Chapter 3

Common solution to generalized vector equilibrium problem and fixed point problems for relatively nonexpansive mappings in Banach space

3.1 Introduction

Throughout the chapter unless otherwise stated, let $E$ be a real Banach space with its dual space $E^*$ and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $E$ and $E^*$ and $\| \cdot \|$ denote the norm of $E$ as well as of $E^*$. Let $C$ be nonempty closed and convex subset of $E$ and let $2^E$ denote the set of all nonempty subsets of $E$.

In 2005, Matsushita and Takahashi [117] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping $T$ in a Banach space $E$:

\[
\begin{aligned}
    x_0 &\in C, \text{ chosen arbitrarily}, \\
    y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\
    C_n &= \{ z \in C : \phi(z, y_n) \leq \phi(z, x_n) \}, \\
    Q_n &= \{ z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0 \}, \\
    x_{n+1} &= \Pi_{C_n \cap Q_n} x_0.
\end{aligned}
\] (3.1.1)

They proved that $\{x_n\}$ converges strongly to $\prod_{\text{Fix}(T)} x_0$, where $\prod_{\text{Fix}(T)}$ is the generalized projection from $C$ onto $\text{Fix}(T)$.

In 2009, Takahashi and Zembayashi [163] proposed the following modification of iteration process (3.1.1) for $\text{EP}(1.3.11)$ and $\text{FPP}(1.2.1)$ for a relatively nonexpansive mapping $T$:
\[
\begin{aligned}
x_0 &= x \in C, \quad C_0 = C, \\
y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\
u_n &\in C \text{ such that } F(u_n, y) + \frac{1}{r_n}(y - u_n, Ju_n - Jx_n) \geq 0, \quad \forall y \in C, \\
C_n &= \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
Q_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} &= \Pi_{C_n \cap Q_n} x, \quad n = 0, 1, 2, \ldots,
\end{aligned}
\]

where \( \Pi_{C_n \cap Q_n} \) is the generalized projection from \( E \) onto \( C_n \cap Q_n \). They proved that the sequence \( \{x_n\} \) converges strongly to \( \Pi_{\text{FPP}(1.2.1) \cap \text{EP}(1.3.11)} x \).

Recently, Shan and Huang [148] studied an iterative method for approximating a common solution of GMVEP(1.3.26), VIP(1.3.1) and FPP(1.2.1) for a nonexpansive mapping in Hilbert spaces. But the study of iterative method to approximate a common solution of vector equilibrium problem and fixed point problem in Banach space, has not done so far.

Therefore, motivated by the work of Takahashi and Zembayashi [163] and Shan and Haung [148], and by the ongoing research in this direction, we introduce and study some iterative methods for approximating a common solution of fixed point problems for relatively nonexpansive mappings and the following generalized vector equilibrium problem (in short, GVEP) in Banach space. Let \( F : C \times C \to Y \) be a nonlinear bimapping and let \( \psi : C \to Y \) be a nonlinear mapping, then GVEP is to find \( x^* \in C \) such that

\[
F(x^*, x) + \psi(x) - \psi(x^*) \in P, \quad \forall x \in C,
\]

where \( P \) be a pointed, proper, closed and convex cone of a Hausdorff topological space \( Y \) with \( \text{int}P \neq \emptyset \). The solution set of GVEP(3.1.3) is denoted by \( \text{Sol}(\text{GVEP}(3.1.3)) \).

If \( \psi = 0 \) then GVEP(3.1.3) reduces to the strong vector equilibrium problem (SVEP): Find \( x^* \in C \) such that

\[
F(x^*, x) \in P, \quad \forall x \in C.
\]
If $Y = \mathbb{R}$, then $P = [0, +\infty)$ and hence GVEP(3.1.3) reduces to the following generalized equilibrium problem (in short, GEP): Find $x^* \in C$ such that

$$F(x^*, x) + \psi(x) - \psi(x^*) \geq 0, \; \forall x \in C,$$

(3.1.5)

where $\psi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function. GEP(3.1.5) has been studied by Ceng and Yao [27].

In this chapter, we study the strong and weak convergence of the sequences generated by the proposed iterative schemes for finding a common solution of GVEP(3.1.3) and fixed point problems for two relatively nonexpansive mappings in real Banach space. The results presented in this paper extend and generalize many previously known results in this research area, see instance [163].

### 3.2 Existence of solution

Let $F : C \times C \rightarrow Y$ and $\psi : C \rightarrow Y$ be two mappings. For any $z \in E$, define a mapping $G_z : C \times C \rightarrow Y$ as follows:

$$G_z(x, y) = F(x, y) + \psi(y) - \psi(x) + \frac{e}{r} \langle y - x, Jx - Jz \rangle,$$

(3.2.1)

where $r$ is a positive real number and $e \in \text{int}P$. Further, these mappings have the following assumptions.

**Assumption 3.2.1.** Let $G_z$, $F$, $\psi$ satisfy the following conditions:

(i) For all $x \in C$, $F(x, x) = 0$;

(ii) $F$ is monotone, i.e., $F(x, y) + F(y, x) \in -P$, $\forall x, y \in C$;

(iii) $F(., y)$ is continuous, $\forall y \in C$;

(iv) $F(x, .)$ is weakly continuous and $P$-convex, i.e.,

$$tF(x, y_1) + (1 - t)F(x, y_2) \in F(x, ty_1 + (1 - t)y_2) + P, \; \forall x, y_1, y_2 \in C, \forall t \in [0, 1];$$
(v) $G_z(., y)$ is lower $P$-continuous, $\forall y \in C$ and $z \in E$;

(vi) $\psi(.)$ is $P$-convex and weakly continuous;

(vii) $G_z(x, .)$ is proper $P$-quasiconvex, $\forall x \in C$ and $z \in E$.

Next, we define a mapping $T_r : E \to C$ as follows:

$$T_r(z) = \{ x \in C : F(x, y) + \psi(y) - \psi(x) + e_r \langle y - x, Jx - Jz \rangle \in P, \ \forall y \in C \}, \ (3.2.2)$$

where $e \in \text{int} P$ and $r$ is a positive real number.

Now, we prove the existence of solution of GVEP(3.1.3) and derive some properties of $T_r$.

**Theorem 3.2.1.** Let $E$ be a uniformly smooth and uniformly convex Banach space and let $C$ be a nonempty compact and convex subset of $E$. Assume that $P$ is a pointed, proper, closed and convex cone of a real Hausdorff topological space $Y$ with $\text{int} P \neq \emptyset$.

Let $G_z : C \times C \to Y$ be defined by (3.2.1) and let the mapping $T_r : E \to C$ given by (3.2.2). Let $F : C \times C \to Y$, $\psi : C \to Y$ and $G_z$ satisfy Assumption 3.2.1. Then

(i) $T_r z \neq \emptyset, \ \forall z \in E$;

(ii) $T_r$ is single-valued;

(iii) $T_r$ is firmly nonexpansive type mapping, i.e., for all $z_1, z_2 \in E$,

$$\langle T_r z_1 - T_r z_2, JT_r z_1 - JT_r z_2 \rangle \leq \langle T_r z_1 - T_r z_2, J z_1 - J z_2 \rangle;$$

(iv) $\text{Fix}(T_r) = \text{Sol}(\text{GVEP}(3.1.3))$;

(v) $\text{Sol}(\text{GVEP}(3.1.3))$ is closed and convex.

