Chapter 2

Harmonic Mean Graphs

In this Chapter, we introduce a new concept namely, Harmonic mean labelings of graphs. S.Somasundaram and R.Ponraj introduced the concept of Mean labelings of graphs. On the similar lines, we define the Harmonic mean labelings of graphs and we obtain the Harmonic mean labeling of some standard graphs.

2.1 Introduction

The Mean labeling of graphs was introduced by S.Somasundaram and R.Ponraj [11]. They called a graph $G = (V,E)$ with $p$ vertices and $q$ edges a mean graph if it is possible to label vertices $x \in V$ with distinct elements $f(x)$ from $0, 1, 2, \ldots, q$ in such a way that when each edge $e = uv$ is labeled with $\frac{(f(u)+f(v))}{2}$ if $f(u) + f(v)$ is even and $\frac{(f(u)+f(v)+1)}{2}$ if $f(u) + f(v)$ is odd, then the resulting edge labels are distinct. On similar lines, we introduce Harmonic mean labeling.
A Graph $G = (V, E)$ with $p$ vertices and $q$ edges is called a Harmonic mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, \ldots, q + 1$ in such a way that when each edge $e = uv$ is labeled with $f(uv) = \left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$ or $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ then the edge labels are distinct. In this case $f$ is called Harmonic mean labeling of $G$.

**Remark 2.1.1.** If $G$ is a Harmonic mean graph, then the vertices get labels from $1, 2, \ldots, q + 1$ and the edges get labels from $1, 2, \ldots, q$.

**Remark 2.1.2.** If $G$ is a Harmonic mean graph, then $1$ must be a label of one of the vertices of $G$, since an edge should get label $1$.

**Remark 2.1.3.** If $u$ gets label $1$ then any edge incident with $u$ must get label $1$ or $2$, since $1 \leq \frac{2m}{m+1} < m$. Hence this vertex must have a degree $\leq 2$.

**Remark 2.1.4.** By the above reasoning, we have that if $G$ is a $k$-regular graph ($k > 2$), then $G$ is not Harmonic.

**Remark 2.1.5.** If $p > q + 1$, then the graph $G = (p, q)$ is not a Harmonic mean graph, since we don’t have sufficient labels from $\{1, 2, 3, \ldots, q + 1\}$ for the vertices of $G$. 
The following are simple examples of Harmonic mean graphs.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{harmonic_mean_graphs.png}
\caption{Harmonic mean graphs}
\end{figure}

In the present chapter, we investigate mean labeling of some standard graphs. In section 2.2, we investigate the Harmonic mean labeling of some classes of trees like path, star, comb, etc. In section 2.3, we investigate the Harmonic mean labeling of cycle related graphs. Particularly Harmonic mean labeling of cycle $C_n$, the complete graph $K_n$, complete bipartite graph $K_{2,2}$, Triangular snake $T_n$, Quadrilateral snake $Q_n$, Polygonal chain $G_{m,n}$, $K_n - e$ etc. Also we investigate the Harmonic mean labeling of particular cases of bistar, subdivision of star and caterpillar.
2.2 Harmonic mean labeling of some trees

In this section, we investigate the Harmonic mean labeling of some classes of trees like path, star, comb etc.

**Theorem 2.2.1.** Any path is a Harmonic mean graph

**Proof.** Let $P_n$ be a path $u_1, u_2, \ldots, u_n$.

Define a function $f : V(P_n) \rightarrow \{1, 2, \ldots, q + 1\}$ by $f(u_i) = i$, $1 \leq i \leq n$.

Obviously $f$ is a Harmonic mean labeling. □

**Example 2.2.2.** A Harmonic mean labeling of $P_6$ is given below

```
1 3 5
2 4 6
```

*Figure 2.2*

The star $K_{1,1}$ is same as $P_2$ and $K_{1,2}$ is $P_3$. Therefore $K_{1,1}$ and $K_{1,2}$ are Harmonic mean graphs by Theorem 2.2.1. Now we display the Harmonic mean labeling of $K_{1,3}$.  

28
Remark 2.2.3. Paths are also mean graphs. Mean and Harmonic mean labeling differ in $K_{1,n}$. 

$K_{1,n}$ is a mean graph iff $n \leq 3$ [15]. For Harmonic mean labeling, we have the following.

**Theorem 2.2.4.** $K_{1,n}$ is a Harmonic mean graph if and only if $n \leq 7$.

**Proof.** $K_{1,1}, K_{1,2}$ are Harmonic mean graphs as already mentioned. Let the central vertex of the star be $u$. The other vertices be $v_1, v_2, \ldots, v_n$ respectively.

