CHAPTER 6

MULTI-ITEM INVENTORY MODEL WITH DEMAND
DEPENDENT UNIT COST AND LEAD TIME
CONSTRAINTS

6.1 INTRODUCTION

The EOQ model is a simple mathematical model to deal with inventory management issues in a production inventory system. It is considered to be one of the most popular inventory models used in the industry. An important problem confronting inventory decision maker is to frame an efficient replenishment policy to keep the cost of the inventory system as low as possible. Two major assumptions in the classical EOQ model are that demand is constant and deterministic and that the unit production cost is independent of the production order quantity.

Inventory makes up a significant part of the total manufacturing firm asset, and plays an important role in the supply chain management. For this reason, the effectiveness of the inventory management affects the profitability of a firm. A common problem in inventory system is defining the value of Economic Order Quantity (EOQ) to minimize annual inventory cost.

Usually, inventory systems are characterized by several parameters such as cost coefficients, demands etc. Accordingly, most of the inventory problems under fuzzy environment can be addressed by fuzzifying these

Classical inventory models assume deterministic parameters. However, in the real case there are many uncertainties that should be considered. Most studies on EOQ modelling use probabilistic approach to deal with uncertainty. Probabilistic model assumes uncertain carrying and holding costs with certain probability distribution. However in many inventory problems uncertainties do not have any costs information.

Lead time in traditional inventory models was a fixed input parameter while in recent trends it is a decision variable to be determined by the model. This former type of models provides a better reflection of today’s competitive market.

In this chapter, a multi-item inventory model with demand dependent unit cost and leading time crashing cost dependent on lead time has been formulated with number of orders and production cost as constraints. Unit production cost has been considered to be fuzzy in nature. The crisp problem for minimizing the annual total cost has been solved by Karush Kuhn-Tucker conditions method. Mathematical derivations and analysis have been made for a single item. Sensitivity analysis has been included with illustration.
6.2 SCOPE AND PURPOSE OF LEADTIME

The traditional inventory model considers the ideal case in which depletion of inventory is caused by a constant demand rate. However, to keep sales higher, the inventory level would need to remain high. Of course, this would also result in higher holding or procurement cost. Also, in many real situations, during a longer-shortage period some of the customers may refuse the management. For instance, for fashionable commodities and high tech products with short product life cycle, the willingness for a customer to wait for backlogging is diminishing with the length of the waiting time.

6.3 ASSUMPTIONS

The following assumptions have been considered in the developed model:

(1) Time horizon is finite

(2) No shortages are allowed

(3) Unit production cost is inversely related to the demand rate.

i.e \( p_i = AD_i^{-\beta} \), \( i = 1,2,3,\ldots,n \), where \( A > 0 \), \( \beta \geq 1 \).

(4) Lead time crashing cost is related to the lead time by a function of the form \( R(t) = \alpha t^{-b} \), \( i=1,2,\ldots,n \), \( \alpha > 0 \), 0 < \( b \leq 0.5 \)

where \( \alpha \), \( b \) are real constants, selected to provide the best fit of the estimated cost function.
6.4 OBJECTIVE FUNCTION OF THE INVENTORY MODEL

The objective is to minimize the total expected annual cost, which is the sum of production, order, inventory carrying and lead time crashing costs which according to the basic assumptions of the EOQ model is:

\[ TC(D_j, Q_j, L_j) = \sum_{j=1}^{n} \left\{ p_jD_j + \frac{S_jD_j}{Q_j} + \left[ \frac{Q_j^2}{2} + K\sigma \sqrt{L_j} \right] H_j + \frac{D_j}{Q_j} R(L_j) \right\} \quad (6.1) \]

Substituting \( p_j \) and \( R(L_j) \) in Equation (6.1) gives

\[ TC(D_j, Q_j, L_j) = \sum_{j=1}^{n} \left[ AD_j^{-\beta} + \frac{S_jD_j}{Q_j} + \left[ \frac{Q_j^2}{2} + K\sigma \sqrt{L_j} \right] H_j + \frac{D_j}{Q_j} \alpha L_j^{-k} \right] \quad (6.2) \]

6.5 CONSTRAINTS OF THE MODEL

To derive the optimal total cost in an inventory problem, there are some restrictions on available resources.

(i) Investment amount on total production cost cannot be infinite, it may have an upper limit on the maximum investment

\[ i.e \quad \sum_{j=1}^{n} p_jQ_j \leq B \]

\[ \Rightarrow \sum_{j=1}^{n} AD_j^{-\beta} Q_j \leq B \]

(ii) An upper limit on the number of orders that can be made in a time cycle on the system

\[ i.e \quad \sum_{j=1}^{n} \frac{D_j}{Q_j} \leq \ell \]
6.6 **OBJECTIVE FUNCTION OF THE INVENTORY PROBLEM**

The annual expected total cost was composed of four components namely expected purchase cost, expected ordering cost, expected holding cost and lead time crashing costs.

