CHAPTER 3

MULTI-ITEM INVENTORY MODEL WITH STORAGE
SPACE AND PRODUCTION COST CONSTRAINTS

3.1 INTRODUCTION

EOQ model has played an important role in the field of control theory. When the EOQ model is applied to some practical problems which encounter in real situation, it is difficult to know the values of carrying cost, shortage cost, setup cost and demand quantity exactly. Only approximate values can be found. Generally, uncertainties are considered as randomness and are handled by probability theory in conventional inventory models.

Inventory control is an important field for both real world applications and research purpose. The most widely used inventory model is the Economic Order Quantity model, in which the successive operations are classified as supply and demand. The first quantitative treatment of inventory was the simple EOQ model. This model was developed by Harris (1915). Later, Hadley & Whitin (1963) analyzed many inventory systems.

The basic objective of inventory control is to reduce investment in inventories and ensuring that production process does not suffer at the same time. In general the classical inventory problems are designed by considering that the demand rate of an item is constant and deterministic and that the unit price of an item is considered to be constant and independent in nature. But in practical situation, unit price and demand rate of an item may be related to
each other. When the demand of an item is high, an item is produced in large numbers and fixed costs of production are spread over a large number of items. Hence the unit cost of the item decreases. That is the unit price of an item inversely relates to the demand of that item. So demand rate of an item may be considered as a decision variable.

In this chapter, a multi-item inventory model with and without shortages and unit cost dependent on demand along with two constraints such as limited storage space and production cost has been formulated. The unit cost is considered here in fuzzy environment and the model has been solved by Karush Kuhn-Tucker conditions method. Finally, a conclusion is given in the last section.

3.2 KARUSH KUHN-TUCKER CONDITIONS

Taha (2007) discussed how to solve the optimum solution of nonlinear programming problem subject to inequality constraints by using the Kuhn-Tucker conditions. The development of the Kuhn-Tucker conditions is based on the Lagrangean method.

Suppose that the problem is given by

Minimize \( y = f(X) \)

subject to \( g_i(x) \geq 0, \ i = 1, 2, \ldots, m. \)

The non-negative constraints \( x \geq 0, \) if any, are included in the \( m \) constraints. The inequality constraints may be converted into equations by using non-negative slack variables. Let \( s_i^2 \) be the slack quantity subtracted from the \( i^{th} \) constraint \( g_i(x) \geq 0. \)

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m), \ g(x) = (g_1(x), g_2(x), \ldots, g_m(x)) \) and \( s^2 = (s_1^2, s_2^2, \ldots, s_m^2). \)
Then the Lagrangean functions are given by

\[ G(x, s, \lambda) = f(X) - \lambda [g(x) - s^2]. \]

Taking the partial derivatives of \( G \) with respect to \( x, s, \) and \( \lambda \) gives the conditions that are also sufficient if the objective function and solution space satisfy the conditions given as follows:

<table>
<thead>
<tr>
<th>Sense of optimization</th>
<th>Required conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Objective function</td>
</tr>
<tr>
<td>Maximization</td>
<td>Concave</td>
</tr>
<tr>
<td>Minimization</td>
<td>Convex</td>
</tr>
</tbody>
</table>

The conditions for establishing the sufficiency of the Kuhn-Tucker conditions are summarized as follows:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Kuhn-Tucker conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Max ( z = f(X) ) subject to ( h(X) \leq 0 ) ( X \geq 0, i = 1, 2, \ldots, m )</td>
<td></td>
</tr>
</tbody>
</table>
|         | \[ \frac{\partial}{\partial x_j} f(X) - \sum_{i=1}^{m} \lambda_i \frac{\partial}{\partial x_j} h^i(X) = 0 \]
|         | \( \lambda_i h^i(X) = 0, h^i(X) \leq 0, i = 1, 2, \ldots, m \) \( \lambda_i \geq 0, i = 1, 2, \ldots, m \) |
| 2. Min \( z = f(X) \) subject to \( h(X) \geq 0 \) \( X \geq 0, i = 1, 2, \ldots, m \) |
|         | \[ \frac{\partial}{\partial x_j} f(X) - \sum_{i=1}^{m} \lambda_i \frac{\partial}{\partial x_j} h^i(X) = 0 \]
|         | \( \lambda_i h^i(X) = 0, h^i(X) \geq 0, i = 1, 2, \ldots, m \) \( \lambda_i \geq 0, i = 1, 2, \ldots, m \) |

