Chapter: 3
Reconstruction Techniques

This chapter addresses the basic reconstruction problem, it also discusses the necessary mathematical framework required to solve the problem of reconstruction by algebraic and analytical approaches.

3.1 Image Reconstruction Problem

As mentioned in Chapter 2, the key problem arising in computerized tomography is image reconstruction from projections obtained from the X-ray scanner given geometry. The human beings are heterogeneous in nature hence the mathematical equation for the attenuation can expressed as [1-4]:

\[ I(U) = I(0) \cdot e^{-\int_0^U \mu(x,y) \, du} \] (3.1)

where \( I(0) \) is the initial X-ray intensity; \( I(U) \) is the X-ray intensity after passing through the distance \( U \); \( \mu(x, y) \) is the function defining the spatial distribution of the attenuation coefficient in the sample.

\( \mu(x, y) \) provides the information regarding the spatial distribution of the attenuation coefficient, which in turn provides information regarding the arrangement of various organs inside the body. Applying logarithm to both side of the equation 3.1 results into:

\[ p = \ln \left( \frac{I(0)}{I(U)} \right) = \int_0^U \mu(x, y) \, du \] (3.2)

where \( p \) is the quantity ratio of X-ray intensity directed at a given point in body to the radiation intensity after passing through the body. In conventional X-ray films, less darkening of the film signifies more attenuation of the X-ray radiation.
“Projection” is to create image of the internal body organ. Hence, it is necessary to form beam of X-rays, which will form the image on the screen after X-rays pass through the object. In order to obtain image of cross-section of object in the plane of the projection, the parameter of quantity \( p \) must be obtained as shown in the figure 3.1:

![Simple Projection Systems](image)

**Figure 3.1: Simple Projection Systems**

The first parameter is the variable \( s \) which is describing the axis perpendicular to the direction of X-ray, the value of \( s = 0 \) defines the principal axis of projection. The second parameter is the \( \alpha \), at which, at given movement, the projection is made. This relation can be mathematically expressed as:

\[
p(s, \alpha) = \int_{\mathcal{D}} \mu(x, y) du \quad (3.3)
\]

where the parameter ranges from \(-\infty < s < \infty, 0 \leq \alpha < \pi\).
The above equation is also called Radon transform [1-4]. Mathematically, it can be expressed as:

\[ R: \mu(x, y) \in \mathbb{H} \cup p(s, \alpha) \in \mathbb{R} \quad (3.4) \]

In Computed tomography, Radon transform is performed physically by the attenuation of the X-rays as they pass through the object. The sensors in the gantry record the change intensity of X-rays after passing through the object.

As shown in the figure 3.2, X-ray intensity at point on the screen corresponds to a single value of \( p(s, \alpha) \). Only the material lying in the path of the ray arriving at that point is responsible for the attenuation of the radiation as the radiation is in the form of a parallel beam. It follows the equation that attenuation takes place along the straight line defined by the parameter \( u \), where the total path length is \( U \).

**Figure 3.2: Basic Geometry of Scanner**

One of the major issues with this relationship is its dependence of the attenuation function on the spatial variable \((x, y)\). The integration of variable takes place along the line at a distance \( s \) from projection axis. So, the fixed coordinate system \((x, y)\)
must be converted into moving coordinate system \((s, u)\) that is rotated by an angle \(\alpha\) with respect to the \((x, y)\) system. This relationship can be derived as under:

The trigonometric relationship between the moving coordinate system and fixed coordinate systems can be expressed as under:

From the figure 3.3 it can be interpreted that:

\[ s = s' + s'' \quad (3.5) \]

The \(x\)-relationship in the fixed coordinate system \((x, y)\) can be written as:

\[ \frac{s'}{x} = \cos \alpha \quad (3.6) \]

![Figure 3.3: Detail of Trigonometric Relationship](image)

Similarly, \(y\)-relationship in the fixed coordinate system \((x, y)\) can be written as:

\[ \frac{s''}{x} = \cos \left( \frac{\pi}{2} - \alpha \right) = \sin \alpha \quad (3.7) \]
Hence,  
\[ s = x\cos\alpha + y\sin\alpha \]  
(3.8)

Similarly,  
\[ u = u' + u'' \]  
(3.9)