**Case (i):** $2 < n \leq 7$, assign 5 to $u$ and $i(1 \leq i \leq 4)$ to $v_i$. Then label $5 + i$ to $v_{4+i}$, $1 \leq i \leq 3$. Clearly this labeling pattern is a Harmonic mean labeling, which is displayed below.
$K_{1,4}$

$K_{1,5}$

$K_{1,6}$

*Figure 2.4*
Case (ii): Assume \( n > 7 \) and suppose \( K_{1,n} \) has Harmonic mean labeling

Here we consider the following subcases

Subcase (ii) (a):

Let the label of the vertex \( u \) be 2

The other vertices \( v_1, v_3, v_4, v_5, v_6, v_7, \ldots \) are labeled 1, 3, 4, 5, 6, 7, \ldots
respectively. Here the edge labels of $uv_4$ and $uv_5$ are from 2 and 3 and the edge label of $uv_6$ is 3 itself. This is not possible.

**Subcase (ii) (b):** The label of the central vertex $u$ be 3. The other vertices $v_1, v_2, v_4, v_5, v_6, v_7, \ldots$ are labeled 1, 2, 4, 5, 6, 7, \ldots respectively.

![Figure 2.6](image)

Here the edge labels of $uv_4$ and $uv_5$ are from 3 and 4 and the edge label of $uv_6$ is 4 itself, which is not possible.

**Subcase (ii) (c):** Let the label of the central vertex be 4. The other vertices $v_1, v_2, v_3, v_5, v_6, v_7, \ldots$ are labeled 1, 2, 3, 5, 6, 7, \ldots respectively.
Here the edge labels of $uv_7$ and $uv_8$ are from 5 and 6. This is not possible.

**Subcase (ii) (d):** A similar argument holds for the case when $u$ is labeled 5.

**Subcase (ii) (e):** Let the label of $u \geq 6$. 

---

*Figure 2.7*

*Figure 2.8*
In this case, there will be no edge with label 1 or no edge with label 2. From all these, we conclude that $K_{1,n}, n > 7,$ is not a Harmonic mean graph.

**Definition 2.2.5.** The graph obtained by joining a single pendant edge to each vertex of path is called a *comb*.

**Theorem 2.2.6.** Combs are Harmonic mean graph.

**Proof.** Let $G$ be a comb with $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Let $P_n$ be a path. Let us take $P_n = v_1v_2 \ldots v_n$ and join a vertex $u_i$ to $v_i$, $1 \leq i \leq n$. Define a function $f: V(G) \to \{1, 2, \ldots, q + 1\}$ by $f(v_i) = 2i - 1, 1 \leq i \leq n$ and $f(u_i) = 2i, 1 \leq i \leq n$. The label of the edge $u_iv_i$ is $2i - 1, 1 \leq i \leq n$ and the label of the edge $v_iv_{i+1}$ is $2i, 1 \leq i \leq n - 1$. This gives a Harmonic mean labeling for a comb.  

\[ \Box \]
Example 2.2.7. A Harmonic mean labeling of the comb obtained from $P_4$ is

\begin{figure}[h]
\centering
\begin{tikzpicture}
  
  \foreach \x in {1,2,3,4,5,6,7,8} {
    \filldraw (\x,0) circle (2pt) node[below] {$\x$};
  }

  \foreach \y in {1,2,3,4} {
    \filldraw (0,\y) circle (2pt) node[left] {$\y$};
  }

  \foreach \y in {1,2,3,4} {
    \draw (\y,0) -- (\y+1,0);
  }

  \foreach \x in {1,2,3} {
    \draw (0,\x) -- (1,\x);
  }

  \foreach \x in {4,5,6} {
    \draw (2,\x) -- (3,\x);
  }

  \foreach \x in {7,8} {
    \draw (4,\x) -- (5,\x);
  }

\end{tikzpicture}
\caption{Figure 2.10}
\end{figure}

Now we find a special tree generated from path.

Theorem 2.2.8. Let $A$ be the collection of paths $P_n^i$ where $n$ is odd. $P_n^i = u_1^i u_2^i \ldots u_n^i$, $(1 \leq i \leq m)$. Let $G$ be the graph obtained from $A$ with $V(G) = \bigcup_{i=1}^{n} V(P_n^i)$ and $E(G) = \bigcup_{i=1}^{n} E(P_n^i) \cup \{u_{i+\frac{1}{2}}, u_{i+\frac{1}{2}} : 1 \leq i \leq m - 1\}$. Then $G$ is a Harmonic mean graph.