Kuhn-Tucker conditions also known as Karush Kuhn-Tucker (KKT) conditions was first developed by Karush in 1939 as part of his M.S thesis at the University of Chicago. The same conditions were developed independently in 1951 by Kuhn and Tucker.
3.3 MATHEMATICAL FORMULATION AND SOLUTION OF THE MODEL WITH SHORTAGES

Let the amount of stock for the $i^{th}$ item ($i = 1, 2, ..., n$) be $R_i$ at time $t = 0$. In the interval $(0, T_i (= t_{i1} + t_{i2}))$, the inventory level gradually decreases to meet the demand. By this process the inventory level reaches zero level at time $t_{i1}$ and then shortages are allowed to occur in the interval $(t_{i1}, T_i)$.

Figure 3.1 shows an inventory model with backorder. $R_i$ is the maximum inventory quantity and $Q$ is the order quantity for one period. Also, $t_{i1}$ denotes the length of time required for the $R_i$ units to be demanded. The length of time during one period over which backorder will be incurred will be $t_{i1}$.

The cycle then repeats itself given by Figure 3.1.

![Figure 3.1 Inventory level of the $i^{th}$ item](image)

Let $q_i(t)$ be the on-hand inventory at time $t (0 \leq t \leq T)$. In this model, uniform replenishment rate starts with inventory level $q_i$. The inventory level decreases with demand. Ultimately the inventory reaches 0 at the end of the cycle time $t_{i1}$.
The differential equations describing the inventory level \( q_i(t) \) of \( i^{th} \) item in the interval \( 0 \leq t \leq T_i \) is given by

\[
\frac{dq_i(t)}{dt} = \begin{cases} 
-D_{ij}, & \text{for } 0 \leq t \leq t_{ii} \\
-D_{ij}, & \text{for } t_{ii} \leq t \leq T_i 
\end{cases}
\]

with the conditions \( q_i(0) = R_i (= Q_i - M_i), q_i(T_i) = -M_i \) and \( q_i(t_{ii}) = 0 \).

For each period a fixed amount of shortage is allowed and there is a penalty cost \( m_i \) per item of unsatisfied demand per unit time.

For \( 0 \leq t \leq t_{ii} \)

\[
\int_{0}^{t} dq_i(t) = -\int_{0}^{t} D_{ij} dt
\]

\( q_i(t) - q_i(0) = -D_{ij} t \)

Hence \( q_i(t) = R_i - D_{ij} t \)

For \( t_{ii} \leq t \leq T_i \)

\[
\int_{t_{ii}}^{t} dq_i(t) = -\int_{t_{ii}}^{t} D_{ij} dt
\]

\( q_i(t) - q_i(t_{ii}) = -D_{ij} (t - t_{ii}) \)

Hence \( q_i(t) = D_{ij} (t_{ii} - t) \)

Thus \( q_i(t) = R_i - D_{ij} t, \) for \( 0 \leq t \leq t_{ii} \)

\( D_{ij} (t_{ii} - t) \) for \( t_{ii} \leq t \leq T_i \)

Also \( D_{ij} t_{ii} = R_i \)

\( M_i = D_{ij} t_{ii} \)

\( Q_i = D_{ij} T_i \)
The holding cost is associated with carrying (or holding) inventory. This cost generally includes the costs such as rent for space, used for storage, interest on the money locked-up, insurance of stored equipment, production, taxes, depreciation of equipment and furniture used, etc. This is obtained by evaluating the integral in the interval \((0,t_i)\).

\[
\text{Holding cost} = H \int_0^{h_i} q_i(t) \, dt
\]

\[
= \frac{H_i (Q_i - M_i)^2}{2D_i}
\]

\[
= \frac{H_i T_i (Q_i - M_i)^2}{2Q_i}
\]

Since \(Q_i = D_i T_i\)