So, y-relationship in the fixed coordinate system \((x, y)\) can be written as:

\[ \frac{u'}{x} = \cos\alpha \]  
(3.10)

For the x-relationship in the fixed coordinate system \((x, y)\) can be written as:

\[ \frac{u''}{x} = -\sin\alpha \]  
(3.11)

Hence,  
\[ u = -x\sin\alpha + y\cos\alpha \]  
(3.12)

Applying the above relationships to the basic reconstruction equation 3.1, following equation can be obtained:

\[ p(s, \alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x, y)\delta(x\cos\alpha + y\sin\alpha - s)dx\,dy \]  
(3.13)

So, it is evident that to determine the projection function for a particular point \(s\) on the screen for particular angle of rotation \(\alpha\) of the scanner, the sum of values of attenuation coefficients of the object along the path of the ray is needed.
3.2. Reconstruction Methods

There are two methods to find the solution for reconstruction problem. These methods are analytical and iterative. The basic analytical technique includes back projection method and iterative method includes Algebraic Reconstruction Technique (ART) [5-6].

3.2.1 Algebraic Reconstruction Techniques

Algebraic Reconstruction Techniques use the finite series expansion [6-10]. In this method, the solution is characterized by assumption that reconstructed images consist of a finite number of elements. In this method, discretisation takes place before the introduction of discreet form of algorithm. The area of interest is divided into blocks of identical size. These blocks are defined as having a uniform radiation attenuation coefficient. The geometrical centre of each block will be considered as corresponding to one pixel of the reconstructed digital image. The topology of the reconstructed image is as shown in the figure 3.4:

![Figure 3.4: Projection Geometry for Algebraic Reconstruction Technique](image-url)
From the figure, each block in the image is identified horizontally by the coordinates $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Hence, the uniform attenuation coefficient can be represented by $\mu_{ij}$. This makes it independent of the geometry of projection system. Here, each projection value is obtained at an angle $\alpha^p_{\psi}$ and measured at a point on the screen at a distance $s_i$ away from the axis of the projection which is represented by the discreet form of the projection function:

$$\tilde{P}^p = (l, \psi) \equiv P^p(l \Delta^p_s, \psi \Delta^p_s) \quad (3.14)$$

where $l$ is the detector number, in the matrix; $\psi$ is the projection number, $\Delta^p_s$ is the distance between the individual detectors on the screen, $\Delta^p_s$ is the angle, through which the lamp-screen arrangement is rotated after each projection. The radon transform in the discreet form can be represented as:

$$\tilde{P}^p(l, \psi) = R(\mu(x, y)) \quad (3.15)$$

The algebraic approach, in addition to this, assumes that attenuation coefficient distribution $\mu(x, y)$ can be represented approximately as a finite linear combination of basis function and constant coefficient which can be written as:

$$\mu(x, y) \cong \hat{\mu}(i, j) = \sum_{i=1}^{I} \sum_{j=1}^{J} \mu_{ij} Q_{ij}(x, y) \quad (3.16)$$

where $Q_{ij}(x, y)$ is elements of set of basic function; $\mu_{ij}$ is a constant coefficient with the block $(i, j)$. Considering the equation 3.16, equation 3.15 can be reformulated as under:

$$\tilde{P}^p(l, \psi) \cong R(\mu(x, y)) = R \left[ \sum_{i=1}^{I} \sum_{j=1}^{J} \mu_{ij} Q_{ij}(x, y) \right] \quad (3.17)$$

$$\tilde{P}^p(l, \psi) \cong \sum_{i=1}^{I} \sum_{j=1}^{J} \mu_{ij} R(Q_{ij}(x, y)) \quad (3.18)$$
For algebraic method, this equation can be written as:

$$\hat{P}\psi(l,\psi) \equiv \sum_{i=1}^{l} \sum_{j=1}^{j} \mu_{ij} \chi_{ij}(l,\psi) \quad (3.19)$$

where $\chi_{ij}(l,\psi)$ can be interpreted physically as the contribution of a given image block with parameters $(i,j)$ to formulation of the projection value identified by the pair $(l,\psi)$, measured at the screen.

