Chapter 3

Hermitian forms and the $u$-invariant

Throughout this chapter $k$ and $K$ denote fields of characteristic not equal to 2.

Let $A$ be a central simple $K$-algebra with an involution $\sigma$. Let $\varepsilon \in \{\pm 1\}$ and

$$S(\sigma, \varepsilon) = \{x \in A | \sigma(x) = \varepsilon x\}.$$

Let $r = \dim_k S(\sigma, \varepsilon)$ and $k = \{\lambda \in K | \sigma(\lambda) = \lambda\}$.

By a theorem of Mahmoudi ([8]) we have

$$u(A, \sigma, \varepsilon) \leq \frac{r(r + 1)}{2m^2[K:k]} u(k)$$

where $m$ is the degree of $A$ over $K$.

In this chapter we give a better bound for $u(A, \sigma, \varepsilon)$ when the degree of $A$ is at most 4.

Let $K = k(\sqrt{d})$ be a quadratic field extension of $k$ and $\theta$ a $k$-automorphism of $K$. Note that there is only one non-trivial $k$-automorphism $K$. If $\theta$ is non-trivial, then $\theta(\sqrt{d}) = -\sqrt{d}$. Let $\eta \in \{\pm 1\}$ be such that $\theta(\sqrt{d}) = \eta \sqrt{d}$. Let $A_0$ be a central simple algebra $k$ with an involution $\tau$ of first kind. Let $A = A_0 \otimes K$ and $\sigma = \tau \otimes \theta$. Then $A$ is a central simple algebra over $K$ and $\sigma$ is an involution on $A$. The involution $\sigma$ on $A$ is of first kind if $\eta = 1$ and second kind if $\eta = -1$. 
Identify $A_0$ as a subalgebra of $A$. Then we have $A = A_0 \oplus A_0 \sqrt{d}$. Let $\pi_i : A \to A_0$ be the projections given by $\pi_1(x + y \sqrt{d}) = x$ and $\pi_2(x + y \sqrt{d}) = y$ for all $x, y \in A_0$. Let $h : V \times V \to A$ be an $\varepsilon$-hermitian form over $(A, \sigma)$. Let $h_i = \pi_i h : V \times V \to A_0$. Then $h(x, y) = h_1(x, y) + h_2(x, y) \sqrt{d}$. We have

$$h_1(x, y) + h_2(x, y) \sqrt{d} = h(x, y) = \varepsilon \sigma(h(y, x)) = \varepsilon \tau(h_1(y, x)) + \varepsilon \eta h_2(x, y) \sqrt{d}.$$ 

Now it is easy to check that $h_1$ is an $\varepsilon$-hermitian form over $(A_0, \tau)$ and $h_2$ is an $\eta \varepsilon$-hermitian form over $(A_0, \tau)$. The assignments $h \mapsto h_1$ induces homomorphisms

$$\pi_1 : W^\varepsilon(A, \sigma) \to W^\varepsilon(A_0, \tau)$$

and

$$\pi_2 : W^\varepsilon(A, \sigma) \to W^{\eta \varepsilon}(A_0, \tau).$$

Let $h_0 : V_0 \times V_0 \to A_0$ be an $\varepsilon$-hermitian space over $(A_0, \tau)$. Let $V = V_0 \otimes_{A_0} A$. Then we can write $V = V_0 \oplus V_0 \sqrt{d}$. Define $h : V \times V \to A$ by

$$h(x_1 + y_1 \sqrt{d}, x_2 + y_2 \sqrt{d}) = h_0(x_1, x_2) + \eta h_0(y_1, y_2) + (\eta h_0(x_1, y_2) + h_0(y_1, x_2)) \sqrt{d}.$$ 

Then it can be checked that $h$ is an $\varepsilon$-hermitian form over $(A, \sigma)$ and the assignment $h_0 \mapsto h$ induces a homomorphism $\rho : W^\varepsilon(A_0, \tau) \to W^\varepsilon(A, \sigma)$.

