Chapter: 5
Problem Formulation
This chapter provides an extended mathematical framework for the formulating the problem of reconstruction for the different beam profiles along the various approaches to solve the problem. The results obtained after the implementation are also discussed.

5.1 Problem Formulation for Parallel Beam Scanner
Parallel Beam profile is most used beam profile for all generation of CT scanner. Hence, the problem for reconstruction needs to be modelled and implemented. [1, 2]

5.1.1. Geometry of Parallel Beam Scanner
Figure 5.1 shows pictorial representation of the parallel beam scanner, which is the moveable part of the scanner and is consisting of an emitter of X-rays and a screen on which radiation detectors are placed. This revolves around the body being examined. The detectors measure the attenuated intensity and it is evaluated relative to the original radiation intensity as defined in equation 3.13. This value of projection is represented by $p^p(s, \alpha^p)$ where $\alpha^p$ is the angle at which the projection is carried out and $s$ is the position of a particular place on the screen.

Figure 5.1: Basic Geometry of Parallel Beam Scanner
This method of scanning allows obtaining the image of the attenuation coefficient distribution for one cross-section of the body under examination. As shown in the figure 5.2, the scanner geometry is in the plane x and y coordinates, that is in the plane perpendicular to z-axis. Using the standard definition of Radon transform [1] it can be derived:

\[
p^p(s, \alpha^p) = \int_{-\infty}^{\infty} \mu(s \cos \alpha^p - u \sin \alpha^p, s \sin \alpha^p + u \cos \alpha^p) \, du \quad (5.1)
\]

This allows easy interpretation of value at each point on the screen. The above equation in the frequency domain can be represented by:

\[
p^p(s, \alpha^p) = F_1 \{ p^p(s, \alpha^p) \} = \int_{-\infty}^{\infty} p^p(s, \alpha^p) e^{-j2\pi fs} \, ds \quad (5.2)
\]

As mentioned in equation 3.36, the practical implementation of the above problem can be solved by using two approaches as below:

- Convolution and back projection method;
- Filtration and back projection method.
5.1.2. Convolution and Back projection method

Out of two approaches, reconstruction by convolution and back projection is most popular [3-5] due to its simplicity and its implementation. In this approach, filtering takes place in s-domain and this can be mathematically expressed as:

\[
\mu(x, y) = \int_0^{\pi} F_1^{-1}(P(f, \alpha^p) \cdot f \cdot \text{sign}(f)) \alpha^p \ (5.3)
\]

By applying the Fourier transform, it is converted into form:

\[
\mu(x, y) = \int_0^{\pi} F_1^{-1}(P(f, \alpha^p) \ast F_1^{-1}(f \cdot \text{sign}) \alpha^p \ (5.4)
\]

This will lead to:

\[
\mu(x, y) = \int_0^{\pi} P^p(s, \alpha^p) \ast F_1^{-1}(f \cdot \text{sign}) \alpha^p \ (5.5)
\]

Comparison of equation 3.36 and 5.5 will result into:

\[
\hat{p}^p(x \cos \alpha^p + y \sin \alpha^p, \alpha^p) = P^p(s, \alpha^p) \ast F_1^{-1}(f \cdot \text{sign}) \alpha^p \ (5.6)
\]

By using Hebert transformation, it can be stated:

\[
\hat{p}^p \left( \frac{1}{2\pi} \frac{dP^p(\hat{s}, \alpha^p)}{d\hat{s}} \frac{1}{s - \hat{s}} d\hat{s} \right) = \hat{p}^p(s, \alpha^p) \ (5.7)
\]