As shown in the figure 3.5, as the ray passes through the test object, all the squares through which part of the ray passes are taken into consideration. The next step is to consider the contribution made by each image block to the way in which ray $(l,\psi)$ passes through in the course of making a series of projections. The value of each contribution $\chi_{ij}(l,\psi)$ varies between 0 and 1 which can be interpreted as when the ray passes through the block is 1 otherwise 0. Hence, values $\chi_{ij}(l,\psi)$ for all projection angles obtained by using equation 3.19 can formulate a system of linear equations.

![Figure 3.5: Determination of the Image Block.](image-url)
Thus, the problem of reconstruction can be solved by algebraic method. From implementation point of view, dimensionality \((I \times J)\) of the array of \(\mu_{ij}\) cannot be implemented as one needs to transform this matrix into vector \(\mu\) with dimension \(I \times J\). One approach to this is by placing successively of the column \(J = 1, \ldots, J\) of the array \(\mu_{ij}\) into the vector \(\mu\). This can be mathematically represented as:

\[
P = \chi \mu \tag{3.20}
\]

where \(P\) the projection is vector with dimensions \(L \times \Psi\); \(\chi\) is the matrix of values \(\chi_{ij}(l, \psi)\) with dimensions \(L \times \Psi \times I \times J\). Hence, the problem is reduced to estimating the value of matrix \(\mu\) based on the value of matrix \(P\). The biggest disadvantage of this method is the complexity of the calculations caused by enamours size of the matrix \(\chi\). For a typical image with dimension of 256 X256, the number of calculation will be 51200 X 65536.

### 3.2.2 Analytical Method

Analytical Method is popularly known as Back-projection method \([11-14]\) and is mathematically expressed as:

\[
B(x, y) = \int_0^\pi P^p(x \cos \alpha^p + y \sin \alpha^p, \alpha^p) \, d\alpha^p \tag{3.21}
\]

This equation assigns each point in space \((x, y)\), as a sum of all projection function values, which correspond to rays going though each point in the course of obtaining projections. Hence, equation 3.21 contains information about attenuation coefficient at that point. But, one of the limitations of this model would produce an indistinct image because back-projection is not same as inverse Radon transform. The image defined by function \(\hat{\mu}(x, y) = B(x, y)\), obtained in this way, would be distorted so much that it will make medical interpretation impossible. The process of obtaining the image \(\mu(x, y)\) by projection and back projection is shown in the figure 3.5:
The relationship between the attenuation function \( \mu(x, y) \), obtained by projection and the attenuation coefficient function \( \mu(x, y) \) of the cross section of the object can be mathematically analysed as follow:

\[
\bar{\mu}(x, y) = B(x, y) (3.22)
\]

\[
\bar{\mu}(x, y) = \int_0^\pi P^p(x \cos \alpha^p + y \sin \alpha^p, \alpha^p) d\alpha^p (3.23)
\]

Substituting the value from equation 3.13 in equation 3.23, a new equation is obtained as under:

\[
\bar{\mu}(x, y) = \int_0^\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x, \dot{y}) \delta(x \cos \alpha + y \sin \alpha - s) d\dot{x} d\dot{y} d\alpha^p (3.24)
\]

Coordinates \((x, \dot{y})\) refer to all the points in the reconstructed image, variable \(s\) only to those points which for a particular projection, lie on the same straight line as the reconstructed image point specified by coordinates \((x, y)\). Substituting the formula for the distance \(s\) of the reconstructed image point, following equation is obtained:
\[ \bar{\mu}(x, y) = \int_0^\infty \left( \int_{-\infty}^\infty \int_{-\infty}^\infty \mu(\hat{x}, \hat{y}) \delta(\hat{x} \cos \alpha + \hat{y} \sin \alpha - x \cos \alpha^p - y \sin \alpha^p) d\hat{x} d\hat{y} \right) d\alpha^p \] (3.25)

Further simplifying,

\[ \bar{\mu}(x, y) = \int_0^\infty \left( \int_{-\infty}^\infty \int_{-\infty}^\infty \mu(\hat{x}, \hat{y}) \delta(x - x \cos \alpha^p + (y - y) \sin \alpha^p) d\hat{x} d\hat{y} \right) d\alpha^p \] (3.26)

