: The wavelet transform or wavelet analysis is probably the most recent solution to overcome the shortcomings of the Fourier transform. The wavelet transform of a one dimensional function is two dimensional. One imposes some additional condition on the wavelet function in order to make the wavelet transform decrease quickly with decreasing scale.

Let \( \psi \in L^2(R) \) and the dilated-translated function is defined as

\[
\psi(a,b) = \frac{1}{\sqrt{a}} \psi \left( \frac{t-b}{a} \right) \neq 0, \quad a, b \in R
\]

The above function is obtained from \( \psi \) by first dilating by factor \( a \) and then by translated by \( b \). Wavelet transform gives information about the finer details of the signal.

Continuous Wavelet Transform:

The CWT of a function \( f(t) \in L^2(R) \) at a scale \( a \) and position \( b \) is given by

\[
W_\psi \left[ f \right](a,b) = \langle f, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} f(t) \psi_{a,b}(t) dt = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \psi \left( \frac{t-b}{a} \right) dt
\]
Compactly Supported Wavelets

The scaling function is given by
\[ \phi(t) = \sum_{k=0}^{N-1} a_k \phi(2t - k), \quad a_k = \sqrt{2} h_k \]
and the associated wavelet function \( \psi(t) = \sum_{k=2-N}^{1} (-1)^k a_{-k} \phi(2t - k) \).

Index \( N = 2^p ) \), \( p \) a positive integer. The coefficients \( h_k \) satisfy the following conditions:

1. \( a_k = 0 \) for \( k \not\in \{0, 1, 2, \ldots, N - 1\} \)

2. Normalization \( \int \phi dt = 1 \), i.e., \( \sum_{k=0}^{N-1} a_k = 2 \).

3. Orthonormality of Translation

\[ \int \phi(t-k) \phi(t-m) dt = \delta_{k,m}, \text{ i.e., } \sum_{k=0}^{N-1} a_k a_{k-2m} = 2 \delta_{0m}, 1 - \frac{N}{2} \leq m \leq \frac{N}{2} - 1 \]

4. Moment \( \int t^m \psi(t) dt = 0 \), i.e. \( \sum_{k=0}^{N-1} (-1)^k k^m a_k = 0, \quad m = 0, 1, 2, \ldots, \frac{N}{2} - 1 \)

The functions, consisting of translations and dilations of wavelet function \( \psi(2^j t - k) \), form a complete and orthonormal basis of \( L^2(R) \).

Definition 1. Let \( L^2(R) \) be a vector space of square integrable function i.e.
\[ L^2(R) = \left\{ f : R \rightarrow C : \int_R |f(x)|^2 dx < \infty \right\} \text{ for } f, g \in L^2(R), \]

Definition 2. The inner product of \( f, g \in L^2(R) \), is defined by \( \langle f, g \rangle = \int_R f(t) g(t) dt \)

In particular if \( \|f\| = \|f\|_2 = \left( \int_R |f(t)|^2 dt \right)^{1/2} \), we can say that \( f \) is square integrable.

Definition 3. Let \( f : R \rightarrow C \) be a function. The support of \( f(\text{supp } f) \), is the closure of the set \( \{ t \in R : f(t) \neq 0 \} \). We say that \( f \) has compact support if \( \text{supp } f \) is a compact set.
In other words, \( f \) has compact support if there exists \( r < \infty \) such that \( \text{supp } f \subseteq [-r, r] \), s.t \( f(t) = 0 \forall t \) satisfying \( |t| > r \).

1.3 Multi-resolution Analysis

In the development of wavelets and multi-resolution analysis, one needs to make modest assumptions on the refinement function so the theory develops smoothly (Daubechies, 1992; Strang, 1996).

Multi-resolution analysis provides a powerful framework for analyzing functions at various levels of detail or resolution (Mallat, 1989). Multi-resolution analysis entails a sequence of nested closed approximation subspaces \( V_m (m \in \mathbb{Z}) \), family of Haar wavelets utilizes the concept of MRA. The increasing sequence \( \{V_n\}_{n \in \mathbb{Z}} \) of subset of \( L^2(R) \) with scaling function \( \phi \) is called MRA if it satisfies the conditions:

(i) Monotonicity \( \cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots \)

(ii) The spaces \( V_j \) satisfy \( \bigcap_{-\infty}^{\infty} V_n = \{0\} \) and \( \bigcup_{-\infty}^{\infty} V_n = L^2(R) \)

(iii) \( f(t) \in V_0 \) iff \( f(2^n t) \in V_n \) \( \forall n \in \mathbb{Z} \) the space \( V_f \) are scaled versions of the central space \( V_0 \)

(iv) There exists \( \phi \in V_0 \) s.t. \( \{\phi(t-k) : k \in \mathbb{Z}\} \) is a Riesz basis in \( V_0 \).