The values of \( \hat{p}^p(s, \alpha^p) \) obtained in this process need to be subjected to the process of back-projection in order to reconstruct the final image. Mathematically, it can be expressed as:

\[
\mu(x, y) = B(P^p(s, \hat{p}^p(s, \alpha^p) \ast F_1^{-1}(|f|) \ (5.8)
\]
If it is assumed that $\tilde{\mu}(x, y)$ is a function approximating to the $\mu(x, y)$, then equation 5.8 can be modified as:

$$\mu(x, y) = \int_0^\pi (\int_{-\infty}^{\infty} \mathbb{H}(f). \text{rect}\left(\frac{f}{2f_0}\right). \mathcal{P}(f. \alpha^p). f. \text{sgn}(f). e^{-j2\pi f s} df). d\alpha^p \quad (5.9)$$

where $f_0$ is cut off frequency function. Hence, finally it can be written as:

$$\tilde{\mu}(x, y) = \int_0^\pi \mathcal{P}(s, \alpha^p) * h^{XX}(s). d\alpha^p \quad (5.10)$$

where $h^{XX}(s)$ is the point spread function of the selected convolution kernel. Considering the practical implementation, equation 5.10 needs to be converted into discrete form as there are limited number of projections which are carried out during each revolution of X-ray tube and limited resolution, at which the radiation intensities are measured. The angles at which discrete projections are carried out are represented by:

$$\alpha^p_i = \psi \Delta^p \quad (5.11)$$
where \( \psi = 0, \ldots , \Psi - 1 \) and the detectors are placed at equal distance \( s_l = l \Delta s \), hence the index variable \( l = -\frac{L-1}{2}, \ldots , L - \frac{1}{2} \). The equation 5.10 can be implemented as shown in figure 5.4:

5.1.3. Filtration and Back projection Method

In this approach, the filtering is carried out in the frequency domain [6-8] as contrast to s-domain in convolution approach. Applying Fourier transform to individual projection, the equation 5.4 can be written as:

\[
\mu(x, y) = \int_0^\pi (F_1^{-1}(F_1(P_0(s, \alpha^p)))) \cdot f \cdot \text{sign}(f) \cdot d\alpha^p \quad (5.12)
\]

So,

\[
P_0(x \cos \alpha^p + y \sin \alpha^p, \alpha^p) = F_1^{-1}(F_1(P_0(s, \alpha^p))) \cdot f \cdot \text{sign}(f) \quad (5.13)
\]

Hence,

\[
P_0(s, \alpha^p) = F_1^{-1}(|f| \cdot F_1(P_0(s, \alpha^p))) \quad (5.14)
\]
Mathematically, it can be expressed as:

$$\mu(x, y) = B(F^{-1}_1 \left( f \cdot F_1(P^p(s, \alpha^p)) \right))$$ (5.15)

As seen in section 5.1.2, for the limited baud spectrum, we approximate $\tilde{\mu}(x, y)$ for the function $\mu(x, y)$. Mathematically, it can be written as:

$$\tilde{\mu}(x, y) = \int_0^\infty \int_{-\infty}^{\infty} P(f, \alpha^p) * H^{XX}(f) e^{j2\pi f t} df \, d\alpha^p$$ (5.16)

where $H^{XX}(f)$ is spectrum of selected convolution kernel. Hence, the sequence of operation to implement reconstruction algorithm can be written as:

$$\tilde{\mu}(x, y) = B[F^{-1}_1 \{P^p(s, \alpha^p) \cdot H^{XX}(f)\}]$$ (5.17)

For practical implementation, the above form is converted into discreet form as discussed in section 5.1.2 and flow chart for the implementation is shown in figure 5.5.

![Flow chart for Filtration Back projection Method](image-url)
5.2 Problem Formulation for Fan Beam Scanner

One of the major disadvantages of parallel beam scanner is the lateral movement of the source and detector which leads to artefacts in the reconstructed image. This can be overcome by using fan beam shaped detectors array as discussed in chapter 3. The reconstruction problem can be formulated on the same line as for the parallel beam scanner. Before problem formulation, it is necessary to understand the geometry of scanner as discussed in the next section.

5.2.1. Geometry of Fan beam Scanner

As the name suggests, the beam profile is in fan shaped as opposite to parallel beam as can be seen in figures 5.6. [9]

![Figure 5.6: Basic shape of Fan Beam Profile](image)

It is evident that axis of rotation of the system and the axis of symmetry of radiation beam plays very vital role in geometrical relationship. The axis of rotation is directed along the line perpendicular to the cross section. A ray emitted from the tube at a given angle of rotation and reaching a particular detector can be identified
by parameter \((\beta, \alpha f)\), where \(\beta\) is the angle that the ray makes with the principal axis of the radiation beam and \(\alpha f\) is angle of rotation.