**Proof.** (i) Let $g(x, y) = G_z(x, y)$ and $\Phi(y) = 0$, for all $x, y \in C$ and $z \in E$. It is easy to observe that $g(x, y)$ and $\Phi(y)$ satisfy all the conditions of Lemma 1.2.17. Then there
exists a point $x \in C$ such that
\[ G_z(x, y) + \Phi(y) - \Phi(x) \in P, \quad \forall y \in C, \]
and thus $T_r z \neq \emptyset$, $\forall z \in E$.

(ii) For each $z \in E$, $T_r z \neq \emptyset$, let $x_1, x_2 \in T_r z$. Then
\[
F(x_1, y) + \psi(y) - \psi(x_1) + \frac{e}{r} \langle y - x_1, Jx_1 - Jz \rangle \in P, \quad \forall y \in C, \tag{3.2.3}
\]
and
\[
F(x_2, y) + \psi(y) - \psi(x_2) + \frac{e}{r} \langle y - x_2, Jx_2 - Jz \rangle \in P, \quad \forall y \in C. \tag{3.2.4}
\]
Letting $y = x_2$ in (3.2.3) and $y = x_1$ in (3.2.4), and then adding, we have
\[ F(x_1, x_2) + F(x_2, x_1) + \frac{e}{r} \langle x_2 - x_1, Jx_1 - Jx_2 \rangle \in P. \]
Since $F$ is monotone, $e \in \text{int} P$, $r > 0$ and $P$ is closed and convex cone, we have
\[ \langle x_2 - x_1, Jx_1 - Jx_2 \rangle \geq 0. \]
Since $E$ is strictly convex, preceding inequality implies $x_1 = x_2$. Hence $T_r$ is single-valued.

(iii) For any $z_1, z_2 \in E$, let $x_1 = T_r z_1$ and $x_2 = T_r z_2$. Then
\[
F(x_1, y) + \psi(y) - \psi(x_1) + \frac{e}{r} \langle y - x_1, Jx_1 - Jz_1 \rangle \in P, \quad \forall y \in C, \tag{3.2.5}
\]
and
\[
F(x_2, y) + \psi(y) - \psi(x_2) + \frac{e}{r} \langle y - x_2, Jx_2 - Jz_2 \rangle \in P, \quad \forall y \in C. \tag{3.2.6}
\]
Letting $y = x_2$ in (3.2.5) and $y = x_1$ in (3.2.6), and then adding, we have
\[ F(x_1, x_2) + F(x_2, x_1) + \frac{e}{r} \langle x_2 - x_1, Jx_1 - Jx_2 - Jz_1 + Jz_2 \rangle \in P. \]
Again, since $F$ is monotone, $e \in \text{int}P$, $r > 0$ and $P$ is closed and convex cone, we have

\[
\langle x_2 - x_1, Jx_2 - Jx_1 \rangle \leq \langle x_2 - x_1, Jz_2 - Jz_1 \rangle,
\]

or

\[
\langle T_r z_1 - T_r z_2, JT_r z_1 - JT_r z_2 \rangle \leq \langle T_r z_1 - T_r z_2, Jz_1 - Jz_2 \rangle. \quad (3.2.7)
\]

Hence $T_r$ is firmly nonexpansive-type mapping.

(iv) Let $x \in \text{Fix}(T_r)$. Then

\[
F(x, y) + \psi(y) - \psi(x) + \frac{e}{r} \langle y - x, Jx - Jx \rangle \in P, \quad \forall y \in C,
\]

and so

\[
F(x, y) + \psi(y) - \psi(x) \in P, \quad \forall y \in C.
\]

Thus $x \in \text{Sol}(GVEP(3.1.3)).$

Let $x \in \text{Sol}(GVEP(3.1.3))$. Then

\[
F(x, y) + \psi(y) - \psi(x) \in P, \quad \forall y \in C,
\]

and so

\[
F(x, y) + \psi(y) - \psi(x) + \frac{e}{r} \langle y - x, Jx - Jx \rangle \in P, \quad \forall y \in C.
\]

Hence $x \in \text{Fix}(T_r)$. Thus $\text{Fix}(T_r) = \text{Sol}(GVEP(3.1.3)).$

(v) By the definition of $\phi$, for any $z_1, z_2 \in C$ we have

\[
\phi(T_r z_1, T_r z_2) + \phi(T_r z_2, T_r z_1) = 2\|T_r z_1\|^2 - 2\langle T_r z_1, JT_r z_2 \rangle - 2\langle T_r z_2, JT_r z_1 \rangle + 2\|T_r z_2\|^2
\]

\[
= 2\langle T_r z_1, JT_r z_1 - JT_r z_2 \rangle + 2\langle T_r z_2, JT_r z_2 - JT_r z_1 \rangle
\]

\[
= 2\langle T_r z_1 - T_r z_2, JT_r z_1 - JT_r z_2 \rangle,
\]
and

\[
\phi(T_r z_1, z_2) + \phi(T_r z_2, z_1) = \phi(T_r z_1, z_1) - \phi(T_r z_2, z_2)
\]

\[
= \|T_r z_1\|^2 - 2\langle T_r z_1, Jz_2 \rangle + \|z_2\|^2
\]

\[
+ \|T_r z_2\|^2 + \|z_1\|^2 - 2\langle T_r z_2, Jz_1 \rangle
\]

\[
- \|T_r z_2\|^2 + 2\langle T_r z_2, Jz_2 \rangle - \|z_2\|^2
\]

\[
- \|T_r z_2\|^2 + 2\langle T_r z_2, z_1 \rangle - \|z_1\|^2
\]

\[
= 2\langle T_r z_1, Jz_1 - Jz_2 \rangle - 2\langle T_r z_2, Jz_1 - Jz_2 \rangle
\]

\[
= 2\langle T_r z_1 - T_r z_2, Jz_1 - Jz_2 \rangle.
\]

Thus, it follows from (3.2.7) and preceding two equations that

\[
\phi(T_r z_1, z_2) + \phi(T_r z_2, z_1) \leq \phi(T_r z_1, z_2) + \phi(T_r z_2, z_1) - \phi(T_r z_2, z_2).
\]

Hence, for \(z_1, z_2 \in C\), we have

\[
\phi(T_r z_1, z_2) + \phi(T_r z_2, z_1) \leq \phi(T_r z_1, z_2) + \phi(T_r z_2, z_1).
\]

Taking \(z_2 = u \in \text{Fix}(T_r)\), we have

\[
\phi(u, T_r z_1) \leq \phi(u, z_1).
\]

Next, we show that \(\hat{\text{Fix}}(T_r) = \text{Sol}(\text{GVEP}(3.1.3))\). Indeed, let \(p \in \hat{\text{Fix}}(T_r)\). Then there exists \(\{z_n\} \subset E\) such that \(z_n \rightharpoonup p\) and \(\lim_{n \to \infty} (z_n - T_r z_n) = 0\). Moreover, we get \(T_r z_n \rightharpoonup p\). Hence, we have \(p \in C\). Since \(J\) is uniformly continuous on bounded sets, we have

\[
\lim_{n \to \infty} \frac{\|Jz_n - JT_r z_n\|}{r} = 0, \quad r > 0.
\]  

(3.2.8)

From the definition of \(T_r\), we have

\[
F(T_r z_n, y) + \psi(y) - \psi(T_r z_n) + \frac{e}{r} \langle y - T_r z_n, JT_r z_n - Jz_n \rangle \in P, \quad \forall y \in C
\]

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\[ 0 \in F(y, T_r z_n) - \psi(y) + \psi(T_r z_n) - \frac{e}{r} (y - T_r z_n, JT_r z_n - J z_n) + P, \quad \forall y \in C. \]