1.4 Haar Wavelet Applications

Among the wavelet families, which are defined by an analytical expression, special attention deserves the Haar wavelets. In 1910, Alfred Haar introduced the notion of wavelets. The Haar wavelets are the simplest orthonormal wavelet with compact support. In mathematics, the Haar wavelet is a certain sequence of functions. It is now recognized as the first known wavelet. Haar used these functions to give an example of a countable orthonormal system for the space of square integrable functions on the real line. The study of wavelets, and even the term "wavelet", did not come until much later. The Haar wavelet
is also the simplest possible wavelet. The technical disadvantage of the Haar wavelet is that it is not continuous, and therefore not differentiable.

The basic and simplest form of Haar wavelet is the Haar scaling function that appears in the form of a square wave over the interval \([0,1]\), denoted with \(h_1(t)\) and generally written as

\[
\begin{cases} 
1 & t \in [0,1) \\
0 & \text{elsewhere} 
\end{cases}
\]

The above expression, called Haar father wavelet, is the zero\(^{\text{th}}\) level wavelet which has no displacement and dilation of unit magnitude.

A good feature of the Haar wavelets is the possibility to integrate analytically arbitrary times. The Haar wavelets are very effective for treating singularities, since they can be interpreted as intermediate boundary conditions. Haar wavelets reduce the differential equations to the problem of solving a system of algebraic equations, thus greatly simplifying the problem.

Haar wavelets are used in image digital processing, in physics for characterization of Brownian motion, quantum field theory, numerical analysis and many other fields in recent years. The Haar transform is one of the earliest examples of what is known now as a compact, dyadic, orthonormal wavelet transform. Haar functions appear very attractive in many applications as for example, image coding, edge extraction and binary logic design. For applications of the Haar wavelet transform in logic design, efficient ways of calculating the Haar spectrum from reduced forms of Boolean functions are needed. Such methods
were introduced for calculation of the Haar spectrum from disjoint cubes and different types of decision diagrams. Optimal control theory is certainly the field of most extensive applications of Haar wavelets since its appearance, in view of its ability to model hereditary phenomena with long memory. This theory has also several applications, e.g. in structural dynamics, space flights, chemical engineering and economy. There is a significant interest in applications, which includes: process and manufacturing, aerospace and defence, marine and automotive systems, structural and mechanical design, robotics and manufacturing systems, chemical, petrochemical and industrial processes, electric power generation and distribution systems, energy systems and management, operations research and business, socio-economic models, biological and biomedical systems, environmental control, water treatment and ecology management, electrical and electronic systems and health care and support. It also covers a wide range of interdisciplinary and complex systems problems, where multi-agent software solutions, intelligent sensors and either dynamic or static optimization plays a major role.

The Haar wavelets are frequently used in signal processing. The digital images require huge amount of storage space because they have redundant data. The low internet connection can take a considerable amount of time to download the large amount of data. The wavelet transform increases the speed of this procedure. When someone clicks on an image to download, the computer recalls the wave transformed matrix from the computer memory. It proceeds with all approximation coefficients and then detailed coefficients.

Development of the mathematical theory of wavelets initially caused certain excitement in the scientific computations community; however, we will show that not all applications of wavelets result in immediate advantages, particularly in terms of efficiency. Great care should be taken with respect to claims of efficiency.
1.5 Literature Review on Applications of Haar Wavelets for Solving Differential, Integral and Integro-Differential Equations

The goals of approximation theory and numerical computation are similar, even though approximation theory is less concerned with computational issues. In numerical computation, information usually comes in a different, less explicit form. For example, the target function may be the solution to an integral equation or boundary value problem and the numerical analyst needs to translate this into more direct information about the target function. Nevertheless, the two subjects of approximation and computation are inexorably intertwined and it is impossible to understand fully the possibilities in numerical computation without a good understanding of the elements of constructive approximation.

It is noteworthy that the developments of approximation theory and numerical computation followed roughly the same line. The early methods utilized approximation from finite-dimensional linear spaces. In the beginning, these were typically spaces of polynomials, both algebraic and trigonometric.