Proof. Define $f : V(G) \to \{1, 2, \ldots, q + 1\}$ by $f(u_1^i) = i$, $1 \leq i \leq n$ and $f(u_j^k) = f(u_n^{k-1}) + (j - 1)$, $2 \leq k \leq m$, $1 \leq j \leq n$. Obviously $f$ is a Harmonic mean labeling. \qed
Example 2.2.9. A Harmonic mean labeling of $G$ with $m = 7$, $n = 5$ is given below.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.11.png}
\caption{Figure 2.11}
\end{figure}
2.3 Results on cycle related graphs

In this section, we investigate the Harmonic mean labeling of some graphs which contain cycles. Particularly Harmonic mean labeling of cycle $C_n$, complete graph $K_n$, complete bipartite graph $K_{2,2}$, triangular snake $T_n$, quadrilateral snake $Q_n$, polygonal chain $G_{m,n}$, alternate triangular snake $A(T_n)$, alternate quadrilateral snake $A(Q_n)$, $K_n - e$ were discussed.

**Theorem 2.3.1.** Any cycle is a Harmonic mean graph

**Proof.** Let $C_n$ be the cycle of length $n$. Let the cycle be $u_1u_2\ldots u_nu_1$. Define a function $f : V(C_n) \to \{1, 2, \ldots, q+1\}$ by $f(u_i) = i$, $1 \leq i \leq n$. The edges are labeled with $f(u_iu_{i+1}) = i + 1$, $1 \leq i \leq n - 1$ and $f(u_nu_1) = 1$. Then $f$ is a Harmonic mean labeling. \qed
Example 2.3.2. A Harmonic mean labeling \( C_7 \) is given below.

\[ \]

![Figure 2.12](image1)

Now we investigate the Harmonic mean labeling of complete graphs. Clearly \( K_1 \) is a Harmonic mean graph. Harmonic mean labeling of \( K_2, K_3 \) are given below.

\[ \]

![Figure 2.13](image2)
By Theorem 2.2.1 and Theorem 2.3.1, \(K_2, K_3\) are Harmonic mean graphs.

Now we prove the following

**Theorem 2.3.3.** If \(n > 3\), \(K_n\) is not a Harmonic mean graph.

**Proof.** Suppose \(K_n, n > 3\) is a Harmonic mean graph. By the definition of a Harmonic mean graph, the label of the edge \(uv\) is \(\left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil\) or \(\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor\). For any two positive integers \(a, b\) with \(a \leq b\), we have \(a \leq \frac{2ab}{(a+b)} \leq b\). Hence to get the edge label 1, we need a vertex \(u\) with label 1. There are at least 3 more vertices \(u_1, u_2, u_3\) incident with \(u\). This is not possible by Remark 2.1.3. Hence \(K_n, n > 3\) is not a Harmonic mean graph. \(\square\)

**Remark 2.3.4.** Cycles \(K_2, K_3\) are also mean graphs and \(K_n, n > 3\) is not a mean graph. By Remark 2.1.2 and Remark 2.1.3, \(K_{n,n}\) where \(n > 2\), is not a Harmonic mean graph. However for \(n = 2\) we have the following.

**Example 2.3.5.** A Harmonic mean labeling of \(K_{2,2}\) is given below.
Next we display the Harmonic mean labeling of $K_{2,3}$.

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure2.15.png}
\caption{Figure 2.15}
\end{figure}

**Remark 2.3.6.** In mean labeling of graphs, $K_{2,n}$ is a Harmonic mean graph for all values of $n$.

Next we study a class of graphs called snakes.
Definition 2.3.7. A triangular snake $T_n$ is obtained from a path $v_1v_2\ldots v_n$ by joining $v_i$ to $v_{i+1}$ to a new vertex $w_i$ for $1 \leq i \leq n-1$. That is, every edge of path is replaced by a triangle $C_3$.

Theorem 2.3.8. Triangular snake $T_n$ is a Harmonic mean graph.

Proof. Let $T_n$ be a triangular snake as in definition 2.3.7. Define $f : V(T_n) \to \{1, 2, \ldots, q + 1\}$ by $f(v_i) = 3i - 2$, $1 \leq i \leq n$. $f(w_i) = 3i - 1$, $1 \leq i \leq n$. The label of the edge $v_iv_{i+1}$ is $3i - 1$, $1 \leq i \leq n$. The label of the edge $v_iw_i$ is $3i - 2$, $1 \leq i \leq n$. The label of the edge $v_{i+1}w_i$ is $3i$, $1 \leq i \leq n$. This makes $T_n$ a Harmonic mean graph. \kern1em \square

Example 2.3.9. The Harmonic mean labeling of $T_5$ is given below.