Let \( y_t = (1 - t)p + ty, \forall t \in (0, 1]. \) Since \( y \in C \) and \( p \in C, \) we get \( y_t \in C \) and hence

\[ 0 \in F(y_t, T_r z_n) - \psi(y_t) + \psi(T_r z_n) - \frac{e}{r} (y_t - T_r z_n, JT_r z_n - J z_n) + P. \tag{3.2.9} \]

Since \( F(x, \cdot) \) and \( \psi(\cdot) \) are weakly continuous for all \( x \in C, \) then it follows from (3.2.8) and (3.2.9) that

\[ 0 \in F(y_t, p) - \psi(y_t) + \psi(p) + P. \tag{3.2.10} \]

Further, it follows from Assumption 3.2.1 (i), (iv), (vi) that

\[ tF(y_t, y) + (1 - t)F(y_t, p) + t\psi(y) + (1 - t)\psi(p) - \psi(y_t) \in F(y_t, y_t) + \psi(y_t) - \psi(y_t) + P \]

\[ \in P, \]

or \( -t[F(y_t, y) + \psi(y) - \psi(y_t)] - (1 - t)[F(y_t, p) + \psi(p) - \psi(y_t)] \in -P. \tag{3.2.11} \]

Using (3.2.10) in (3.2.11), we have

\[ -t[F(y_t, y) + \psi(y) - \psi(y_t)] \in -P \]

\[ F(y_t, y) + \psi(y) - \psi(y_t) \in P. \]

Letting \( t \to 0, \) we obtain

\[ F(p, y) + \psi(y) - \psi(p) \in P, \quad \forall y \in C, \]

i.e., \( p \in \text{Sol}(\text{GVEP}(3.1.3)). \) So, we get \( \text{Fix}(T_r) = \text{Sol}(\text{GVEP}(3.1.3)) = \hat{\text{Fix}}(T_r). \) Therefore \( T_r \) is a relatively nonexpansive mapping. Further, it follows from Lemma 1.2.9 that \( \text{Sol}(\text{GVEP}(3.1.3)) = \text{Fix}(T_r) \) is closed and convex. This completes the proof. \( \square \)

Next, we have the following consequence of Theorem 3.2.1.

**Lemma 3.2.1.** Let \( E, C, F, \psi, G_z \) be same as in Theorem 3.2.1 and let \( r > 0. \) Then,
for $x \in E$ and $q \in \text{Fix}(T_r)$, we have

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

### 3.3 Iterative methods

Now, we prove a strong convergence theorem for approximating a common solution of GVEP(3.1.3) and the fixed point problems of two relatively nonexpansive mappings in a Banach space.

**Theorem 3.3.1.** Let $E$ be a uniformly smooth and uniformly convex Banach space and let $C$ be a nonempty compact and convex subset of $E$. Assume that $P$ is a pointed, proper, closed and convex cone of a real Hausdorff topological space $Y$ with $\text{int}P \neq \emptyset$. Let the mappings $F : C \times C \to Y$ and $\psi : C \to Y$ satisfy Assumption 3.2.1 and let $S, T$ be relatively nonexpansive mappings from $C$ into itself such that $\Gamma := \text{Fix}(T) \cap \text{Fix}(S) \cap \text{Sol}(\text{GVEP}(3.1.3)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the scheme:

$$x_0 = x \in C,$$

$$y_n = J^{-1}(\delta_n Jx_n + (1 - \delta_n) JTz_n),$$

$$z_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSx_n),$$

$$u_n \in C \text{ such that } F(u_n, y) + \psi(y) - \psi(u_n) + \frac{e}{r} \langle y - u_n, Ju_n - Jy_n \rangle \in P, \ \forall y \in C,$$

$$H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\},$$

$$W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\},$$

$$x_{n+1} = \prod_{H_n \cap W_n} x, \ \text{ for every } n \in N \cup \{0\},$$

where $e \in \text{int}P$, $J$ is the normalized duality mapping on $E$ with inverse $J^{-1}$, and $r \in [a, \infty)$ for some $a > 0$. Assume that $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the conditions:

(i) $\limsup_{n \to \infty} \delta_n < 1$;

(ii) $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1$. 

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Then \( \{x_n\} \) converges strongly to \( \prod_{\Gamma} x \), where \( \prod_{\Gamma} x \) is the generalized projection of \( E \) onto \( \Gamma \).

**Proof.** Since \( S \) and \( T \) are relatively nonexpansive mappings from \( C \) into itself, it follows from Lemma 1.2.9 and Theorem 3.2.1 (v) that \( \Gamma \) is closed and convex. Now, we show that \( H_n \cap W_n \) is a closed and convex. From the definition of \( W_n \), it is obvious that \( W_n \) is closed and convex. Further, from the definition of \( \phi \), we observe that \( H_n \) is closed and convex. So, \( H_n \cap W_n \) is a closed convex subset of \( E \) for all \( n \in N \cup \{0\} \).

Let \( u \in \Gamma \). It follows from Theorem 3.2.1 that (3.2.2) is equivalent to \( u_n = T_y y_n \) for all \( n \in N \cup \{0\} \), and \( T_y \) is relatively nonexpansive. Since \( S \) and \( T \) are relatively nonexpansive, we have

\[
\phi(u, u_n) = \phi(u, T_y y_n) \\
\leq \phi(u, y_n) \\
\leq \phi(u, J^{-1}(\delta_n Jx_n + (1 - \delta_n)JTz_n)) \\
= \|u\|^2 - 2\langle u, \delta_n Jx_n + (1 - \delta_n)JTz_n \rangle + \|\delta_n Jx_n + (1 - \delta_n)JTz_n\|^2 \\
\leq \|u\|^2 - 2\delta_n \langle u, Jx_n \rangle - 2(1 - \delta_n)\|u, JTz_n\| + \delta_n \|x_n\|^2 + (1 - \delta_n)\|Tz_n\|^2 \\
\leq \delta_n \phi(u, x_n) + (1 - \delta_n)\phi(u, Tz_n) \\
\leq \delta_n \phi(u, x_n) + (1 - \delta_n)\phi(u, z_n),
\]

and

\[
\phi(u, z_n) = \phi(u, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n)) \\
= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1 - \alpha_n)JSx_n \rangle \\
+ \|\alpha_n Jx_n + (1 - \alpha_n)JSx_n\|^2 \\
\leq \|u\|^2 - 2\alpha_n \langle u, Jx_n \rangle - 2(1 - \alpha_n)\langle u, JSx_n \rangle + \alpha_n \|x_n\|^2 + (1 - \alpha_n)\|Sx_n\|^2 \\
\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n)\phi(u, Sx_n)
\]
\[ \leq \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, x_n) \]
\[ \leq \phi(u, x_n). \]  

(3.3.2)

Using (3.3.2) in (3.3.1), we have

\[ \phi(u, u_n) \leq \delta_n \phi(u, x_n) + (1 - \delta_n) \phi(u, x_n) \]
\[ \leq \phi(u, x_n). \]

Hence, we have \( u \in H_n \). This implies that \( \Gamma \subset H_n, \forall n \in N \cup \{0\} \).