According to the basic assumptions and notation, the EOQ model is

\[
\text{Min} \text{TC}(D, Q, L) = \sum_{i=1}^{n} \left[ AD_i^{-\beta} + \frac{SD_i}{Q_i} + \left( \frac{Q_i}{2} + K \sigma \sqrt{L_i} \right) H_i + \frac{D_i}{Q_i} \alpha L_i^{-\theta} \right]
\]

subject to the inequality constraints

\[
\sum_{i=1}^{n} AD_i^{-\beta}Q_i \leq B
\]

\[
\sum_{i=1}^{n} \frac{D_i}{Q_i} \leq t
\]

6.7 **FUZZIFICATION OF COST PARAMETER**

In this paper the unit production cost \( p_i \) is defined under fuzzy environment. The membership function for the fuzzy variable \( p_i \) is defined as follows.

\[
\mu_{p_i}(X) = \begin{cases} 
1, & p_i \leq L_{i_1} \\
\frac{U_{i_1} - p_i}{U_{i_1} - L_{i_1}}, & L_{i_1} \leq p_i \leq U_{i_1} \\
0, & p_i \geq U_{i_1} 
\end{cases}
\]

Here \( U_{i_1} \) and \( L_{i_1} \) are upper limit and lower limit of \( p_i \) respectively.
6.8 KARUSH KUHN-TUCKER CONDITIONS APPROACH FOR A SINGLE ITEM

To solve the problem using Karush Kuhn-Tucker conditions method for a single item, it can be formulated as:

\[
\text{MinUC}(D, Q, L) = AD^{-\beta} + \frac{SD}{Q} + \left[ \frac{Q}{2} + K\sigma \sqrt{L} \right] H + \frac{D}{Q} \alpha L^{\beta}
\]

subject to the inequality constraints

\[
AD^{-\beta} Q \leq B \quad (6.4)
\]

\[
\frac{D}{Q} \leq I \quad (6.5)
\]

This is a minimization problem for a single item without shortage under two constraints. It can be solved by using Karush Kuhn-Tucker approach.

By using Karush Kuhn-Tucker method the above function can be restated as

\[
G = AD^{-\beta} + \frac{SD}{Q} + \left[ \frac{Q}{2} + K\sigma \sqrt{L} \right] H + \frac{D}{Q} \alpha L^{\beta} - \lambda_1 \left( B - AD^{-\beta} Q - s^2 \right) - \lambda_2 \left( I - DQ^{-1} - s^2 \right)
\]

(6.6)

Differentiating (6.6) partially with respect to D, Q and L we get the following equations

\[
\frac{\partial G}{\partial D} = (1 - \beta)AD^{-\beta} - SQ^{-1} - \alpha Q^{-1} L^{\beta} - \beta \lambda_1 AD^{-\beta} Q + \lambda_2 Q^{-1}
\]

\[
\frac{\partial G}{\partial Q} = -SDQ^{-2} + 0.5H - \alpha DQ^{-2} L^{\beta} + \lambda_1 AD^{-\beta} - \lambda_2 DQ^{-2}
\]
\[
\frac{\partial G}{\partial L} = 0.5K\sigma L^{\alpha-5}H - \alpha bDQ^{-1}L^{\beta-1}
\]

The necessary conditions for KKT conditions are

\[
\frac{\partial G}{\partial D} = 0 \Rightarrow (1-\beta)AD^{\beta} + SQ^1 + \alpha Q^1L^\beta - \beta_1AD^{\beta-1}Q + \lambda_2Q^{-1} = 0 \quad (6.7)
\]

\[
\frac{\partial G}{\partial Q} = 0 \Rightarrow -SDQ^{-2} + 0.5H - \alpha DQ^{-2}L^\beta + \lambda_1AD^{\beta-1}Q - \lambda_2DQ^{-2} = 0 \quad (6.8)
\]

\[
\frac{\partial G}{\partial L} = 0 \Rightarrow 0.5K\sigma L^{\alpha-5}H - \alpha bDQ^{-1}L^{\beta-1} = 0 \quad (6.9)
\]

Solving the above set of Equations (6.7), (6.8) and (6.9) gives the optimal solution of the decision variables D, Q and L.

For solving these equations the Newton Raphson method has been applied to obtain the solution of the transcendental equation.

6.9 \hspace{1em} NUMERICAL EXAMPLE

The decision variables namely the optimal order quantity \(Q_1\), optimal demand rate \(D_1\) and optimal lead time \(L_1\) whose values determine the minimum annual relevant total cost are computed for different values of \(\beta\). The optimum solutions are shown in the Table 6.2.