![Figure 2.16](image_url)

Definition 2.3.10. A Quadrilateral snake $Q_n$ is obtained from a path $u_1u_2\ldots u_n$ by joining $u_i$, $u_{i+1}$ to new vertices $v_i$, $w_i$ respectively and joining $v_i$ and $w_i$. That is, every edge of a path is replaced by a cycle $C_4$. 41
**Theorem 2.3.11.** Any Quadrilateral snake $Q_n$ is a Harmonic mean graph.

**Proof.** Define $f : V(Q_n) \rightarrow \{1, 2, \ldots, q + 1\}$ by $f(u_i) = 4i - 3$, $1 \leq i \leq n$, $f(v_i) = 4i - 2$, $1 \leq i \leq n$ and $f(w_i) = 4i$, $1 \leq i \leq n$. The label of the edge $u_iu_{i+1}$ is $4i - 2$, $1 \leq i \leq n$. The label of the edge $u_iv_i$ is $4i - 3$, $1 \leq i \leq n$. The label of the edge $u_{i+1}w_i$ is $4i$, $1 \leq i \leq n$. The label of the edge $v_iw_i$ is $4i - 1$, $1 \leq i \leq n$. This gives a Harmonic mean labeling of $Q_n$. \hfill \Box

**Example 2.3.12.** Harmonic mean labeling of $Q_3$ is given below

![Figure 2.17](image)

**Definition 2.3.13.** A polygonal chain $G_{m,n}$ is a connected graph all of whose $m$ blocks are polygons on $n$ sides.

**Theorem 2.3.14.** Polygonal chain $G_{m,n}$ are Harmonic mean graph for all $m$ and $n$

**Proof.**
Figure 2.18
In $G_{m,n}$ chain, let $u_1 u_2 u_4 u_6 \ldots u_{n-4} u_{n-2} u_{n+1} u_{n-1} u_{n-3} u_{n-5} \ldots u_7 u_5 u_3 u_1$ be the first cycle. The second cycle is connected to the first cycle at the vertex $u_{n+1}$. Let $u_{n+1} u_{n+2} u_{n+4} \ldots u_{n+2} u_{n+4} \ldots u_{2n+1} u_{2n-1} u_{2n-3} \ldots u_{n+7} u_{n+5} u_{n+3} u_{n+1}$ be the second cycle. The third cycle is connected to the second cycle at the vertex $u_{2n+1}$. Let the third cycle be $u_{2n+1} u_{2n+2} u_{2n+4} \ldots u_{3n+1} u_{3n-1} u_{3n-3} \ldots u_{2n+5} u_{2n+3} u_{2n+1}$. In general the $r^{th}$ cycle is connected to the $(r-1)^{th}$ cycle at the vertex $u_{rn+1}$. Let the $r^{th}$ cycle be $u_{rn+1} u_{rn+2} u_{rn+4} u_{rn+6} \ldots u_{(r+1)n-4} u_{(r+1)n-2} u_{(r+1)n+1} u_{(r+1)n-1} u_{(r+1)n-3} u_{(r+1)n-5} u_{(r+1)n-7} \ldots u_{rn+5} u_{rn+3} u_{rn+1}$. 

44
Magnified figure of the $r^{th}$ cycle is given below.

Assume the graph has $m$ cycles. Define a function $f : V(G_{m,n}) \to \{1, 2, \ldots, q+1\}$ by $f(v_i) = i, 1 \leq i \leq mn-1, f(v_n) = mn+1$. Then the label of the edge is given below $f(u_{mn+1}u_{mn+2}) = mn + 1, f(u_{mn+i}u_{mn+i+2}) = mn + i + 1, f(u_{(m+1)n-2}u_{(m+1)n+1}) = \ldots$
\[(m + 1)n - 1, f(u_{(m+1)n+1}u_{(m+1)n-1}) = (m + 1)n.\] Since the graph \(G_{m,n}\) has distinct edge labels, \(G_{m,n}\) is a Harmonic mean graph.

\[\square\]

**Example 2.3.15.** Harmonic mean labeling of \(G_{4,9}\) chain is given below.
Figure 2.20

Now we define the following
Definition 2.3.16. An Alternate Triangular Snake $A(T_n)$ is obtained from a path $u_1u_2\ldots u_n$ by joining $u_i$ and $u_{i+1}$ (alternatively) to new vertex $v_i$. That is every alternate edge of a path is replaced by $C_3$.