Next, we show by induction that \( \Gamma \subset H_n \cap W_n, \forall n \in N \cup \{0\} \). From \( W_0 = C \), we have \( \Gamma \subset H_0 \cap W_0 \). Suppose that \( \Gamma \subset H_k \cap W_k \), for some \( k \in N \cup \{0\} \). Then there exists \( x_{k+1} \in H_k \cap W_k \) such that \( x_{k+1} = \prod_{H_k \cap W_k} x \). From the definition of \( x_{k+1} \), we have, for all \( z \in H_k \cap W_k \),

\[ \langle x_{k+1} - z, Jx - Jx_{k+1} \rangle \geq 0. \]

(3.3.3)

Since \( \Gamma \subset H_k \cap W_k \), we have

\[ \langle x_{k+1} - z, Jx - Jx_{k+1} \rangle \geq 0, \forall z \in \Gamma, \]

and hence \( z \in W_{k+1} \). So, we have \( \Gamma \subset W_{k+1} \). Therefore, we have \( \Gamma \subset H_{k+1} \cap W_{k+1} \). Thus, we have that \( \Gamma \subset H_n \cap W_n \) for all \( n \in N \cup \{0\} \). This means that \( \{x_n\} \) is well-defined.

From the definition of \( W_n \), we have \( x_n = \prod_{W_n} x \) and hence, from Lemma 1.2.8, we have

\[ \phi(x_n, x) = \phi(\prod_{W_n} x, x) \leq \phi(u, x) - \phi(u, \prod_{W_n} x) \leq \phi(u, x), \forall u \in \Gamma \subset W_n. \]

Thus \( \{\phi(x_n, x)\} \) is bounded. Therefore \( \{x_n\} \) and \( \{Sx_n\} \) are bounded.

Since \( x_{n+1} = \prod_{H_n \cap W_n} x \in H_n \cap W_n \subset W_n \) and \( x_n = \prod_{W_n} x \), from the definition of \( \prod_{W_n} \), we have

\[ \phi(x_n, x) \leq \phi(x_{n+1}, x), \forall n \in N \cup \{0\}. \]
Thus \( \{\phi(x_n, x)\} \) is nondecreasing. So, the limit of \( \{\phi(x_n, x)\} \) exists. By the construction of \( W_n \), we have \( W_m \subseteq W_n \) and \( x_m = \prod_{W_m} x \in W_n \) for any positive integer \( m \geq n \). It follows that

\[
\phi(x_m, x_n) = \phi(x_m, \prod_{W_n} x) \\
\leq \phi(x_m, x) - \phi(\prod_{W_n} x, x) \\
= \phi(x_m, x) - \phi(x_n, x). \tag{3.3.4}
\]

Letting \( m, n \to \infty \) in (3.3.4), we have \( \phi(x_m, x_n) \to 0 \). It follows from Lemma 1.2.10 that \( \|x_m - x_n\| \to 0 \) as \( m, n \to \infty \). Hence \( \{x_n\} \) is a Cauchy sequence. Since \( E \) is a Banach space and \( C \) is a closed and convex, one can assume that \( x_n \to \hat{x} \in C \) as \( n \to \infty \). From (3.3.4), we have

\[
\phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x) - \phi(x_n, x), \quad \forall n \in N \cup \{0\},
\]

which implies

\[
\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0.
\]

Further, from \( x_{n+1} = \prod_{H_n \cap W_n} x \in H_n \), we have

\[
\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \in N \cup \{0\},
\]

and hence

\[
\lim_{n \to \infty} \phi(x_{n+1}, u_n) = 0.
\]

Since

\[
\lim_{n \to \infty} \phi(x_{n+1}, x_n) = \lim_{n \to \infty} \phi(x_{n+1}, u_n) = 0
\]

and \( E \) is uniformly convex and smooth then, from Lemma 1.2.10, we have

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \|x_{n+1} - u_n\| = 0,
\]
and hence, we have
\[
\lim_{n \to \infty} \| x_n - u_n \| = 0.
\]
Since \( J \) is uniformly norm-to-norm continuous on bounded sets, we have
\[
\lim_{n \to \infty} \| Jx_n - Ju_n \| = 0,
\]
because \( E \) is uniformly smooth Banach space and \( E^* \) is uniformly convex Banach space.

Since \( \{x_n\} \) and \( \{Sx_n\} \) are bounded and \( z_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n) \), then we can
easy to see that \( \{z_n\} \) is a bounded sequence and hence \( \{Tz_n\} \) is bounded.

Let \( r = \sup_{n \in \mathbb{N} \cup \{0\}} \{\|x_n\|, \|Tz_n\|, \|Sx_n\|\} \). From Lemma 1.2.11, we have

\[
\phi(u, z_n) = \phi(u, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n))
\]
\[
= \|u\| - 2\langle u, \alpha_n Jx_n + (1 - \alpha_n)JSx_n \rangle + \|\alpha_n Jx_n + (1 - \alpha_n)JSx_n\|^2
\]
\[
\leq \|u\| - 2\alpha_n \langle u, Jx_n \rangle - 2(1 - \alpha_n)\langle u, JSx_n \rangle + \alpha_n \|x_n\|^2
\]
\[
+ (1 - \alpha_n)\|Sx_n\|^2 - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|)
\]
\[
\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n)\phi(u, Sx_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|)
\]
\[
\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n)\phi(u, x_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|)
\]
\[
\leq \phi(u, x_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|).
\]

It follows from (3.3.1) that

\[
\phi(u, u_n) \leq \delta_n \phi(u, x_n) + (1 - \delta_n)\phi(u, z_n)
\]
\[
\leq \delta_n \phi(u, x_n) + (1 - \delta_n)[\phi(u, x_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|)]
\]
\[
\leq \phi(u, x_n) - \alpha_n(1 - \alpha_n)(1 - \delta_n)g(\|Jx_n - JSx_n\|),
\]
or

\[
\alpha_n(1 - \alpha_n)(1 - \delta_n)g(\|Jx_n - JSx_n\|) \leq \phi(u, x_n) - \phi(u, u_n). \tag{3.3.5}
\]

Further, we have

\[
\phi(u, x_n) - \phi(u, u_n) = \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n \rangle
\]
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\[
\begin{align*}
\|x_n\|^2 - \|u_n\|^2 & \leq 2\|u, Jx_n - Ju_n\| \\
\|x_n\| - \|u_n\|\left(\|x_n\| + \|u_n\|\right) & \leq \|x_n - u_n\|\left(\|x_n\| + \|u_n\|\right) + 2\|u\|\|Jx_n - Ju_n\| \\
\|x_n - u_n\|\left(\|x_n\| + \|u_n\|\right) & \leq \|x_n - u_n\|\left(\|x_n\| + \|u_n\|\right) + 2\|u\|\|Jx_n - Ju_n\|.
\end{align*}
\]

and hence, it follows from \(\lim_{n \to \infty} \|x_n - u_n\| = 0\) and \(\lim_{n \to \infty} \|Jx_n - Ju_n\| = 0\) that

\[
\lim_{n \to \infty} (\phi(u, x_n) - \phi(u, u_n)) = 0. \tag{3.3.6}
\]

Using conditions (i)-(ii) and (3.3.6) in (3.3.5), we have

\[
\lim_{n \to \infty} g(\|Jx_n - JSx_n\|) = 0.
\]

Further, it follows from the property of \(g\) that

\[
\lim_{n \to \infty} \|Jx_n - JSx_n\| = 0.
\]

Since \(J^{-1}\) is uniformly norm-to-norm continuous on bounded sets, we have

\[
\lim_{n \to \infty} \|x_n - Sx_n\| = 0. \tag{3.3.7}
\]