Changing the order of integration,

\[ \bar{\mu}(x, y) = \int_{-\infty}^\infty \int_{-\infty}^\infty \mu(\hat{x}, \hat{y}) \delta(x - x \cos \alpha^p + (y - y) \sin \alpha^p) d\hat{x} d\hat{y} d\alpha^p \] (3.27)

Using the substitution, following equation is obtained:

\[ \bar{\mu}(x, y) = \int_{-\infty}^\infty \int_{-\infty}^\infty \mu(\hat{x}, \hat{y}) \frac{1}{\sqrt{(x - \hat{x})^2 + (y - \hat{y})^2}} d\hat{x} d\hat{y} \] (3.28)

Hence, the final equation turns out to be:

\[ \bar{\mu}(x, y) = \mu(x, y) \ast (x^2 + y^2)^{-\frac{1}{2}} \] (3.29)

From the above equation, it is clear that the image that is obtained through back projection lacks information about the actual form of the attenuation function, but is distorted by geometric factor \((x^2 + y^2)^{-\frac{1}{2}}\). This distortion appears to be artefact in the reconstructed image. If a system is shown by only two projections that are performed and it contains artefacts, it takes form of a line lying along the path of the rays as seen in the figure 3.6.

When data processing is done continuously, it is assumed that the object consists of one non-zero point at centre as follows:

\[ \mu(x, y) = \delta(x - x_0, y - y_0) \] (3.30)
where \((x - x_0, y - y_0)\) are the coordinates of the centre of object’s cross section. In frequency domain, the distortion can be represented by:

\[
FUN(f_1, f_2) = (f_1^2 + f_2^2)^{-\frac{1}{2}} (3.31)
\]

**Figure 3.7: a) Sequence of Projection b) Distorted image after Back projection**

Considering the moving coordinate system, equation 3.29 can be re written as:

\[
\tilde{\mu}(r, \phi) = \mu(r, \phi) * \frac{1}{|r|} (3.32)
\]

Following the definition of Fourier Transformation, equation (3.32) can be written as:

\[
P(f, \alpha^p) = \int_{-\infty}^{\infty} P^p(s, \alpha^p) e^{-j2\pi s f} ds (3.33)
\]
The projection for one particular $s$ is the integral over all the points lying in one straight line and using the equation 3.13, following equation can be obtained:

$$P(f, \alpha^p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x, y) \delta(x \cos \alpha + y \sin \alpha - s) e^{-j2\pi fs} ds du \quad (3.34)$$

Modifying the above equation results into:

$$P(f, \alpha^p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(s \cos \alpha^p - u \sin \alpha^p, u \sin \alpha^p + u \cos \alpha^p) e^{-j2\pi fs} ds du \quad (3.35)$$

Converting into $(x, y)$ coordinate system gives:

$$P(f, \alpha^p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x, y) e^{-j2\pi f(x \cos \alpha^p + y \sin \alpha^p)} dx dy \quad (3.36)$$

Applying the definition of two dimensional ‘Fourier transformation’ to equation 3.36 the final form obtained is:

$$P(f, \alpha^p) = \mathcal{M}(f \cos \alpha^p, f \sin \alpha^p) \quad (3.37)$$

From the above equation 3.37, it follow that the frequency spectrum projection carried out at an angle $\alpha^p$ is equal to a section of the two dimensional spectrum of original image. Hence, instead of filtering the whole image in two dimensions, it is enough to filter all projections in one dimension, using the familiar filter form. Also, instead of filtering the image in two dimensions after back projection, each image is filtered separately and followed by back projection. This can be achieved by inverse Radon transform $R^{-1}\{p(s, \alpha^p)\}$. Mathematically it can be expressed as:

$$\mu(x, y) = R^{-1}\{R(\mu(x, y))\} \quad (3.38)$$

This can be further modified as follow [9]

$$\mu(x, y) = R^{-1}p(s, \alpha^p) = \int_{0}^{\pi} \tilde{p}^p(x \cos \alpha^p + y \sin \alpha^p, \alpha^p) d\alpha^p \quad (3.38)$$
Equation 3.38 defines inverse radon transform. The main process in analytical method is to obtain filtered back projection before applying back projection operator to obtain the reconstruction image.