Theorem 2.3.17. Alternative Triangular snakes are Harmonic mean graphs

Proof. Let $A(T_n)$ be the Alternate Triangular snake. Here we consider two different cases.

Case (i): If the triangle starts from $u_2$, then define a function $f : V(A(T_n)) \rightarrow \{1, 2, \ldots, q + 1\}$ by $f(u_1) = 1$, $f(u_2) = 2$, $f(u_i) = 2i - 2$, for all $i = 3, 4 \ldots n$ , $f(v_i) = 2i - 1$, for all $i = 2, 4, 6 \ldots n - 2$. The edge are labeled with $f(u_iu_{i+1}) = 2i - 1$ for all $i = 1, 2 \ldots n - 1$. $f(u_iv_i) = 2i - 2$, for all $i = 2, 4, 6 \ldots n - 2$. $f(v_iu_{i+1}) = 2i$, for all $i = 2, 4, 6 \ldots n - 2$. In this case $f$ is a Harmonic mean labeling.

![Figure 2.21](image)

Case (ii): If the triangle starts from $u_1$ then define a function $f : V(AT_n) \rightarrow \{1, 2, \ldots, q + 1\}$ by $f(u_i) = 2i-1$, for all $i=1, 2 \ldots n$,
\( f(v_i) = 2i, \) for all \( i=1,3\ldots n-1. \) The edges are labeled with \( f(u_iu_{i+1}) = 2i, \) for all \( i=1,2\ldots n, \) \( f(u_iv_i) = 2i-1, \) for all \( i=1,3,5\ldots n-1, \) \( f(v_iv_{i+1}) = 2i+1, \) for all \( i=1, 3,5\ldots n-1. \) Then \( f \) is a Harmonic mean labeling.

![Figure 2.22](image)

From case(i) and case(ii) we conclude that Alternate Triangular snake is a Harmonic mean graph. \( \square \)

**Definition 2.3.18.** An Alternate Quadrilateral snake \( A(Q_n) \) is obtained from a path \( u_1u_2\ldots u_n \) by joining \( u_i, u_{i+1} \) (alternatively) to new vertices \( v_i, w_i \) respectively and then joining \( v_i \) and \( w_i \). That is every alternate edge of a path is replaced by a cycle \( C_4. \)

**Theorem 2.3.19.** Alternate Quadrilateral snakes are Harmonic mean graphs

**Proof.** Let \( A(Q_n) \) be the Alternate Quadrilateral snake. Here we consider two different cases.
Case (i): If the Quadrilateral starts from $u_1$ then define a function $f : V(AQ_n) \rightarrow \{1, 2 \ldots q + 1\}$ by $f(u_1) = 1$, $f(u_2) = 4$, $f(u_i) = f(u_{i-2}) + 5$ for all $i = 3, 4 \ldots n$. $f(v_i) = f(u_i) + 1$, for all $i = 1, 3, 5 \ldots n - 1$. $f(v_i) = f(u_i) - 1$, for all $i = 2, 4 \ldots n$. Edges are labeled with $f(u_1u_2) = 2$, $f(u_2u_3) = 5$, $f(u_iu_{i+1}) = f(u_{i-2}u_{i-1}) + 5$ for all $i = 3, 4 \ldots n - 1$ and $f(v_1v_2) = 3$, $f(v_iv_{i+1}) = f(v_{i-2}v_{i-1}) + 5$ for all $i = 3, 4 \ldots n - 1$, $f(u_1v_1) = 1$, $f(u_2v_2) = 4$, $f(u_iv_i) = f(u_{i-2}v_{i-2}) + 5$ for all $i = 3, 4 \ldots n$. Obviously $f$ is a Harmonic mean labeling.