Next, we have

\[
\begin{align*}
\phi(u, u_n) & \leq \phi(u, y_n) \\
& \leq \phi(u, J^{-1}(\delta_n Jx_n + (1 - \delta_n)JTz_n)) \\
& = \|u\|^2 - 2\langle u, \delta_n Jx_n + (1 - \delta_n)JTz_n \rangle + \|\delta_n Jx_n + (1 - \delta_n)JTz_n\|^2 \\
& \leq \|u\|^2 - 2\delta_n \langle u, Jx_n \rangle - 2(1 - \delta_n)\langle u, JTz_n \rangle \\
& \quad + \delta_n \|x_n\|^2 + (1 - \delta_n)\|Tz_n\|^2 - \delta_n(1 - \delta_n)g(\|Jx_n - JTz_n\|) \\
& \leq \delta_n\phi(u, x_n) + (1 - \delta_n)\phi(u, Tz_n) - \delta_n(1 - \delta_n)g(\|Jx_n - JTz_n\|) \\
& \leq \delta_n\phi(u, x_n) + (1 - \delta_n)\phi(u, z_n) - \delta_n(1 - \delta_n)g(\|Jx_n - JTz_n\|) \\
& \leq \delta_n\phi(u, x_n) + (1 - \delta_n)\phi(u, x_n) - \delta_n(1 - \delta_n)g(\|Jx_n - JTz_n\|)
\end{align*}
\]
\[
\leq \phi(u, x_n) - \delta_n(1 - \delta_n)g(\|Jx_n - JTz_n\|),
\]

or

\[
\delta_n(1 - \delta_n)g(\|Jx_n - JTz_n\|) \leq \phi(u, x_n) - \phi(u, u_n) \to 0 \text{ as } n \to \infty.
\]

Thus

\[
\lim_{n \to \infty} g(\|Jx_n - JTz_n\|) = 0.
\]

It follows from the property of \( g \) that

\[
\lim_{n \to \infty} \|Jx_n - JTz_n\| = 0,
\]

and hence

\[
\lim_{n \to \infty} \|x_n - Tz_n\| = 0. \tag{3.3.8}
\]

Now,

\[
\|Jx_n - Jz_n\| = \|Jx_n - (\alpha_n Jx_n + (1 - \alpha_n)JSx_n)\|
\]

\[
= \|(1 - \alpha_n)(Jx_n - JSx_n)\|
\]

\[
= (1 - \alpha_n)\|Jx_n - JSx_n\|.
\]

Since \( \lim_{n \to \infty} \|Jx_n - JSx_n\| = 0 \), preceding equality implies that

\[
\lim_{n \to \infty} \|Jx_n - Jz_n\| = 0,
\]

and hence

\[
\lim_{n \to \infty} \|x_n - z_n\| = 0. \tag{3.3.9}
\]

It follows from (3.3.8), (3.3.9) and the inequality

\[
\|z_n - Tz_n\| \leq \|z_n - x_n\| + \|x_n - Tz_n\|,
\]

that

\[
\lim_{n \to \infty} \|z_n - Tz_n\| = 0. \tag{3.3.10}
\]

Since \( x_n \to \hat{x} \), it follows from (3.3.7), (3.3.8) and (3.3.10) that \( \hat{x} \) is a fixed points of \( S \)
and \( T \), i.e., \( \hat{x} \in \text{Fix}(T) \cap \text{Fix}(S) \).

On the same lines of proof of Theorem 3.2.1(v), we can easily prove that \( \hat{x} \in \text{Sol}(\text{GVEP}(3.2.3)) \). Then \( \hat{x} \in \Gamma \).

Finally, we prove that \( \hat{x} = \prod_{\Gamma} x \). By taking the limit in (3.3.3), we have

\[
\langle \hat{x} - z, Jx - J\hat{x} \rangle \geq 0, \quad \forall z \in \Gamma.
\]

Further, in view of Lemma 1.2.8, we see that \( \hat{x} = \prod_{\Gamma} x \). This completes the proof. \( \square \)

Now, we prove the weak convergence theorem for approximating a common solution for GVEP (3.2.3) and the fixed point problems of two relatively nonexpansive mappings. First, we prove the following proposition:

**Proposition 3.3.1.** Let \( E \) be a uniformly smooth and uniformly convex Banach space and let \( C \) be a nonempty compact and convex subset of \( E \). Assume that \( P \) is a pointed, proper, closed and convex cone of a real Hausdorff topological space \( Y \) with \( \text{int}P \neq \emptyset \). Let \( F : C \times C \to Y \) and \( \psi : C \to Y \) satisfy Assumption 3.2.1 and let \( S, T \) be relatively nonexpansive mappings from \( C \) into itself such that \( \Gamma \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated by the following scheme:

\[
\begin{align*}
z_1 & \in E, \\
x_n & \in C \text{ such that } F(x_n, y) + \psi(y) - \psi(x_n) + \frac{e}{r}(y - x_n, Jx_n - Jz_n) \in P, \quad \forall y \in C, \\
y_n & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\
z_{n+1} & = J^{-1}(\delta_n Jx_n + (1 - \delta_n)JTy_n),
\end{align*}
\]

for every \( n \in \mathbb{N} \), where \( e \in \text{int}P \), \( J \) is the normalized duality mapping on \( E \) with inverse \( J^{-1} \), and \( r \in [a, \infty) \) for some \( a > 0 \). Assume that \( \{\alpha_n\} \) and \( \{\delta_n\} \) are sequences in \([0,1]\) satisfying the conditions (i)-(ii) of Theorem 3.3.1. Then \( \prod_{\Gamma} x_n \) converges strongly to \( z \in \Gamma \).
Proof. Let \( u \in \Gamma \). Since \( x_n = T_r z_n \) and \( T_r, S, T \) are relatively nonexpansive, we have

\[
\begin{align*}
\phi(u, x_{n+1}) &= \phi(u, T_r z_{n+1}) \\
&\leq \phi(u, z_{n+1}) \\
&\leq \phi(u, J^{-1}(\delta_n J x_n + (1 - \delta_n) J Ty_n)) \\
&= \|u\|^2 - 2\langle u, \delta_n J x_n + (1 - \delta_n) J Ty_n \rangle + \|\delta_n J x_n + (1 - \delta_n) J Ty_n\|^2 \\
&\leq \|u\|^2 - 2\delta_n \langle u, J x_n \rangle - 2(1 - \delta_n) \langle u, JTy_n \rangle \\
&\quad + \delta_n \|x_n\|^2 + (1 - \delta_n) \|Ty_n\|^2 \\
&\leq \delta_n \phi(u, x_n) + (1 - \delta_n) \phi(u, Ty_n) \\
&\leq \delta_n \phi(u, x_n) + (1 - \delta_n) \phi(u, y_n),
\end{align*}
\]

(3.3.11)

and

\[
\begin{align*}
\phi(u, y_n) &= \phi(u, J^{-1}(\alpha_n J x_n + (1 - \alpha_n) JSx_n)) \\
&= \|u\|^2 - 2\langle u, \alpha_n J x_n + (1 - \alpha_n) J Sx_n \rangle + \|\alpha_n J x_n + (1 - \alpha_n) J Sx_n\|^2 \\
&\leq \|u\|^2 - 2\alpha_n \langle u, J x_n \rangle - 2(1 - \alpha_n) \langle u, JSx_n \rangle + \alpha_n \|x_n\|^2 + (1 - \alpha_n) \|Sx_n\|^2 \\
&\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, Sx_n) \\
&\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, x_n) \\
&\leq \phi(u, x_n).
\end{align*}
\]

(3.3.12)