**Lemma 3.1** Let $K/k$, $A_0$, $A$, $\sigma$, $\tau$, $\rho$ and $\pi_2$ as above. Then $\pi_2 \circ \rho = 0$.

**Proof.** Let $(V_0, h_0)$ be an $\varepsilon$-hermitian space over $(A_0, \tau)$. We have $\rho(h_0) = (V, h)$, where $V = V_0 \oplus \sqrt{d} V_0$ and $h$ is as defined above. Let $W = \{x + \sqrt{d} 0 \mid x \in V_0\} \subseteq V$. Then we have

$$\pi_2 \rho(h_0)(x_1 + \sqrt{d} 0, x_2 + \sqrt{d} 0) = \pi_2(h_0(x_1, x_2) + \sqrt{d} 0) = 0.$$
Thus $W \subset W^\perp$. Let $x + \sqrt{y} \in W^\perp$. Then we have $0 = \pi_2 \rho(h_0)(z + \sqrt{d} 0, x + \sqrt{d}y) = \eta h_0(z, y)$ for all $z \in V_0$. Since $h_0$ is non-degenerate, we have $y = 0$. Hence $W = W^\perp$ and $\pi_2 \rho(h_0)$ is hyperbolic. \qed

**Theorem 3.2** Let $K/k$, $A_0$, $A$, $\sigma$, $\tau$, $\rho$ and $\pi_2$ as above. Let $(V, h)$ be an anisotropic $\varepsilon$-hermitian space over $(A, \sigma)$. Suppose that $\pi_2(h) = h' \perp h$ for some hyperbolic space $h$. Then there exist an $\varepsilon$-hermitian space $h_1$ over $(A, \sigma)$ and an $\varepsilon$-hermitian space $h_2$ over $(A_0, \tau)$ such that

$$h = h_1 \perp \rho(h_2) \quad \text{and} \quad \pi_2(h_1) = h'.$$

**Proof.** We prove this by induction on $\dim(h)$. If the $\dim(h) = 0$, i.e. there is no $h$, then we take $h_1 = h$ and we are done. Assume that $\dim(h) = m \geq 1$. In particular $\pi_2(h)$ is isotropic. Then there exists $z \in V$, $z \neq 0$, such that $\pi_2(h)(z, z) = 0$. Let $V_0 = z A_0$ be the $A_0$-submodule of $V$ generated by $z$. Since $\pi_2(h(z, z)) = 0$, we have $h(z, z) \in A_0$. Let $a, b \in A_0$. Then $h(z a, z b) = \sigma(a) h(z, z) = \tau(a) h(z, z) b \in A_0$. Thus the restriction of $h$ to $V_0$ induces an $\varepsilon$-hermitian form $(V_0, h_0)$. Since $h$ is anisotropic, the form $(V_0, h_0)$ is anisotropic and hence non-degenerate. Since $V$ is an $A$-module, we have $V_0 \oplus V_0 \sqrt{d} \subset V$ and $\rho(h_0)$ is the restriction of $h$ to $V_0 \oplus V_0 \sqrt{d}$. Once again using the fact that $h$ is isotropic we have $h = h'_1 \perp \rho(h_0)$. Since $\pi_2(\rho(h_0))$ is hyperbolic (by 3.1), by the Witt’s cancellation, we have $\pi_2(h'_1) = h' \perp h'$ for some hyperbolic space $h'$. Since $\dim(h_1) < \dim(h)$, we have $\dim(h') < \dim(h)$. Hence by induction, we have $h'_1 = h_1 \perp \rho(h'_2)$ with $\pi_2(h_1) = h'$. We have

$$h = h'_1 + \rho(h_0) = h_1 + \rho(h'_2) + \rho(h_0).$$

Let $h_2 = h'_2 + h_0$. Then $h_1$ and $h_2$ have the required properties. \qed
Corollary 3.3 Let $K/k$, $A_0$, $A$, $\sigma$, $\tau$ as above. With the notation as above we have the following exact sequence:

$$W^\epsilon(A_0, \tau) \xrightarrow{\rho} W^\epsilon(A, \sigma) \xrightarrow{\pi_2} W^{-\epsilon}(A_0, \tau).$$

Proof: Follows from (3.1) and (3.2).

Theorem 3.4 Let $k$ be a field of characteristic not equal to $2$ and $K = k(\sqrt{d})$ a quadratic extension of $k$. Let $A_0$ be a central simple algebra over $k$ with an involution $\tau$. Let $A = A_0 \otimes_k K$ and $\sigma = \tau \otimes \theta$, where $\theta$ is an automorphism of $K$. Let $\eta \in \{\pm 1\}$ be such that $\theta(\sqrt{d}) = \eta(\sqrt{d})$. Then we have

$$u(A, \sigma, \epsilon) \leq \frac{1}{2}u(A_0, \tau, \eta\epsilon) + u(A_0, \tau, \epsilon) + 1.$$ 

Proof: If $u(A_0, \tau, \epsilon)$ is not finite, then there is nothing to prove. Assume that $u(A_0, \tau, \epsilon)$ is finite.