![Figure 2.23](image_url)

Case (ii): If the Quadrilateral starts from $u_2$ then define a function $f : V(AQ_n) \rightarrow \{1, 2 \ldots q + 1\}$ by $f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 5$, $f(u_i) = f(u_{i-2}) + 5$ for all $i = 4, 5 \ldots n - 1$. $f(v_1) = 3$, $f(v_2) = 4$, $f(v_i) = f(v_{i-2}) + 5$, for all $i = 3, 4, 5 \ldots n - 1$. Edges are labeled with $f(u_1u_2) = 1$, $f(u_2u_3) = 3$, $f(u_iu_{i+1}) = f(u_{i-2}u_{i-1}) + 5$ for all $i = 3, 4 \ldots n - 1$ $f(u_1v_1) = 1$, $f(u_2v_2) = 4$, $f(u_iv_i) = f(u_{i-2}v_{i-2}) + 5$ for all $i = 3, 4 \ldots n$.
\( f(u_i v_i) = f(u_{i-2} v_{i-2}) + 5 \) for all \( i = 3, 4 \ldots n \),

Hence \( f \) is a Harmonic mean labeling.

From case(i) and case(ii), we conclude that \( A(Q_n) \) is a Harmonic mean graph. \( \square \)

**Theorem 2.3.20.** Let \( P_n \) be the path and \( G \) be the graph obtained from \( P_n \) by attaching \( C_3 \) in both the end edges of \( P_n \). Then \( G \) is a Harmonic mean graph.

**Proof.** Let \( P_n \) be the path \( u_1 u_2 \ldots u_n \) and \( v_1 u_1 u_2 \), \( v_2 u_{n-1} u_n \) be the triangles at the end. Define a function \( f : V(G) \to \{1, 2, \ldots, q + 1\} \) by \( f(u_i) = i + 1, 1 \leq i \leq n - 1 \). \( f(u_n) = n + 3, f(v_1) = 1, f(v_2) = n+2 \). The edges labels are given below \( f(u_1 v_1) = 1, f(u_2 v_1) = 2, f(u_{n-1} v_2) = n + 1, f(u_n v_2) = n + 3, f(u_{n-1} u_n) = n + 2 \). Obviously \( G \) is a Harmonic mean graph. \( \square \)
Example 2.3.21. A Harmonic mean labeling of $G$ obtained from $P_8$ is given below.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{example.png}
\caption{Figure 2.25}
\end{figure}

Further, we investigate square of a path.

Definition 2.3.22. [11] The square $G^2$ of a graph $G$ has $V(G^2) = V(G)$, with $u, v$ adjacent in $G^2$ whenever $d(u, v) \leq 2$ in $G$.

Theorem 2.3.23. The graph $P_n^2$ is a Harmonic mean graph.

Proof. Let $P_n$ be the path $u_1, u_2, \ldots, u_n$. Clearly $P_n^2$ has $n$ vertices and $2n - 3$ edges. Define $f : V(P_n^2) \rightarrow \{1, 2, \ldots, q + 1\}$ by $f(u_i) = 2i - 1, 1 \leq i \leq n - 1$ and $f(u_n) = 2n - 2$. The label of the edge $u_i u_{i+1}$ is $2i - 1, 1 \leq i \leq n - 1$. The label of the edge $u_i u_{i+2}$ is $2i, 1 \leq i \leq n - 2$. The label of the edge $u_{n-1} u_n$ is $2n - 3$. Hence $P_n^2$ is a Harmonic mean graph. \hfill \qed
Example 2.3.24. Harmonic mean labeling of $P^2_8$ is given below.

![Harmonic mean labeling of $P^2_8$](image)

Figure 2.26

Now we investigate a class of graphs generated by cycles.

**Definition 2.3.25.** [11] The graph $C^{(t)}_n$ denotes the one point union of $t$ copies of cycle $C_n$. The graph $C^{(t)}_3$ is called a friendship graph (or) Dutch t-wind mill.

$C^{(1)}_3$ is a cycle, and so is a Harmonic mean graph by Theorem 2.3.1. A Harmonic mean labeling of $C^{(2)}_3$ is given below.
Now we prove the following

**Theorem 2.3.26.** Let $C_n$ be the cycle $u_1u_2\ldots u_nu_1$. Let $G$ be a graph with $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{u_2u_n\}$. Then $G$ is a Harmonic mean graph.

**Proof.** Define a function $f : V(G) \to \{1, 2\ldots q + 1\}$ by $f(u_1) = 1$, $f(u_2) = 4$, $f(u_n) = 3$ and $f(u_i) = i+2$, $3 \leq i \leq n-1$. Correspondingly we assign edge labels as in the following figure.
Then we get distinct edge labels from $1, 2, \ldots q$. Hence $f$ is a Harmonic mean labeling.

Example 2.3.27. A Harmonic mean labeling of $G$ obtained from $C_9$ is given below.
Theorem 2.3.28. Let $C_n(n > 3)$, be the cycle $u_1u_2\ldots u_nu_1$ and let $G$ be the graph with $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{u_2u_{n-1}\}$. Then $G$ is a Harmonic mean graph.