Using (3.3.12) in (3.3.11), we have

\[
\begin{align*}
\phi(u, x_{n+1}) &\leq \delta_n \phi(u, x_n) + (1 - \delta_n) \phi(u, x_n) \\
\phi(u, x_{n+1}) &\leq \delta_n \phi(u, x_n) + (1 - \delta_n) \phi(u, x_n) \\
&\leq \phi(u, x_n).
\end{align*}
\]

(3.3.13)

Therefore, \( \lim_{n \to \infty} \{\phi(u, x_n)\} \) exists and hence \( \{\phi(u, x_n)\} \) is bounded. This implies that \( \{x_n\} \) and \( \{Sx_n\} \) are bounded. Further, it follows from (3.3.12) that \( \{\phi(u, y_n)\} \) is also bounded and hence \( \{y_n\} \) and \( \{Ty_n\} \) are bounded. Define \( w_n = \prod_{\Gamma} x_n \), for every \( n \in \mathbb{N} \).
Then, from $w_n \in \Gamma$ and (3.3.13), we have
\[
\phi(w_n, x_{n+1}) \leq \phi(w_n, x_n) \tag{3.3.14}
\]

Since $\prod_{\Gamma}$ is the generalized projection, from Lemma 1.2.8, we have
\[
\phi(w_{n+1}, x_{n+1}) = \phi(\prod_{\Gamma} x_{n+1}, x_{n+1}) \\
\leq \phi(w_n, x_{n+1}) - \phi(w_n, \prod_{\Gamma} x_{n+1}) \\
= \phi(w_n, x_{n+1}) - \phi(w_n, w_{n+1}) \\
\leq \phi(w_n, x_{n+1}) \tag{3.3.15}
\]

Hence, from (3.3.14), we have
\[
\phi(w_{n+1}, x_{n+1}) \leq \phi(w_n, x_n).
\]

Therefore, $\{\phi(w_n, x_n)\}$ is a convergent sequence. We also have from (3.3.14) that, for all $m \in \mathbb{N}$,
\[
\phi(w_n, x_{n+m}) \leq \phi(w_n, x_n).
\]

From $w_{n+m} = \prod_{\Gamma} x_{n+m}$ and Lemma 1.2.8, we have
\[
\phi(w_n, w_{n+m}) + \phi(w_{n+m}, x_{n+m}) \leq \phi(w_n, x_{n+m}) \leq \phi(w_n, x_n)
\]
and hence
\[
\phi(w_n, w_{n+m}) \leq \phi(w_n, x_n) - \phi(w_{n+m}, x_{n+m}).
\]

Let $r = \sup_{n \in \mathbb{N}} \|w_n\|$. From Lemma 1.2.13, there exists a continuous, strictly increasing, and convex function $g$ with $g(0) = 0$ such that $g(\|x - y\|) \leq \phi(x, y)$ for $x, y \in B_r$. So, we have
\[
g(\|w_n - w_{n+m}\|) \leq \phi(w_n, w_{n+m}) \leq \phi(w_n, x_n) - \phi(w_{n+m}, x_{n+m}).
\]
Since \( \{ \phi(w_n, x_n) \} \) is a convergent sequence, from the property of \( g \) we have that \( \{ w_n \} \) is a Cauchy sequence. Since \( \Gamma \) is closed, \( \{ w_n \} \) converges strongly to \( z \in \Gamma \). This completes the proof.

Now, we are able to prove the following weak convergence theorem.

**Theorem 3.3.2.** Let \( E \) be a uniformly smooth and uniformly convex Banach space and let \( C \) be a nonempty compact and convex subset of \( E \). Assume that \( P \) is a pointed, proper, closed and convex cone of a real Hausdorff topological space \( Y \) with \( \text{int} P \neq \emptyset \). Let \( F : C \times C \to Y \) and \( \psi : C \to Y \) satisfy Assumption 3.2.1 and let \( S, T \) be relatively nonexpansive mappings from \( C \) into itself such that \( \Gamma \neq \emptyset \). Let \( \{ x_n \} \) be a sequence generated by the scheme:

\[
\begin{align*}
z_1 &\in E, \\
x_n &\in C \text{ such that } F(x_n, y) + \psi(y) - \psi(x_n) + \frac{e}{r} \langle y - x_n, Jx_n - Jz_n \rangle \in P, \quad \forall y \in C, \\
y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSx_n), \\
z_{n+1} &= J^{-1}(\delta_n Jx_n + (1 - \delta_n) JTy_n),
\end{align*}
\]

for every \( n \in \mathbb{N} \), where \( e \in \text{int} P \), \( J \) is the normalized duality mapping on \( E \) with inverse \( J^{-1} \), and \( r \in [a, \infty) \), for some \( a > 0 \). Assume that \( \{ \alpha_n \} \) and \( \{ \delta_n \} \) are sequences in \([0,1]\) satisfying the conditions (i)-(ii) of Theorem 3.3.1. If \( J \) is weakly sequentially continuous, then \( x_n \) converges weakly to \( z \in \Gamma \), where \( z = \lim_{n \to \infty} \prod_{\Gamma} x_n \).

**Proof.** As in the proof of Proposition 3.3.1, we have that \( \{ x_n \}, \{ y_n \}, \{ Sx_n \}, \) and \( \{ Ty_n \} \) are bounded sequences. Let \( r = \sup_{n \in \mathbb{N}} \{ \| x_n \|, \| y_n \|, \| Sx_n \|, \| Ty_n \| \} \). Let \( u \in \Gamma \). Since \( x_n = T_{\alpha_n} z_n \) and \( T_{\alpha_n}, S, T \) are relatively nonexpansive, using Lemma 1.2.11, we have

\[
\begin{align*}
\phi(u, y_n) &= \phi(u, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSx_n)) \\
&= \| u \|^2 - 2\langle u, \alpha_n Jx_n + (1 - \alpha_n) JSx_n \rangle + \| \alpha_n Jx_n + (1 - \alpha_n) JSx_n \|^2 \\
&\leq \| u \|^2 - 2\alpha_n \langle u, Jx_n \rangle - 2(1 - \alpha_n)\langle u, JSx_n \rangle + \alpha_n \| x_n \|^2
\end{align*}
\]
\[(1 - \alpha_n)\|Sx_n\|^2 - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|)\]
\[\leq \alpha_n\phi(u, x_n) + (1 - \alpha_n)\phi(u, Sx_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|)\]
\[\leq \alpha_n\phi(u, x_n) + (1 - \alpha_n)\phi(u, x_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|)\]
\[\leq \phi(u, x_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|). \quad (3.3.16)\]

Using (3.3.16) in (3.3.11), we have

\[\phi(u, x_{n+1}) \leq \delta_n\phi(u, x_n) + (1 - \delta_n)[\phi(u, x_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|)]\]
\[\leq \delta_n\phi(u, x_n) + (1 - \delta_n)\phi(u, x_n) - \alpha_n(1 - \alpha_n)(1 - \delta_n)g(\|Jx_n - JSx_n\|)\]
\[\leq \phi(u, x_n) - \alpha_n(1 - \alpha_n)(1 - \delta_n)g(\|Jx_n - JSx_n\|),\]

or
\[\alpha_n(1 - \alpha_n)(1 - \delta_n)g(\|Jx_n - JSx_n\|) \leq \phi(u, x_n) - \phi(u, x_{n+1}). \quad (3.3.17)\]

Since \(\{\phi(u, x_n)\}\) is convergent and using conditions (i)-(ii) in (3.3.17), we have

\[\lim_{n \to \infty} g(\|Jx_n - JSx_n\|) = 0.\]