As in the parallel beam, the motion of the tube detector arrangement is rotational. The angle will be in the range of \([0, 2\pi]\), and thus the corresponding range for the \(\beta\) will be:

\[ \beta_{max} = \beta_{min} = \arcsin \left( \frac{R}{R_f} \right) \] (5.18)

where \(R\) is the radius of the circle and \(R_f\) is the radius of circle describing the focus of the tube. In most of cases, the \(\beta\) lies within the range of \([-\pi/6, \pi/6]\). The projection function for the fan beam profile is mathematically represented as:

\[ P_f(\beta, \alpha f) \] (5.19)

Figure 5.7: Basic Geometrical Relationship for Fan Beam

For the practical implementation, the problem must be addressed into the discreet domain. Hence, the geometry of discreet projection system must be analysed. The fan beam scanner can be divided into two broad categories; equiangular sampling
and equidistance sampling. But, for practical reasons and due to limitation of the X-ray tube, equidistance sampling systems are widely used. In the equidistance sampling system, the projection is obtained at predetermined angles $\alpha^f$. Hence, the angular distance $\beta_\eta$ is determined by the location of the detectors. As the system is equidistance, the angular distance between the detectors is equal and is obtained by $\Delta_\beta$.

The discreet angles at which projections are made can be given by:

$$\alpha^f = \gamma\Delta^f_\alpha \quad (5.20)$$

where $\Delta^f_\alpha$ is the angle through which tube screen system is rotated and $\gamma = 0, 1, ..., \Gamma - 1$ is the sample index for each projection. The location of the arc is defined by an angle:

$$\beta_\eta = \eta\Delta_\beta \quad (5.21)$$

where $\Delta_\beta$ is the distance between the radiation detector, $\eta$ is the index of detector matrix. Hence, the projection value for the fan beam profile can be expressed mathematically as:

$$\bar{p}^f(\eta, \gamma) = P^f(\eta\Delta_\beta, \gamma\Delta^f_\alpha) \quad (5.22)$$

### 5.2.2 Rebinning Reconstruction Method

This method uses re-shorting for [10-12] reconstruction of image. In this method, all the projections $P^f(\beta, \alpha^f)$, which correspond to the hypothetical parallel beam projection $p^p(s, \alpha^p)$ are identified. The rays which would be equivalent to the parallel rays will be considered and collected. The projection data obtained will used for the reconstruction as per the methods described in section 5.1. The flow chart for the implementation of this method is shown in the figure 5.8. Mathematically,

$$p^f(\beta, \alpha^f) = p^p(s, \alpha^p) = p^p(R_\beta \sin \beta, \alpha^f + \beta) \quad (5.23)$$
which is simplified as:

\[ p^p(s, \alpha^p) = p^f(\beta, \alpha^f) = p^f\left(\arcsin \frac{s}{R_f}, \alpha^p - \arcsin \frac{s}{R_f}\right) \] (5.24)

Using equation 5.25, the equivalent ray of fan beam systems to parallel beam system is:

\[ \beta = \arcsin \frac{s}{R_f} \] (5.25)

However, for practical implementation, discreet domain must be considered.

---

**Figure 5.8: Flow chart for Implementation of Rebinning Method**
5.2.3 Direct Fan Beam Reconstruction Method

In opposite to the using the reconstruction methods developed for the parallel beam systems, here a direct approach is used to solve the problem [13,14]. Here, the formula for the parallel beam method is found in which the values obtained from the fan beam can be directly used. In other words, this method can be considered as the extension of the parallel beam reconstruction method. Using the equation 5.5, the relationship between the quantities of two approaches can be expressed as:

\[ s = R_f \cdot \sin \alpha^f \] (5.26)

\[ \alpha^p = \alpha^f + \beta \] (5.27)