Let $h$ be an anisotropic $\epsilon$-hermitian form over $(A, \sigma)$. Suppose that the dimension of $h \geq \frac{1}{2}u(A_0, \tau, \eta\epsilon) + u(A_0, \tau, \epsilon) + 1$. We know that $\dim \pi_2(h) = 2 \dim(h)$. Thus $\dim \pi_2(h) \geq u(A_0, \tau, \eta\epsilon) + 2u(A_0, \tau, \epsilon) + 2$. Since any $\eta\epsilon$-hermitian space of dimension bigger that $u(A_0, \tau, \eta\epsilon)$ is isotropic, we have $\pi_2(h) = h' + h$ with $\dim h' \leq u(A_0, \tau, \eta\epsilon)$. In particular, then $\dim(h) \geq 2u(A_0, \tau, \epsilon) + 2$. By (3.2), there exist an $\epsilon$-hermitian space $h_1$ over $(A, \sigma)$ and an $\epsilon$-hermitian space $h_2$ over $(A_0, \tau)$ such that

$$h = h_1 \perp \rho(h_2) \quad \text{and} \quad \pi_2(h_1) = h'.$$

Since $\dim(\rho(h_2)) = \dim(h_2)$, we have $\dim h_2 \geq u(A_0, \tau, \epsilon) + 1$. Hence $h_2$ is isotropic. Therefore $\rho(h_2)$, in particular $h$, is isotropic. Which is a contradiction. Hence $\dim(h) \leq \frac{1}{2}u(A_0, \tau, \eta\epsilon) + u(A_0, \tau, \epsilon)$. This proves the theorem. \qed
**Theorem 3.5** Let $k$ be a field of characteristic not equal to 2 and $K = k(\sqrt{d})$ a quadratic extension of $k$. Let $A_0$ be a central simple algebra over $k$ with an involution $\tau$. Let $A = A_0 \otimes_k K$ and $\sigma = \tau \otimes id$, where $id$ is the identity map of $K$. Then

$$u(A, \sigma, \varepsilon) \leq \frac{3}{2} u(A_0, \tau, \varepsilon).$$

**Proof.** By (3.4), we have $u(A, \sigma, \varepsilon) \leq \frac{1}{2} u(A_0, \tau, \eta \varepsilon) + u(A_0, \tau + \varepsilon)$. Since $\sigma = \tau \otimes id$, we have $\eta = 1$. Therefore $u(A, \sigma, \varepsilon) \leq \frac{3}{2} u(A_0, \tau, \varepsilon)$. \qed

**Theorem 3.6** Let $k$ be a field of characteristic not equal to 2 and $K/k$ a quadratic extension. Let $A_0$ be a central simple algebra over $k$ with an involution $\tau$. Let $A = A_0 \otimes_k K$ and $\sigma = \tau \otimes -$, where $-$ is the non-trivial automorphism of $K/k$. Then

$$u(A, \sigma, \varepsilon) \leq \text{minimum}\{u(A_0, \tau, \varepsilon) + \frac{1}{2} u(A_0, \tau, -\varepsilon), u(A_0, \tau, -\varepsilon) + \frac{1}{2} u(A_0, \tau, \varepsilon)\}.$$

**Proof:** By (3.4), we have $u(A, \sigma, \varepsilon) \leq \frac{1}{2} u(A_0, \tau, \eta \varepsilon) + u(A_0, \tau + \varepsilon)$. Since $\sigma = \tau \otimes -$, we have $\eta = -1$. Therefore $u(A, \sigma, \varepsilon) \leq \frac{1}{2} u(A_0, \tau, -\varepsilon) + u(0_0, \tau + \varepsilon)$.

Since $\sigma$ is an involution of second kind we have, $u(A, \sigma, \varepsilon) = u(A, \sigma, -\varepsilon)$ (see chapter 1). Thus $u(A, \sigma, \varepsilon) \leq \text{minimum}\{u(A_0, \tau, \varepsilon) + \frac{1}{2} u(A_0, \tau, -\varepsilon), u(A_0, \tau, -\varepsilon) + \frac{1}{2} u(A_0, \tau, \varepsilon)\}$. \qed

**Corollary 3.7** Let $k$ be a field of characteristic not equal to 2 and $K/k$ a quadratic extension. Let $H$ be a quaternion algebra over $K$ with an involution $\sigma$ of second kind. Then $u(H, \sigma, \varepsilon) \leq \frac{7}{8} u(k)$. 
Proof: Let $H$ be a quaternion algebra over $K$ with an involution $\sigma$ of second kind. By a theorem of Albert (see [10]), there exists a quaternion subalgebra $H_0$ over $k$ such that $H = H_0 \otimes_k K$ and $\sigma = \tau \otimes -$, $\tau$ is the canonical involution on $H_0$ and $-$ the non-trivial automorphism of $K/k$. By (3.6), we have $u(H, \sigma, \varepsilon) \leq u(H_0, \tau, 1) + \frac{1}{2}u(H_0, \tau, -1)$. By (1.9), we have $u(H_0, \tau, 1) \leq \frac{1}{4}u(k)$ and $u(H_0, \tau, -1) \leq \frac{5}{4}u(k)$. Hence $u(H, \sigma, \varepsilon) \leq \frac{1}{4}u(k) + \frac{1}{2} \times \frac{5}{4}u(k) = \frac{7}{8}u(k)$. 