From the property of \(g\), we have

\[\lim_{n \to \infty} \|Jx_n - JSx_n\| = 0.\]

Since \(J^{-1}\) is uniformly norm-to-norm continuous on bounded sets, we have

\[\lim_{n \to \infty} \|x_n - Sx_n\| = 0. \quad (3.3.18)\]

Next, we have

\[\phi(u, x_{n+1}) \leq \phi(u, z_{n+1})\]
\[\leq \phi(u, J^{-1}(\delta_n Jx_n + (1 - \delta_n)JTy_n))\]
\[= \|u\|^2 - 2\langle u, \delta_n Jx_n + (1 - \delta_n)JTy_n \rangle + \|\delta_n Jx_n + (1 - \delta_n)JTy_n\|^2\]
\[ \leq ||u||^2 - 2\delta_n \langle u, Jx_n \rangle - 2(1 - \delta_n)\langle u, JT y_n \rangle + \delta_n ||x_n||^2 \\
+ (1 - \delta_n)||Ty_n||^2 - \delta_n(1 - \delta_n)g(||Jx_n - JT y_n||) \]
\[ \leq \delta_n \phi(u, x_n) + (1 - \delta_n)\phi(u, Ty_n) - \delta_n(1 - \delta_n)g(||Jx_n - JT y_n||) \]
\[ \leq \delta_n \phi(u, x_n) + (1 - \delta_n)\phi(u, y_n) - \delta_n(1 - \delta_n)g(||Jx_n - JT y_n||) \]
\[ \leq \delta_n \phi(u, x_n) + (1 - \delta_n)\phi(u, x_n) - \delta_n(1 - \delta_n)g(||Jx_n - JT y_n||) \]
\[ \leq \phi(u, x_n) - \delta_n(1 - \delta_n)g(||Jx_n - JT y_n||), \]

or
\[ \delta_n(1 - \delta_n)g(||Jx_n - JT y_n||) \leq \phi(u, x_n) - \phi(u, x_{n+1}). \quad (3.3.19) \]

Since \( \{\phi(u, x_n)\} \) is convergent and using condition (i) in (3.3.19), we have
\[ \lim_{n \to \infty} g(||Jx_n - JT y_n||) = 0. \]

From the property of \( g \), we have
\[ \lim_{n \to \infty} ||Jx_n - JT y_n|| = 0, \]

and hence
\[ \lim_{n \to \infty} ||x_n - Ty_n|| = 0. \quad (3.3.20) \]

Now,
\[ ||Jx_n - Jy_n|| = ||Jx_n - (\alpha_n Jx_n + (1 - \alpha_n)JS x_n)|| \]
\[ = ||(1 - \alpha_n)(Jx_n - JS x_n)|| \]
\[ = (1 - \alpha_n)||Jx_n - JS x_n||, \]
which implies
\[ \lim_{n \to \infty} ||Jx_n - Jy_n|| = 0, \]

and hence
\[ \lim_{n \to \infty} ||x_n - y_n|| = 0. \quad (3.3.21) \]
It follows from (3.3.20), (3.3.21) and the inequality \( \| y_n - Ty_n \| \leq \| y_n - x_n \| + \| x_n - Ty_n \| \)
that
\[
\lim_{n \to \infty} \| y_n - Ty_n \| = 0.
\] (3.3.22)

Since \( \{ x_n \} \) and \( \{ y_n \} \) are bounded and \( \lim_{n \to \infty} \| x_n - y_n \| = 0 \), there exist subsequences \( \{ x_{n_k} \} \) and \( \{ y_{n_k} \} \) of \( \{ x_n \} \) and \( \{ y_n \} \), respectively, such that \( x_{n_k} \rightharpoonup \hat{x} \in C \) and \( y_{n_k} \rightharpoonup \hat{x} \in C \).

It follows from (3.3.18) and (3.3.22) that \( \hat{x} \in \text{Fix}(S) \cap \text{Fix}(T) = \text{Fix}(S) \cap \text{Fix}(T) \), i.e., \( \hat{x} \in \text{Fix}(S) \cap \text{Fix}(T) \).

Next, we show that \( \hat{x} \in \text{Sol}(\text{GVEP}(3.1.3)) \). Let \( r = \sup_{n \in \mathbb{N}} \{ \| x_n \|, \| z_n \| \} \). From Lemma 1.2.13, there exists a continuous, strictly increasing and convex function \( g_1 \) with \( g_1(0) = 0 \) such that \( g_1(\| x - y \|) \leq \phi(x, y), \quad \forall x, y \in B_r \).

Since \( x_n = T_r z_n \), we have from Lemma 3.2.1 that, for \( u \in \Gamma \),
\[
g_1(\| x_n - z_n \|) \leq \phi(x_n, z_n) \leq \phi(u, z_n) - \phi(u, x_n) \\
\leq \phi(u, x_{n-1}) - \phi(u, x_n).
\]
Since \( \{ \phi(u, x_n) \} \) converges, we have
\[
\lim_{n \to \infty} g_1(\| x_n - z_n \|) = 0.
\]

From the property of \( g_1 \), we have
\[
\lim_{n \to \infty} \| x_n - z_n \| = 0.
\]

Since \( J \) is uniformly norm-to-norm continuous on bounded sets, we have
\[
\lim_{n \to \infty} \| Jx_n - Jz_n \| = 0.
\]
From \( r \geq a \), we have
\[
\lim_{n \to \infty} \frac{\| Jx_n - Jz_n \|}{r} = 0.
\]
By \( x_n = T_r z_n \), we have

\[
F(T_r z_n, y) + \psi(y) - \psi(T_r z_n) + \frac{e}{r} \langle y - T_r z_n, JT_r z_n - J z_n \rangle \in P, \quad \forall y \in C
\]

\[
0 \in F(y, T_r z_n) - \psi(y) + \psi(T_r z_n) + \frac{e}{r} \langle y - T_r z_n, JT_r z_n - J z_n \rangle + P, \quad \forall y \in C.
\]

Replacing \( n \) by \( n_i \), we have

\[
0 \in F(y, T_r(z_{n_i})) - \psi(y) - \psi(T_r z_{n_i}) + \frac{e}{r} \langle y - T_r z_{n_i}, JT_r z_{n_i} - J z_{n_i} \rangle + P, \quad \forall y \in C.
\]

Let \( y_t = (1 - t)\hat{x} + ty, \ \forall t \in (0, 1] \). Since \( y \in C \) and \( \hat{x} \in C \), we get \( y_t \in C \) and thus

\[
0 \in F(y_t, T_r(z_{n_i})) - (\psi(y_t) - \psi(T_r(z_{n_i}))) - \frac{e}{r} \langle y_t - T_r z_{n_i}, JT_r z_{n_i} - J z_{n_i} \rangle + P
\]

\[
= F(y_t, T_r(z_{n_i})) - (\psi(y_t) - \psi(T_r(z_{n_i}))) - e \langle y_t - T_r z_{n_i}, \frac{JT_r z_{n_i} - J z_{n_i}}{r} \rangle + P.
\]

Since \( \|JT_r z_{n_i} - J z_{n_i}\| \to 0 \) and using the property of \( F \), we have

\[
0 \in F(y_t, \hat{x}) - (\psi(y_t) - \psi(\hat{x})) + P. \quad (3.3.23)
\]