Proof. Define a function $f : V(G) \to \{1, 2, \ldots q + 1\}$ by $f(u_1) = 1$, $f(u_2) = 5$, $f(u_{n-1}) = 3$, $f(u_n) = 2$ and $f(u_i) = i + 3$, $3 \leq i \leq n - 2$. Correspondingly we assign edge labels as in the following figure.
Then we get distinct edge labels from 1, 2, \ldots q. Hence $f$ is a Harmonic mean labeling.

Example 2.3.29. A Harmonic mean labeling of $G$ obtained from $C_{14}$ is given below.
Remark 2.3.30. In the case of $C_3$, we already have a Harmonic mean labeling (Theorem 2.3.1).

Theorem 2.3.31. The crown $C_n \odot K_1$ is a Harmonic mean graph for all $n \geq 3$. 
Proof. Let $C_n$ be the cycle $u_1u_2 \ldots u_nu_1$ and $v_i$ be the pendant vertices adjacent to $u_i$, $1 \leq i \leq n$. Define a function $f : V(C_n \odot K_1) \rightarrow \{1, 2 \ldots q + 1\}$ by $f(u_i) = 2i$ and $f(v_i) = 2i - 1$, $1 \leq i \leq n$. Obviously $f$ is a Harmonic mean labeling.

Example 2.3.32. A Harmonic mean labeling of $C_6 \odot K_1$ is given below

Next we prove the following

Theorem 2.3.33. $C_n \odot \overline{K}_2$ is a Harmonic mean graph for all $n \geq 3$.

Proof. Let $C_n$ be the cycle $u_1u_2 \ldots u_nu_1$ and $v_i, w_i$ be the pendant vertices adjacent to $u_i$, $1 \leq i \leq n$. Define a function $f : V(C_n \odot \overline{K}_2) \rightarrow$
\[\{1, 2 \ldots q + 1\} \text{ by } f(u_i) = 3i - 1, \ 1 \leq i \leq n, \ f(v_i) = 3i - 2, \ 1 \leq i \leq n\]
\[\text{and } f(w_i) = 3i, \ 1 \leq i \leq n. \]  Then the edge labels are all distinct.

Hence \(C_n \odot \overline{K}_2\) is a Harmonic mean graph.

\[\square\]

**Example 2.3.34.** Here we display the Harmonic mean labeling of \(C_7 \odot \overline{K}_2\).

![Diagram](image)

In the similar manner, we can see the Harmonic mean labeling \(C_n \odot \overline{K}_3\). Harmonic mean labeling of \(C_8 \odot \overline{K}_3\) in given below.
Now we discuss the Harmonic mean labeling of $K_n - e$.

**Definition 2.3.35.** $K_n - e$ is a graph obtained by deleting an edge $e$ from $K_n$.

**Remark 2.3.36.** $K_2 - e$ is a set of two isolated vertices. Here $p = 2$, $q + 1 = 1$. Since $p > q + 1$ and $q = 0$, which is not a label in a Harmonic mean graph. By Remark 2.1.5, $K_2 - e$ is not a Harmonic mean graph.
**Remark 2.3.37.** \( K_3 - e = P_2 \), which is path of length 2. By Theorem 2.2.1, \( K_3 - e \) is Harmonic.

**Remark 2.3.38.** \( K_4 - e \) is also Harmonic Now we display the Harmonic mean labeling of \( K_4 - e \).

![Figure 2.35](image)

**Theorem 2.3.39.** If \( n > 4 \), \( K_n - e \) is not a Harmonic mean graph.

**Proof.** Since each vertex of \( K_n - e \), \( n > 4 \) has degree greater than 2, \( K_n - e \), \( n > 4 \) is not Harmonic by Remark 2.1.4.

**Remark 2.3.40.** In mean labeling also we have that if \( n > 4 \), \( K_n - e \) is not a Harmonic mean graph [11].

**Definition 2.3.41.** [25] Let \( v \) be a vertex of a graph \( G \). Then the duplication of \( v \) is a graph \( G(v) \) obtained from \( G \) by adding a new vertex \( v' \) with \( N(v') = N(v) \).
Theorem 2.3.42. The graph obtained by duplicating an arbitrary vertex of $C_n$ admits a Harmonic mean labeling.