The penalty cost for running out of stock (i.e., when an item cannot be supplied on the customer’s demand) is known as shortage cost. This cost includes the loss of potential profit through sales of items and loss of goodwill, in terms of permanent loss of customers and it is associated lost profit in future sales. Thus the shortage cost is obtained by evaluating the integral in the interval \((t_{i-1}, T_i)\). This is because the shortage occurs only when all the existing stocks are exhausted. Hence

\[
\text{Shortage cost} = m_i \int_{t_{i-1}}^{T_i} [-q_i(t)] \, dt
\]

\[
= -m_i \left[ T_i q_i(T_i) - t_i q_i(t_i) \right] - \int_{t_{i-1}}^{T_i} [t(-D_i)] \, dt
\]
Since \( dq(t) = -D_i dt \) for \( t_{i-1} \leq t \leq T_i \) the shortage cost becomes

\[
\text{Shortage cost} = -m_i \int_{t_{i-1}}^{T_i} [D_i - 0] dt = m_i \int_{t_{i-1}}^{T_i} D_i dt
\]

\[
= m_i T_i - m_i \frac{D_i}{2} [\bar{T}_i^2 - \bar{t}_i^2]
\]

But \( T_i = \frac{Q_i}{D_i} \)

\[
\bar{t}_i = \frac{R_i}{D_i} = \frac{Q_i - M_i}{D_i} \quad \text{and} \quad Q_i = R_i + M_i
\]

Therefore

\[
\text{Shortage cost} = m_i M_i T_i - m_i \frac{D_i}{2} \left[ \frac{Q_i^2}{D_i^2} - \frac{R_i^2}{D_i^2} \right]
\]

\[
= m_i M_i T_i - m_i \frac{D_i}{2} \left[ (R_i + M_i)^2 - R_i^2 \right]
\]

\[
= m_i \frac{M_i^2}{2D_i}
\]

\[
= m_i \frac{M_i^2 T_i}{2Q_i} \quad \text{since} \quad D_i = \frac{Q_i}{T_i}
\]

The production cost for each item of the system is given by

Production cost = \( p_i Q_i \)

Also the setup cost for each item is given by

Setup cost = \( S_i \)
The total cost = production cost + setup cost + holding cost + shortage cost

\[ TC = p_i Q_i + S_i + \frac{H_i T_i (Q_i - M_i) y_i^2}{2 Q_i} + \frac{m_i M_i^2 T_i}{2 Q_i} \]

The total cost of the \( i^{th} \) item in crisp model is

\[ TC(p_i, Q_i, M_i) = p_i Q_i + S_i + \frac{H_i T_i (Q_i - M_i) y_i^2}{2 Q_i} + \frac{m_i M_i^2 T_i}{2 Q_i} \]

for \( i = 1, 2, \ldots, n \).

The total average cost of the \( i^{th} \) item of the system is then given by

\[ TC(p_i, Q_i, M_i) = p_i D_i + \frac{S_i D_i}{Q_i} + \frac{H_i (Q_i - M_i) y_i^2}{2 Q_i} + \frac{m_i M_i^2}{2 Q_i} \]

In general the classical inventory problems are designed by considering that the demand rate of the item is constant and deterministic and that the unit price of an item is considered to be constant and independent in nature. Silver & Peterson (1985) formulated an inventory model by considering the demand rate and unit price as constants and independent of each other. But in practical situation, unit price and demand rate of an item may be related to each other. When the demand of an item is high, an item is produced in large numbers and fixed costs of production are spread over a large number of items. Hence the unit cost of the item decreases, i.e., the unit price of an item inversely relates to the demand of that item. Jung & Klein (2001) formulated the Economic Order Quantity problem with this idea and solved using geometric programming method. In the present research, the inventory model with demand dependent unit price is solved using Karush Kuhn-Tucker method.
3.4 ASSUMPTIONS OF THE INVENTORY MODEL

A multi-item inventory model with shortages has been developed under the following assumptions.

- Replenishment is instantaneous
- Lead time is zero
- Unit price is related to the demand as

\[ p_i = A_i^\beta D_i^{-\beta} \]

where \( A_i (> 0) \) and \( \beta (\beta > 1) \) are constants and real numbers selected to provide the best fit of the estimated price function. \( A_i > 0 \) is an obvious condition since both \( D_i \) and \( p_i \) must be non-negative.

3.5 OBJECTIVE FUNCTION OF THE MODEL

The objectives of the problem can be explained as follows:

According to the purpose of modelling, the total cost of materials must minimize in the system which has three segments with shortage cost.