### 3.3 Display of Images

After applying the appropriate reconstruction algorithm to the project data, the next logical step is display of the image. The projection data obtained depends on the density of the object. In case of human organism, if the tissues are diseased, the attenuation of the X-ray will be different as compared to the healthy tissues. Mathematically, the value of attenuation can be expressed as:

\[ \mu: (x, y) \in \mathbb{R}^2 \xrightarrow{H} \mu(x, y) \in [\mu_{\text{min}}, \mu_{\text{max}}] \] (3.39)

The discreet representation of equation (5.33) is:

\[ \hat{\mu}: [1, \ldots, I] \times [1, \ldots, J] \in \mathbb{R}^2 \xrightarrow{H} \mu(x, y) \in [\mu_{\text{min}}, \mu_{\text{max}}] \] (3.40)

One of the important factors which effects the display of image is luminance and defined by the equation [16] as:

\[ \text{lum}(x, y) = \int_{0}^{\infty} \Lambda(\lambda) I_{\text{light}}(x, y, \lambda) d\lambda \] (3.41)

where \( \lambda \) is wavelength of the light and \( I_{\text{light}}(x, y, \lambda) \) is the distribution of the light emitted by the object and \( \Lambda(\lambda) \) is the function of efficiency of the visual system.

The analogue luminance of images need to converted to discreet format before displaying it on the computer screen[15]. For converting the image to discreet form and display, two processes must be undertaken Discretisation and Quantatization as described in figure 3.7.
First, through the sampling process, the luminance is transformed as:

\[ \hat{\text{lum}}[i,j] = \text{lum}(x,y) \cdot \text{comb}(x,y, \Delta x, \Delta y) \quad (3.42) \]

where \( \Delta x = \frac{1}{2f_{x0}} \) is the horizontal raster discretisation; \( \Delta y = \frac{1}{2f_{y0}} \) is the vertical raster discretisation, \( f_{x0} \) is horizontal cut-off frequency and \( f_{y0} \) is vertical cut-off frequency.

\[ \text{comb}(x,y, \Delta x, \Delta y) \triangleq \sum_{i=1}^{I} \sum_{j=1}^{J} \delta(x - i\Delta x, y - j\Delta y) \quad (3.43) \]

where \( I, J \) are the number of image points sampled vertically and horizontally respectively. Combining equation 5.36 and 5.37:

\[ \hat{\text{lum}}[i,j] = \sum_{i=1}^{I} \sum_{j=1}^{J} \delta(x - i\Delta x, y - j\Delta y) \quad (3.44) \]

In frequency domain, equation 5.38 can be expressed as:

\[ \mathcal{LUM}(f_x, f_y) = \mathcal{LUM}(f_x, f_y) \cdot \mathcal{COH}(f_x, f_y) \quad (3.45) \]
After sampling process, the next process is quantisation. But before displaying it to screen, a non linear transformation needs to be applied [15]. For medical purpose, this transformation is Hounsfield scale or CT number [16, 17], which is in honour of CT scanner inventor. The range of value of this scale is from – 1000 to 3000, which makes it necessary to apply window. The window selection is nothing but the selection of scale by two parameters window centre C and window width W. Some of the CT number common used to view human tissue are as follow as for example for human bone is C=1000 HU and W=2500 HU and C=-600 HU and W=1700 HU for the lung imaging.

During the comprehensive topographic examination a series of actions are performed which produces the set of images from the slices of the tissue. During the scan the patient is laid on the table and table will move into the gantry as per the requirement set by the radiologist. The places where the image slices are planned are indicated as Field of view (FOV) markers.

In some of the cases in order to enhance he contrast of the iodine based dye is injected in the body of the patient [18] especially in case of tumours in soft tissue. After obtaining the set of tomography images the next step is to diagnosis by radiologist often it is necessary to measure the distance between the tissues and enlarge the particular tissue which in medical terms is known as region of interest (ROI).

### 3.4 Concluding Remarks

Thus, chapter discusses various aspects of basic reconstruction problem along with the two approaches analytical and algebraic to solve the problem. This chapter also provide necessary frame work for the newly developed approach which is discussed in subsequent chapters along with techniques to display the reconstructed images.