It follows from Assumption 3.2.1 (i), (iv), (vi) that

\[
tF(y_t, y) + (1 - t)F(y_t, \hat{x}) + t\psi(y) + (1 - t)\psi(\hat{x}) - \psi(y_t)
\]

\[
\in F(y_t, y_t) + \psi(y_t) - \psi(y_t) + P
\]

\[
\in P,
\]

or

\[
-t[F(y_t, y) + \psi(y) - \psi(y_t) - (1 - t)[F(y_t, \hat{x}) + \psi(\hat{x}) - \psi(y_t)] \in -P. \quad (3.3.24)
\]

Using (3.3.23) in (3.3.24), we have

\[
-t[F(y_t, y) + \psi(y) - \psi(y_t)] \in -P
\]

\[
F(y_t, y) + \psi(y) - \psi(y_t) \in P.
\]
Letting $t \to 0$, we obtain

$$F(\hat{x}, y) + \psi(y) - \psi(\hat{x}) \in P, \quad \forall y \in C.$$ 

Therefore $\hat{x} \in \text{Sol}(\text{GVEP}(3.1.3))$. As in the proof of Theorem 3.3.1, we have $\hat{x} \in \text{Sol}(\text{GVEP}(3.1.3))$. Hence $\hat{x} \in \Gamma$. Let $w_n = \prod_{\Gamma} x_n$. From Lemma 1.2.8, and $\hat{x} \in \Gamma$, we have

$$\langle w_{n_k} - \hat{x}, Jx_{n_k} - Jw_{n_k} \rangle \geq 0.$$ 

It follows from Proposition 3.1 that $\{w_n\}$ converges strongly to $z \in \Gamma$. Since $J$ is weakly sequentially continuous, we have

$$\langle z - \hat{x}, J\hat{x} - Jz \rangle \geq 0 \text{ as } k \to \infty.$$ 

On the other hand, since $J$ is monotone, we have

$$\langle z - \hat{x}, J\hat{x} - Jz \rangle \leq 0.$$ 

Hence, we have

$$\langle z - \hat{x}, J\hat{x} - Jz \rangle = 0.$$ 

From the strict convexity of $E$, we have $z = \hat{x}$. Therefore, $\{x_n\}$ converges weakly to $\hat{x} \in \Gamma$, where $\hat{x} = \lim_{n \to \infty} \prod_{\Gamma} x_n$. This completes the proof. 

3.4 Consequences

Now, we have the following consequences:

**Corollary 3.4.1.** Let $E$ be a uniformly smooth and uniformly convex Banach space and let $C$ be a nonempty compact and convex subset of $E$. Assume that $P$ is a pointed, proper, closed and convex cone of a real Hausdorff topological space $Y$ with $\text{int}P \neq \emptyset$. Let the mapping $F : C \times C \to Y$ and satisfy Assumption 3.2.1 and let $S, T$ be relatively nonexpansive mappings from $C$ into itself such that $\Gamma := \text{Fix}(T) \cap \text{Fix}(S) \cap$
Sol(SVEP(3.1.4)) ≠ ∅. Let \( \{x_n\} \) be a sequence generated by the scheme:

\[
\begin{align*}
x_0 &= x \in C, \\
y_n &= J^{-1}(\delta_n Jx_n + (1 - \delta_n)JTz_n), \\
z_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\
u_n &\in C \text{ such that } F(u_n, y) + \frac{e}{r}(y - u_n, Ju_n - Jy) \in P, \quad \forall y \in C, \\
H_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \prod_{H_n \cap W_n} x, \quad \text{for every } n \in N \cup \{0\},
\end{align*}
\]

where \( e \in \text{int}P, J \) is the normalized duality mapping on \( E \) with inverse \( J^{-1} \), and \( r \in [a, \infty) \) for some \( a > 0 \). Assume that \( \{\alpha_n\} \) and \( \{\delta_n\} \) are sequences in \([0,1]\) satisfying the conditions (i)-(ii) of Theorem 3.3.1. Then \( \{x_n\} \) converges strongly to \( \prod_{\Gamma} x \), where \( \prod_{\Gamma} x \) is the generalized projection of \( E \) onto \( \Gamma \).

**Proof.** The proof follows by taking \( \psi = 0 \) in Theorem 3.3.1. \qed

**Corollary 3.4.2.** Let \( E \) be a uniformly smooth and uniformly convex Banach space and let \( C \) be a nonempty compact and convex subset of \( E \). Assume that \( P \) is a pointed, proper, closed and convex cone of a real Hausdorff topological space \( Y \) with \( \text{int}P \neq \emptyset \). Let \( F : C \times C \to Y \) be a mapping satisfy Assumption 3.2.1 and let \( S, T \) be relatively nonexpansive mappings from \( C \) into itself such that \( \Gamma := \text{Fix}(T) \cap \text{Fix}(S) \cap \text{Sol}(\text{SVEP}(3.1.4)) \neq \emptyset \).

Let \( \{x_n\} \) be a sequence generated by the scheme:

\[
\begin{align*}
z_1 &\in E, \\
x_n &\in C \text{ such that } F(x_n, y) + \frac{e}{r}(y - x_n, Jx_n - Jz_n) \in P, \quad \forall y \in C, \\
y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\
z_{n+1} &= J^{-1}(\delta_n Jx_n + (1 - \delta_n)JTy_n),
\end{align*}
\]

for every \( n \in N \), where \( e \in \text{int}P, J \) is the normalized duality mapping on \( E \) with inverse \( J^{-1} \), and \( r \in [a, \infty) \), for some \( a > 0 \). Assume that \( \{\alpha_n\} \) and \( \{\delta_n\} \) are sequences in \([0,1]\)
satisfying the conditions (i)-(ii) of Theorem 3.3.1. If \( J \) is weakly sequentially continuous, then \( x_n \) converges weakly to \( z \in \Gamma \), where \( z = \lim_{n \to \infty} \prod_{\Gamma} x_n \).

**Proof.** The proof follows by taking \( \psi = 0 \) in Theorem 3.3.2. \( \square \)

The following Corollary is due to Takahashi and Zembayashi [163].

**Corollary 3.4.3.** Let \( E \) be a uniformly smooth and uniformly convex Banach space and let \( C \) be a nonempty compact and convex subset of \( E \). Let \( F : C \times C \to \mathbb{R} \) be a mapping satisfy Assumption 3.2.1 and let \( S \) be relatively nonexpansive mappings from \( C \) into itself such that \( \Gamma := \text{Fix}(S) \cap \text{Sol(EP(1.3.11))} \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated by the scheme:

\[
\begin{align*}
x_0 &= x \in C, \\
y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\
u_n \in C \text{ such that } F(u_n, y) + \frac{1}{r}(y - u_n, Ju_n - Jy_n) \geq 0, \quad \forall y \in C, \\
H_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} &= \prod_{H_n \cap W_n} x, \quad \text{for every } n \in N \cup \{0\},
\end{align*}
\]

where \( J \) is the normalized duality mapping on \( E \) with inverse \( J^{-1} \), and \( r \in [a, \infty) \) for some \( a > 0 \). Assume that \( \{\alpha_n\} \) be sequence in \([0,1]\) satisfying the condition (ii) of Theorem 3.3.1. Then \( \{x_n\} \) converges strongly to \( \prod_{\Gamma} x \), where \( \prod_{\Gamma} x \) is the generalized projection of \( E \) onto \( \Gamma \).

**Proof.** The proof follows by taking \( Y = \mathbb{R}, \ P = [0, \infty), \ T = I \), identity mapping, \( \delta_n = 0 \) and \( \psi = 0 \) in Theorem 3.3.1. \( \square \)

**Remark 3.4.1.** The method presented in this paper can be used to extend the results of Shan and Huang [148] and Petrot et al. [136].