The annual total cost according to the basic assumptions of the EOQ model is:

The total cost = production cost + setup cost + holding cost + shortage cost

\[
TC(p_i, Q_i, M_i) = p_i P_i + \frac{S_i D_i}{Q_i} + \frac{H_i (Q_i - M_i)^2}{2Q_i} + \frac{m_i M_i^2}{2Q_i}
\]

(3.1)
Substituting for \( p_i \) in (3.1) gives

\[
TC(D, Q, M, \alpha) = A_i \beta D_i^{\alpha - \beta} + \frac{SD_i}{Q_i} + \frac{H_i (Q_i - M_i)^2}{2Q_i} + \frac{m_i M_i^2}{2Q_i}
\]

for \( i = 1, 2, 3, \ldots, n \)

The objective function of the inventory model to minimize the total cost is given by

\[
\text{Min} TC(D, Q, M, \alpha) = \sum_{i=1}^{n} \left[ A_i \beta D_i^{\alpha - \beta} + \frac{SD_i}{Q_i} + \frac{H_i (Q_i - M_i)^2}{2Q_i} + \frac{m_i M_i^2}{2Q_i} \right] (3.2)
\]

### 3.6 CONSTRAINTS OF THE MODEL

The proposed model has been formulated with the following constraints.

There are some restrictions on the available resources in inventory problems that cannot be ignored to derive the optimal total cost.

- There is a limitation on the available warehouse floor space where the items are to be stored.

  \[
i.e \quad \sum_{i=1}^{n} w_i Q_i \leq W \tag{3.3}
\]

- Investment amount on total production cost cannot be infinite. it may have an upper limit on the maximum investment.

  \[
i.e \quad \sum_{i=1}^{n} p_i Q_i \leq B
\]

  \[
\Rightarrow \sum_{i=1}^{n} A_i \beta D_i^{\alpha - \beta} Q_i \leq B \tag{3.4}
\]
3.7 FUZZY INVENTORY MODEL

When $p_i$'s are fuzzy decision variables, the above crisp model under fuzzy environment reduces to

$$\text{MinTC}\left(\tilde{p}_i, D_i, Q_i, M_i\right) = \sum_{i=1}^{n} \left[ A_i^\beta D_i^{1-\beta} + \frac{SD_i}{Q_i} + \frac{H_i(Q_i - M_i)^2}{2Q_i} + \frac{mM_i^2}{2Q_i}\right]$$

subject to the constraints

$$\sum_{i=1}^{n} w_i Q_i \leq W$$

$$\sum_{i=1}^{n} A_i^\beta D_i^{1-\beta} Q_i \leq B$$

where $\tilde{p}_i = A_i^\beta D_i^{1-\beta}$

[Here cap `~` denotes the fuzzification of the parameters]

3.8 KARUSH KUHN-TUCKER CONDITIONS FOR SOLVING THE INVENTORY MODEL

The objective function of an inventory model is

$$\text{MinTC}(D_i, Q_i, M_i) = \sum_{i=1}^{n} \left[ A_i^\beta D_i^{1-\beta} + \frac{SD_i}{Q_i} + \frac{H_i(Q_i - M_i)^2}{2Q_i} + \frac{mM_i^2}{2Q_i}\right]$$

subject to the constraints

$$\sum_{i=1}^{n} w_i Q_i \leq W$$
\[ \sum_{i=1}^{n} A_i^{\beta} D_i^{-\beta} Q_i \leq B \]

Here the decision variables are the demand \( D_i \), lot size \( Q_i \) and the shortage level \( M_i \). The problem is to solve the above inventory model with these decision variables subject to the inequality constraints (3.3) and (3.4) in order to minimize the total cost function. The problem is solved for a single item.

For a single item the objective function and the constraints can be written as follows.

\[
\text{Min} \, TC(D, Q, M) = A^{\beta} D^{-\beta} + \frac{SD}{Q} + \frac{H(Q-M)^2}{2Q} + \frac{mM^2}{2Q}
\]

subject to the inequality constraints

\[ wQ \leq W \]

\[ A^{\beta} D^{-\beta} Q \leq B \]

To minimize the objective function, the Lagrangean function has been constructed by introducing the variables \( s_1 \) and \( s_2 \) as follows

\[
G = A^{\beta} D^{-\beta} + SDQ^{-1} + 0.5H(Q-M)^2Q^{-1} + 0.5mM^2Q^{-1} - \lambda_1 \left( W - wQ - s_1^2 \right) - \lambda_2 \left( B - A^{\beta} D^{-\beta} Q - s_2^2 \right)
\]

### 3.9 MEMBERSHIP FUNCTION

The membership function for the fuzzy variable \( p_i \) is defined as follows
\[ \mu_i(x) = \begin{cases} 
1, & p_i \leq L_i \\
\frac{U_i - p_i}{U_i - L_i}, & L_i \leq p_i \leq U_i \\
0, & p_i \geq U_i 
\end{cases} \]

Here \( U_i \) and \( L_i \) are upper limit and lower limit of \( p_i \), respectively.