The parameter values of the model are given in Table 6.1.
Table 6.1 Input data for different values of parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of items</td>
<td>$n$</td>
<td>1</td>
</tr>
<tr>
<td>Constant</td>
<td>$A_1$</td>
<td>15</td>
</tr>
<tr>
<td>Setup cost of the item 1</td>
<td>$S_1$</td>
<td>$200</td>
</tr>
<tr>
<td>Holding cost of the item 1</td>
<td>$H_1$</td>
<td>$0.80</td>
</tr>
<tr>
<td>Total investment cost</td>
<td>$B$</td>
<td>$200</td>
</tr>
<tr>
<td>Lower limit of the unit cost of 1</td>
<td>$L_{1_1}$</td>
<td>$6$</td>
</tr>
<tr>
<td>Upper limit of the unit cost of 1</td>
<td>$U_{1_1}$</td>
<td>$8$</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>$\sigma$</td>
<td>6 unit/year</td>
</tr>
<tr>
<td>constant</td>
<td>$k$</td>
<td>2</td>
</tr>
<tr>
<td>Real constant</td>
<td>$\alpha$</td>
<td>1</td>
</tr>
<tr>
<td>Real constant</td>
<td>$b$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 6.2 The optimal solution of $D_1$, $Q_1$ & $L_1$ for varying $\beta$ and $\alpha=1$, $b=0.1$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$D_1$</th>
<th>$Q_1$</th>
<th>$L_1$</th>
<th>Min TC</th>
<th>$p_t$</th>
<th>$\mu_{p_t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>1.415</td>
<td>26.69</td>
<td>$1.12\times10^5$</td>
<td>32.074</td>
<td>7.492</td>
<td>0.254</td>
</tr>
<tr>
<td>2.4</td>
<td>1.370</td>
<td>28.377</td>
<td>$9.58\times10^6$</td>
<td>30.843</td>
<td>7.046</td>
<td>0.477</td>
</tr>
<tr>
<td>2.6</td>
<td>1.347</td>
<td>28.919</td>
<td>$9.02\times10^6$</td>
<td>30.375</td>
<td>6.914</td>
<td>0.543</td>
</tr>
<tr>
<td>2.8</td>
<td>1.342</td>
<td>30.376</td>
<td>$8.26\times10^6$</td>
<td>29.990</td>
<td>6.582</td>
<td>0.709</td>
</tr>
</tbody>
</table>

The optimal solution is $D_1 = 1.342$, $Q_1 = 30.376$, $L_1 = 8.26\times10^6$ and Min TC = $29,990$ which corresponds to maximum membership function $0.709$. It has been seen that as $\beta$ value increases, the lot size $Q$ also increases. But conversely the demand $D$, lead time $L$, the unit price $p$ and hence the minimum total costs decreases.
6.10 SENSITIVITY ANALYSIS

The problem has been formulated analytically and has been used to arrive at the optimal solution. Numerical assessment and sensitivity analysis are implemented to illustrate the theoretical model. Hence, from the economical point of view, the proposed model will be useful to the business situations in the present context as it gives better inventory control system. The model presents ample scope for further extension and development.

The solution table shows that both demand rate and the unit price decreases as the parameter $\beta$ increases even though they are inversely related. This is because of the existence of the other parameters like $\alpha$ and $b$.

The Table 6.2 shows the change in optimal inventory policy with change in a model parameter. Now the effect of change in the unit price, demand, lot size and total annual cost with respect to the parameter $\beta$ is shown below.

![Graph](image)

**Figure 6.1 Two dimensional plot of unit price p with respect to $\beta$**

The Figure 6.1 shows that as the value of the parameter $\beta$ increases from 2 to 2.8, the unit price decreases from $7.492$ to $6.582$. 
Figure 6.2 Two dimensional plot of demand $D$ with respect to $\beta$

The Figure 6.2 shows that as the value of the parameter $\beta$ increases from 2 to 2.8, demand decreases from 1.415 to 1.342.

Figure 6.3 Two dimensional plot of lot size $Q$ with respect to $\beta$

The Figure 6.3 shows that as the value of the parameter $\beta$ increases from 2 to 2.8, the lot size also increases from 26.69 to 30.376.
Figure 6.4 Two dimensional plot of minimum total cost with respect to $\beta$

The Figure 6.4 shows that as the value of the parameter $\beta$ increases from 2 to 2.8, the total cost decreases from $32.074 to $29.990.

6.11 SUMMARY

Inventory modelers have so far considered that the unit cost of production is constant. Constant unit cost of production implies a uniform or constant change in the total cost per unit time. This is rarely seen to occur in the real market. In the opinion of the researchers an alternative approach is to consider reliability and demand-dependence of unit production cost which may represent all types of dependence depending on the signs of parameters of the unit production cost function.

An Economic Order Quantity model with reliability and demand dependent unit cost of production is considered in a fuzzy environment for exceptional products, planning horizon is taken as finite and of crisp in nature. But in reality due to rapid change of environment, demand for the items fluctuates every year. Complexities of the models increase when some other
parameters of the model become imprecise in non-stochastic sense that is fuzzy in nature.

The proposed model can be further extended in several ways. We may add both time values of money and quality strategies into consideration. Also we could extend the deterministic model into a stochastic model. Finally, we could generalize the model to allow shortages, deterioration and others. In future the holding cost and shortage cost coefficients will also be fuzzified to solve the model.

Moreover, in this chapter, the model has been formulated with demand dependent unit cost, finite time horizon, and no shortages. The present analysis can be easily extended to other types of inventory models with fully or partially backlogged shortages, fixed time horizon, etc. Another possible extension of this study may consider the assumption of the stochastic demand and deterioration rate.