Proof. Let $C_n = v_1v_2\ldots v_nv_1$ be the cycle. Let $v'_i$ be the duplicated vertex of $v_i$. Define a function $f : V(G(v_i)) \rightarrow \{1, 2, \ldots, q + 1\}$ by $f(v_1) = n + 3$, $f(v'_1) = 1$ and $f(v_i) = i + 3$, $2 \leq i \leq n$. Hence $f$ is harmonic mean labeling of the graph $G(v_i)$.

Example 2.3.43. A Harmonic mean labeling $C_5(v)$ is given below.

![Figure 2.36](image)

In the similar manner we have the following

Definition 2.3.44. Let $e = uv$ be an edge of $G$. Then duplication of an edge $e = uv$ is a graph $G(uv)$ obtained from $G$ by adding a new edge $u'v'$ such that $N(u') = N(u) \cup v' - v$ and $N(v') = N(v) \cup u' - u$. 

63
Now we prove the following

**Theorem 2.3.45.** The graph obtained by duplicating an arbitrary edge in cycle $C_n$ is a Harmonic mean graph.

**Proof.** Let $C_n = v_1v_2\ldots v_nv_1$ be the cycle. Let $e' = u_1'u_2'$ be the duplicated edge of $e = u_1u_2$. Now we define $f : V(G(u_1u_2)) \to \{1, 2, \ldots q + 1\}$ by $f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 3$, $f(u_i) = i + 1$, $4 \leq i \leq n$, $f(u_1') = n$ and $f(u_2') = n + 1$. Hence $f$ is a Harmonic mean labeling of duplicated graph $G(u_1u_2)$. □

**Example 2.3.46.** Harmonic mean labeling of $C_9(u_1u_2)$ is given below.
Next we have the following

**Definition 2.3.47.** [25] Consider two copies of $C_n$, connect a vertex of first copy to a vertex of second copy with a new edge, the new graph obtained is called *joint sum of $C_n$*.

Next we have

**Theorem 2.3.48.** *The joint sum of two copies of $C_n$ admits a Harmonic mean labeling.*
Proof. Let $G$ be the joint sum of the cycle $C_n$. Let $v_1 v_2 \ldots v_n v_1$ be the first copy of $C_n$ and $v_{n+1} v_{n+2} \ldots v_{2n} v_{n+1}$ be the second copy of $C_n$. Without loss of generality we can assume that $v_n$ and $v_{n+1}$ are joined in $G$. Now define a function $f : V(G) \rightarrow \{1, 2, \ldots, q + 1\}$ by

$$f(v_i) = \begin{cases} i & 1 \leq i \leq n \\ n + 1 & i = n + 1 \\ i + 1 & n + 2 \leq i \leq 2n \end{cases}$$

Clearly $f$ is a Harmonic mean labeling. \qed

Example 2.3.49. Joint sum of $C_9$ and its Harmonic mean labeling is given below.
**Definition 2.3.50.** The *Bistar* $B_{m,n}$ is the graph obtained from $K_2$ by joining $m$ pendent edges to one end of $K_2$ and $n$ pendent edges to the other end of $K_2$. The edge of $K_2$ is called the central edge of $B_{m,n}$ and the other vertices of $K_2$ are called the central vertices of $B_{m,n}$.

**Remark 2.3.51.** [11] $B_{n,n}$, $B_{n+1,n}$, $B_{n+2}$ are mean graphs and $B_{m,n}$, $m > n + 2$ are not mean graphs. Also caterpillar and subdivision of star $S(K_{1,n})$, $n \leq 4$ are mean graphs. We have not been successful in settling Harmonic mean labeling for bistar, subdivision of star and caterpillar. However we could give Harmonic mean labeling in particular cases.

Now we have the following

**Example 2.3.52.** Here we displayed the Harmonic labeling of some of the Bistars
Definition 2.3.53. If $x = uv$ is an edge of $G$ and $w$ is not a vertex of $G$, then $x$ is subdivided when it is replaced by the edges $uw$ and $wv$. If every edge of $G$ is subdivided, the resulting graph is the subdivision graph $s(G)$.

Example 2.3.54. Harmonic mean labeling of the graph obtained by subdividing the central edge of $B_{3,4}$ is given below.
Next we display the Harmonic mean labeling of subdivision of some of the stars. Clearly $S(K_{1,1})$ is $P_3$ and $S(K_{1,2})$ is $P_5$. Hence by Theorem 2.2.1, $S(K_{1,1})$ and $S(K_{1,2})$ are mean graphs.

Harmonic mean labeling of subdivision $S(K_{1,3})$ is displayed below.
Next we have the Harmonic mean labeling behavior of the caterpillar.