### 3.10 Numerical Example

To illustrate the proposed inventory model for with and without shortage cases, the following input data are considered in proper units for a single item.

A numerical example has been derived for a single item with the set of input parametric values given in Table 3.1.

#### Table 3.1 The input values of parameters in the mathematical model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of items</td>
<td>( n )</td>
<td>1</td>
</tr>
<tr>
<td>Constant</td>
<td>( A_1 )</td>
<td>20</td>
</tr>
<tr>
<td>Setup cost of the item 1</td>
<td>( S_1 )</td>
<td>$80</td>
</tr>
<tr>
<td>Holding cost of the item 1</td>
<td>( H_1 )</td>
<td>$0.7</td>
</tr>
<tr>
<td>Storage space for the item 1</td>
<td>( w_1 )</td>
<td>3 sq. ft.</td>
</tr>
<tr>
<td>Storage space available</td>
<td>( W )</td>
<td>280 sq. ft.</td>
</tr>
<tr>
<td>Total investment cost</td>
<td>( B )</td>
<td>$40</td>
</tr>
<tr>
<td>Shortage cost per unit item</td>
<td>( m_1 )</td>
<td>$10</td>
</tr>
<tr>
<td>Lower limit of the unit cost of the item 1</td>
<td>( l_{1i} )</td>
<td>$1</td>
</tr>
<tr>
<td>Upper limit of the unit cost of the item 1</td>
<td>( u_{1i} )</td>
<td>$2</td>
</tr>
</tbody>
</table>
For the above data, the objective function becomes

\[
G = 20^\beta D^{\beta} + 80 D Q^\alpha + 0.35(Q-M)^{Q^\alpha} + 5 M^Q \lambda^Q - \lambda\left(280 - 3Q - s^2\right) - \lambda\left(40 - 20^\beta D^{\beta} Q - s^2\right)
\]

\[
G = 20^\beta D^{\beta} + 80 D Q^\alpha + 0.35Q - 0.7M + 5.35M^Q \lambda^Q - \lambda\left(280 - 3Q - s^2\right) - \lambda\left(40 - 20^\beta D^{\beta} Q - s^2\right)
\]

(3.5)

Differentiating (3.5) partially with respect to \(D\), \(Q\) and \(M\) respectively we get

\[
\frac{\partial G}{\partial D} = (1-\beta)20^\beta D^{\beta} + 80Q^\beta + \beta\lambda_2 20^\beta D^{\beta-1} Q
\]

\[
\frac{\partial G}{\partial Q} = -80DQ^{-2} + 0.35 - 5.35Q^{-2} M^2 + 3\lambda_1 + \lambda_2 20^\beta D^{\beta}
\]

\[
\frac{\partial G}{\partial M} = -0.7 + 10 Q^{-1} M
\]

By the Kuhn-Tucker conditions

\[
\frac{\partial G}{\partial D} = 0 \Rightarrow (1-\beta)20^\beta D^{\beta} + 80Q^\beta + \beta\lambda_2 20^\beta D^{\beta-1} Q = 0 \quad (3.6)
\]

\[
\frac{\partial G}{\partial Q} = 0 \Rightarrow -80DQ^{-2} + 0.35 - 5.35Q^{-2} M^2 + 3\lambda_1 + \lambda_2 20^\beta D^{\beta} = 0 \quad (3.7)
\]

\[
\frac{\partial G}{\partial M} = 0 \Rightarrow -0.7 + 10 Q^{-1} M = 0 \quad (3.8)
\]

Hence, an optimal solution has been obtained by solving the Equations (3.6), (3.7) & (3.8) by using Karush Kuhn-Tucker conditions with demand \(D\), lot size \(Q\) and the shortage level \(M\) as the decision variables by varying the parametric value \(\beta\). Also an optimum solution has been obtained by fuzzifying the unit cost and the results are discussed in Table 3.2.
The value of the parameter $\beta$ has been chosen between 2 and 3. The most suitable values are obtained for the parametric values $\beta$ such as 2.4, 2.5, 2.6 and 2.8 by trial and error method that minimizes the objective function.

A sensitivity analysis for optimum solution with shortages corresponding to the parameter $\beta$ is given in Table 3.2.