Converting the equation to polar form, using equation 5.26 and equation 5.27:

\[ r \cos (\alpha^p - \phi) - s = \dot{u} \sin (\beta - \beta) \] (5.28)

where \( \dot{\beta} = \arctan \left( \frac{\alpha^f - \phi}{R_f + r \sin (\alpha^f - \phi)} \right) \). The following sequence of equation based on Radon transforms can be derived:

\[ \mu(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} N(f_1, f_2) \cdot e^{j2\pi(f_1x + f_2y)} df_1 df_2 \] (5.29)

Converting above into polar form:

\[ \mu(x, y) = \frac{1}{2} \int_{0}^{\pi} \int_{-\infty}^{\infty} |f| P(f, \alpha^p) \cdot e^{j2\pi(x \cos \alpha^p + y \sin \alpha^p)} d\alpha^p \] (5.30)

As the system is using fan beam profile, the limit of integration changes:

\[ \mu(x, y) = \frac{1}{2} \int_{0}^{2\pi} \int_{-\infty}^{\infty} |f| P(f, \alpha^p) \cdot e^{j2\pi(x \cos \alpha^p + y \sin \alpha^p)} d\alpha^p \] (5.31)

Converting above equation 5.31 into frequency domain, it is reformulated to:
As discussed in above sections, the above method can be implemented in discreet domain. The implementation steps are shown in the figure 5.9

![Flow chart for implementation of Direct Fan Beam Method](image)

**Figure 5.9: Flow chart for implementation of Direct Fan Beam Method**

### 5.3 Implementation

As discussed in previous sections most of CT scanner either uses the Parallel beam profile or Fan beam profile. But it is evident that if scanner uses the fan beam profile but in one form or other reconstruction problem is reformulated to parallel beam reconstruction problem. Hence, it is logical to address parallel beam reconstruction problem and implement the same. As stated in Chapter 4, MATLAB®(R) platform is used to implement the research.

In order to implement and test the reconstruction method user defined phantom with the resolution 256X256 is used [15], the detail description is discussed in Chapter 7 in detail. The user defined phantom is shown in the figure 5.10:
Before applying the method discussed in previous section is interesting, to view reconstructed image by only applying the back projection algorithm the result obtained can be viewed in the figure 5.11:

![Figure 5.11: Reconstructed User defined Phantom using Back projection](image)

Implementing the method discussed in section 5.1.2 on the MATLAB®(R) platform the phantom reconstructed is shown in the figure 5.12:
On similar lines the method discussed in section 5.1.3 the result obtained is shown in figure 5.12. From the result it can be concluded that by employing the methods discussed in the chapter the quality of the image has improved significantly and thus making the diagnosis more accurate.
The details analysis and comparative analysis of the results obtained by the implementation of this work is being discussed in Chapter 8. The function files that are developed to implement the existing techniques are listed in table 5.1:

<table>
<thead>
<tr>
<th>Sr. No</th>
<th>Name of Function File</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>ct_reconstruction</td>
<td>It is main function file which takes required input from the user and produces the output.</td>
</tr>
<tr>
<td>02</td>
<td>phtantom_data</td>
<td>It creates the phantom data as per the choice of the user.</td>
</tr>
<tr>
<td>02</td>
<td>radon</td>
<td>It converts the generated phantom data into projection data.</td>
</tr>
<tr>
<td>03</td>
<td>simple_backprojection</td>
<td>It performs back projection operation on the pre-processed image.</td>
</tr>
<tr>
<td>04</td>
<td>filtered_backprojections_con</td>
<td>It performs convolution operation on the projection data</td>
</tr>
<tr>
<td>05</td>
<td>filtered_backprojection_fd</td>
<td>It performs filtration operation in frequency domain on projection data.</td>
</tr>
</tbody>
</table>

Table 5.1: Table listing the MATLAB® Function Files Developed for Implementation.

5.4 Concluding Remarks
Chapter discusses mathematical derivation for existing reconstruction techniques and the step wise implementation and testing of these techniques was done on MALTLAB platform and corresponding results obtained as shown in the above shown figures.