Corollary 3.8 Let $k$ be a field of characteristic not equal to 2. Let $H_1$ and $H_2$ be two quaternion algebras over $k$ and $\tau$ the involution given by the tensor product of canonical involutions on $H_1$ and $H_2$. Then $u(H_1 \otimes H_2, \tau, 1) \leq \frac{29}{16}u(k)$. 

Proof: Since $H_2$ is a quaternion algebra, there exist $\lambda$ and $\mu$ as in the paragraph before (1.5.1). Thus by (1.5.3), we have $u(H_1 \otimes H_2, \tau, 1) \leq \frac{1}{2}u(H_1 \otimes k(\lambda), \tau_2, -1) + u(H_1 \otimes k(\lambda), \tau_1, 1)$. Since $\tau_1$ is an involution of second kind, by (3.7), we have $u(H_1 \otimes k(\lambda), \tau_1, \varepsilon) \leq \frac{7}{8}u(k)$. Since $\tau_2$ is an involution of first kind, by (3.5), we have $u(H_1 \otimes k(\lambda), \tau_2, -1) \leq \frac{3}{2}u(H_1, -, -1)$. By (1.5.4), we have $u(H_1, -, -1) \leq \frac{5}{4}$. Hence $u(H_1 \otimes k(\lambda)) \leq \frac{15}{8}$. Therefore, we have $u(H_1 \otimes H_2, \tau, 1) \leq \frac{1}{2} \times \frac{15}{8}u(k) + \frac{7}{8}u(k) = \frac{29}{16}u(k)$.

Corollary 3.9 Let $k$ be a field of characteristic not equal to 2. Let $H_1$ and $H_2$ be two quaternion algebras over $k$ and $\tau$ the involution given by the tensor product of canonical involutions on $H_1$ and $H_2$. Then $u(H_1 \otimes H_2, \tau, -1) \leq \frac{17}{16}u(k)$. 
**Proof:** As in the proof of (3.8), we have
\[
\begin{align*}
u(H_1 \otimes H_2, \tau, -1) & \leq \frac{1}{2}u(H_1 \otimes k(\lambda), \tau_2, 1) + u(H_1 \otimes k(\lambda), \tau_1, 1) \\
& \leq \frac{1}{2} \times \frac{7}{8}u(k(\lambda)) + \frac{7}{8}u(k) \\
& \leq \frac{7}{8} \times \frac{3}{2}u(k) + \frac{7}{8}u(k) \\
& = \frac{17}{16}u(k).
\end{align*}
\]

\[\square\]

**Corollary 3.10** Let \(k\) be a field of characteristic not equal to 2. Let \(A\) be a central simple \(k\)-algebra of degree 4 with an orthogonal involution \(\tau\). Then
\[
u(A, \tau, +1) \leq \frac{29}{16}u(k)\) and \(u(A, \tau, -1) \leq \frac{17}{16}u(k)\).

**Proof.** Since \(A\) is a central simple \(k\)-algebra of degree 4 with an involution of first kind, we have \(A \simeq H_1 \otimes H_2\). Let \(\sigma\) be the tensor product of the canonical involutions on \(H_1\) and \(H_2\). Then \(\sigma\) is an orthogonal involution on \(H_1 \otimes H_2\) (cf. [7], 8.1.3). Hence we have \(u(A, \tau, \varepsilon) = u(A, \sigma, \varepsilon)\) (cf. §1.5). The corollary follows from (3.8) and (3.9).

**Remark 3.11.** Let \(H\) be a quaternion algebra over \(K\) and \(\sigma\) an involution of second kind with fixed field \(k\). Then by ([8]), we have \(u(H, \sigma, \varepsilon) \leq \frac{5}{4}u(k)\). Let \(A\) be a central simple \(k\)-algebra of degree 4, with an orthogonal involution \(\tau\). Then, by ([8]), we have \(u(A, \tau, 1) \leq \frac{55}{16}u(k)\) and \(u(A, \tau, -1) \leq \frac{21}{16}u(k)\). Hence our bounds are sharper.