**Table 3.2 Optimal results for the model with shortages**

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$p_t$</th>
<th>$\mu_{p_t}$ value</th>
<th>$D_t$</th>
<th>$Q_t$</th>
<th>$M_t$</th>
<th>Expected Total cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4</td>
<td>1.2853</td>
<td>0.7147</td>
<td>18.014</td>
<td>62.092</td>
<td>4.061</td>
<td>55.151</td>
</tr>
<tr>
<td>2.5</td>
<td>1.3053</td>
<td>0.6947</td>
<td>17.978</td>
<td>62.899</td>
<td>4.114</td>
<td>54.872</td>
</tr>
<tr>
<td>2.6</td>
<td>1.3278</td>
<td>0.6722</td>
<td>17.934</td>
<td>63.627</td>
<td>4.161</td>
<td>54.633</td>
</tr>
<tr>
<td>2.8</td>
<td>1.3806</td>
<td>0.6194</td>
<td>17.824</td>
<td>64.834</td>
<td>4.240</td>
<td>54.197</td>
</tr>
</tbody>
</table>

In Table 3.2, a study of expected total cost with demand and lot size including shortages is given for different values of $\beta$. We can conclude that when demand decreases, lot size increases but the annual total cost decreases.

From the above table it follows that $p_t = $1.2853 has the maximum membership value 0.7147. Hence the required optimum solution is $Q_t = 62.092$, $D_t = 18.014$, $M_t = 4.061$ and Minimum expected total cost = $55.151$.

### 3.11 SENSITIVITY ANALYSIS

Sensitivity analysis is studied to see how far the output of the model is affected by changes or errors in its input parameters based on the numerical example. The results are illustrated with the help of numerical example. A model with and without shortage is considered in this chapter.
A stochastic inventory model is one of the most fundamental of all inventory models. The importance of the model is that it is still one of the most widely used inventory model in industry, and served as a basis for more sophisticated inventory models.

By solving the inventory model using Kuhn-Tucker condition method the values of imprecise variable for decision making are obtained. The optimum values of the decision variables and the total cost corresponds to the maximum membership function value 0.7147. Hence the optimum solution is $Q_t = 62.092$, $D_t = 18.014$, $M_t = 4.061$ and $TC = 55,151$.

Results due to different values of $\beta$ for the model has been calculated and depicted in the following Figures 3.2, 3.3, 3.4, 3.5 and 3.6.

![Figure 3.2 Effect of change in demand level with respect to $\beta$](image)

The above Figure 3.2 shows that as the value of the parameter $\beta$ increases from 2.4 to 2.8, the value of the demand decreases from 18.014 to 17.824.
Figure 3.3 Effect of change in unit price with respect to $\beta$

The above Figure 3.3 shows that as the value of the parameter $\beta$ increases from 2.4 to 2.8, the value of the unit price increases from $1.2853$ to $1.3806$.

Figure 3.4 Effect of change in ordering quantity with respect to $\beta$

The above Figure 3.4 shows that as the value of the parameter $\beta$ increases from 2.4 to 2.8, the value of the lot size increases from 62.092 to 64.834.
Figure 3.5 Effect of change in shortage level with respect to $\beta$

The above Figure 3.5 shows that as the value of the parameter $\beta$ increases from 2.4 to 2.8, the value of the shortage level increases from 4.061 to 4.240.

Figure 3.6 Effect of change in total cost with respect to $\beta$

The above Figure 3.6 shows that as the value of the parameter $\beta$ increases from 2.4 to 2.8, the value of the annual total cost decreases from $\$55.151$ to $\$54.197$. 
3.12 INVENTORY MODEL WITHOUT SHORTAGES AS A SPECIAL CASE

The without shortage situation can be obtained by substituting $M = 0$ in the previous model. In this case, the total cost function depends only on the demand and lot size.