The original definition of wavelets uses functions defined on the whole real line. In practical cases we want to find the wavelet decomposition of a finite sequence or of a function defined only on an interval. The wavelet method was first applied to solving differential and integral equations in the 1990s. Their shortcoming is that an explicit expression is lacking. This obstacle makes the differentiation and integration of these wavelets very complicated. Numerical difficulties appear in the treatment of nonlinearities, where integrals of products of wavelets and their derivatives must be computed. This can be done by introducing the connection coefficients in Galerkin method (Latto et al., 1991), but this method is applicable only for a narrow class of equations. Mishra and Sabina (2011) also used the connection coefficients in Galerkin method for solving differential equations. The complexity of the wavelet solutions has induced some pessimistic estimates. Strang and Nguyen (1996, p. 394) write “The competition with other methods is severe. We do not necessarily predict that wavelets will win”.
There are some possibilities to come out from this impasse. First the piecewise constant Haar function can be regularized with interpolation splines, this technique has been applied in several papers by Cattani (2001; 2004). The second possibility is to make use of the integral method, by which the highest derivative appearing in the differential equation is expanded into the Haar series. This approximation is integrated while Lepik (2005). The boundary conditions are incorporated by using integration constants; this approach has been realized for the Haar wavelets by Chen and Hsiao (1997). The main idea of this technique is to convert a differential equation into an algebraic one. For the Chen and Hsiao method the choice of solution steps is essential: if the step is too small the coefficient matrix may be nearly singular and its inversion brings to instability of the solution. Cattani (2004) observed that computational complexity can be reduced if the interval of integration is divided into some segments, this method he called the reduced Haar transform. The number of collocation points in each segment is considerably smaller as in the case of Chen and Hsiao method. It is assumed that the highest derivative is constant in each segment, therefore this method is called “piecewise constant approximation”. This method has been applied by Hsiao and Wang (1999). Kiliçman and Zhou (2007) had introduced the Kronecker operational matrices for fractional calculations and some applications. Some of the books include: Chui (1991), Daubechies (1992), Frazier (1999), Hernandez and Weiss (1996), Berrus (1997), Goswami (1999) and Sweldens (1995). When the solution of a system contains components which change at significantly different rates to give changes in the independent variable, the system is said to be “stiff”. Hsiao dealt state analysis of linear time delayed systems via Haar wavelets and carried out comparative study of the same with finite difference method (FDM). Hsiao and Wang (1999) proposed a key idea to transform the time-varying function and its product with the states into a Haar product matrix. Chen and Hsiao (1997) established Haar wavelet method for solving lumped and distributed parameter systems and also wavelet approach to optimizing dynamic systems. Razzaghi and Ordokhani (2001) introduced the rationalized Haar(RH) functions to solve variational problems and differential equations. Maleknejad and Mirzaee (2005) presented the RH wavelet for solving linear integral equations. Hariharan and Kannan (2010) applied a wavelet method for a comparative study of reaction-diffusion equation. Schwab and Stevenson (2008) developed the adaptive wavelet algorithms for elliptic PDE's on product


Cattani (2001; 2004) established the wave propagation of Shannon wavelets and harmonic wavelet analysis method for Fredholm equation of the second kind. Saeedi et al. (2011) and Islam et al. (2010) have established the convergence analysis of the Haar wavelet method. In order to analyze the convergence of Haar wavelet method, they have showed that the method is convergent for a special class of functions in the sense that the corresponding error tends to zero as \( m \) tends to infinity. Mohammadi and Hosseini (2011) introduced a new Legendre wavelet operational matrix of derivative and its applications in solving the singular ordinary differential equations.

Investigation of various wavelet methods and reviews shows that the Haar wavelet method (HWM) is efficient and powerful in solving wide class of linear and nonlinear differential, integral and integro-differential equations. This review intends to provide the great utility of Haar wavelets to science and engineering problems which owes its origin to 1910. Besides future scope and directions involved in developing Haar wavelet algorithm for solving differential and integral equations are addressed.
This thesis is dedicated to the development of Haar wavelet based techniques and algorithms for solving linear-nonlinear initial and boundary value problems, integral and integro-differential equations. The broad range of numerical problems arising in different fields like as Boundary value problems, Nonlinear oscillators (Van der Pol, Duffing and Duffing Van der Pol oscillators), Generalized Lane-Emden equation which play crucial role in astrophysics, Laminar viscous flow Blasius equation, Benjamin-Bona-Mahony-Burgers equation, Coupled Burgers’ equation and Volterra population model are considered systematically. From the mathematical point of view, these problems represent the differential, integral and integro-differential equations which are treated in the mathematical framework of functional analysis, linear algebra and approximation theory.