An inventory model without shortages can be reduced to

$$\text{Min} TC(D, Q) = \sum_{j=1}^{n} \left[ A_j D_j^{\alpha - \beta} + \frac{S_j D_j}{Q_j} + \frac{H O_j}{2} \right]$$  \hspace{1cm} (3.9)

subject to the constraints

$$\sum_{j=1}^{n} w_i Q_j \leq W$$

$$\sum_{j=1}^{n} A_j D_j^{\alpha - \beta} Q_j \leq B$$

For a single item the inventory model can be stated as

$$\text{Min} TC(D, Q) = A^\beta D^{\alpha - \beta} + \frac{SD}{Q} + \frac{HQ}{2}$$  \hspace{1cm} (3.10)

subject to the constraints

$$wQ \leq W \text{ and}$$

$$A^\beta D^{\alpha - \beta} Q \leq B$$
The Lagrangean function corresponding to this objective function can be written as

\[ G = A^\beta D^{\beta-1} + SDQ^{-1} + 0.5HQ - \lambda_1 \left( W - wQ - s_1^z \right) - \lambda_2 \left( B - A^\beta D^{\beta-1} Q - s_2^z \right) \]  

(3.11)

Differentiating the above function (3.11) partially with respect to D and Q gives the following derivatives.

\[
\frac{\partial G}{\partial D} = (1 - \beta) A^\beta D^{\beta-1} + SQ^{-1} - \lambda_2 \beta A^\beta D^{\beta-2} Q
\]

(3.12)

\[
\frac{\partial G}{\partial Q} = -SDQ^{-2} + 0.5H + \lambda_1 w + \lambda_2 A^\beta D^{\beta-1}
\]

(3.13)

By Karush Kuhn-Tucker conditions

\[
\frac{\partial G}{\partial D} = 0 \Rightarrow (1 - \beta) A^\beta D^{\beta-1} + SQ^{-1} - \lambda_2 \beta A^\beta D^{\beta-2} Q = 0
\]

(3.14)

\[
\frac{\partial G}{\partial Q} = 0 \Rightarrow -SDQ^{-2} + 0.5H + \lambda_1 w + \lambda_2 A^\beta D^{\beta-1} = 0
\]

(3.15)

Solving the Equations (3.14) and (3.15) gives the required optimum solution.

Optimum solution for without shortages are given in the following Table 3.3 for the same set of input values given in Table 3.1.
Table 3.3 Optimal values for various values of $\beta$ for no-shortage case

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$p_l$</th>
<th>$\mu_{pl}$ value</th>
<th>$D_l$</th>
<th>$Q_l$</th>
<th>Expected Total cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4</td>
<td>1.3405</td>
<td>0.6595</td>
<td>17.701</td>
<td>59.538</td>
<td>56.519</td>
</tr>
<tr>
<td>2.5</td>
<td>1.3624</td>
<td>0.6376</td>
<td>17.673</td>
<td>60.268</td>
<td>56.282</td>
</tr>
<tr>
<td>2.6</td>
<td>1.3857</td>
<td>0.6143</td>
<td>17.642</td>
<td>60.967</td>
<td>56.059</td>
</tr>
<tr>
<td>2.8</td>
<td>1.4407</td>
<td>0.5593</td>
<td>17.555</td>
<td>62.223</td>
<td>55.649</td>
</tr>
</tbody>
</table>

In this case, it follows that the minimum annual total cost corresponds to the maximum membership function value 0.6595. Hence the optimal solution satisfying the constraints are $D_l=17.701$, $Q_l=59.538$ and the minimum total cost is equal to $56.519$. The graphical representations are shown in the Figure 3.7, 3.8, 3.9 and 3.10.

![Figure 3.7 Effect of change in demand level with respect to $\beta$](image)

The above Figure 3.7 shows that as the value of the parameter $\beta$ increases from 2.4 to 2.8, the value of the demand decreases from 17.701 to 17.555.
Figure 3.8 Effect of change in ordering quantity with respect to $\beta$

The above Figure 3.8 shows that as the value of the parameter $\beta$ increases from 2.4 to 2.8, the value of the lot size increases from 59.538 to 62.223.

Figure 3.9 Effect of change in unit price with respect to $\beta$

The above Figure 3.9 shows that as the value of the parameter $\beta$ increases from 2.4 to 2.8, the value of the unit price increases from $1.3405$ to $1.4407$. 
Figure 3.10  Effect of change in annual total cost with respect to $\beta$

The above Figure 3.10 shows that as the value of the parameter $\beta$ increases from 2.4 to 2.8, the value of the total cost decreases from $56.519$ to $55.649$.

3.13  SUMMARY

In this chapter, a comparative study of the results for with shortage case and without shortage case is done. In the numerical examples, it is found that the optimum total cost in with shortage case is less than that of the without shortage case.

One of the possible extensions of this chapter is to find specific conditions that guarantee the global optimality of solutions. It is also possible to consider the life cycle to be a stochastic function. When this model will cover the time value of money and inflation, a better reflection of real life situations could be provided.