Chapter 1

Preliminaries

1.1 Introduction

It is well known that the theory of variational inequalities and the theory of equilibrium problems provide powerful techniques for studying a large number of problems arising in optimization, transportation, economics equilibrium and other problems of practical interest.

The theory of variational inequalities was initiated independently by Fichera [57] and Stampacchia [155] in the early 1960’s to study the boundary value problems arising in the elasticity and potential theory, respectively. Since then variational inequalities have been extended in various directions using novel and innovative techniques. It is well-known that the variational inequality theory has played a fundamental and important role in the study of a wide range of problems arising in physics, mechanics, elasticity, optimization, control theory, management science, operations research, economics, transportation and other branches of mathematical and engineering sciences, see for example Baiocchi and Capelo [7], Bensoussan and Lions [10], Cottle et al. [46], Crank [47], Duvaut and Lions [52], Ekeland and Temam [53], Giannessi and Maugeri [61], Glowinski et al. [64], Kikuchi and Oden [102], Kinderlehrer and Stampacchia [103] and Konnov [105]. An important generalization of variational inequality is a vector variational inequality introduced by Giannessi [59]. Vector variational inequalities have many applications in transportation and other areas, see for example Giannessi [60].

Equilibrium problems were initially introduced by Zuhovichki, Poljak and Primak [177],
Fan [54, 55], perhaps motivated by minimax problems appearing in economic equilibrium. A more general result than that in [55] was established by Brézis, Nirenberg and Stampacchia [16]. But, in 1994, the terminology of equilibrium problem was adopted by Blum and Oettli [14]. Since then various generalizations of equilibrium problem considered by Blum and Oettli [14] have been introduced and studied by many authors. It has been shown that the theory of equilibrium problem provides a natural, novel and unified framework for several problems arising in nonlinear analysis, optimization, economics, finance, game theory, physics and engineering. The equilibrium problem includes many mathematical problems as particular cases for examples, mathematical programming problems, complementary problems, variational inequality problems, saddle point problems, Nash equilibrium problems in noncooperative games, minimax inequality problems, minimization problems and fixed point problems, see [14, 50, 62, 100, 112, 119].

Vector equilibrium problem, an important generalization of equilibrium problem, is introduced and studied independently by Kazmi [82, 84], Konnov [104] and Bianchi et al. [13]. Vector equilibrium problems include vector optimization problems, vector variational inequalities as a special case, and have deep connections with some important areas of nonlinear analysis.

The remaining part of this chapter is organized as follows:

In Section 1.2, we review notations, known definitions and results which are essential for the presentation of the results in subsequent chapters.

In Section 1.3, we give brief survey of some classes of variational inequalities and equilibrium problems. Further, we give brief survey of some iterative methods for solving fixed point problems, variational inequalities and equilibrium problems.

1.2 Some definitions and results of nonlinear functional analysis

Throughout the thesis unless otherwise stated, $H$ denotes a real Hilbert space with its dual space $H^*$. We denote the induced norm and inner product of $H$ by $\| \cdot \|$ and
\((\cdot,\cdot)\), respectively. Let \(C\) be a nonempty closed and convex subset of \(H\). Let \(\{x_n\}\) be a sequence in \(H\), then \(x_n \to x\) (respectively, \(x_n \rightharpoonup x\)) will denote strong (respectively, weak) convergence of the sequence \(\{x_n\}\) and \(\mathbb{R}\) denotes the set of all real numbers.

**Definition 1.2.1.** [35] Let \((X, \| \cdot \|)\) be a normed linear space. A mapping \(T : X \to X\) is said to be:

(i) continuous at a point \(x_0 \in X\), if for each \(\epsilon > 0\) there is a real number \(\delta > 0\) such that

\[ x \in X, \| x - x_0 \| < \delta \Rightarrow \| Tx - Tx_0 \| \leq \epsilon; \]

(ii) Lipschitz continuous if there exists a real constant \(k > 0\) such that

\[ \| Tx - Ty \| \leq k \| x - y \|, \quad \forall x, y \in X; \]

(iii) contraction if it is Lipschitz continuous with \(k \in (0, 1)\);

(iv) nonexpansive if it is Lipschitz continuous with \(k = 1\).

**Definition 1.2.2.** [151] Let \(T : C \to C\) be a mapping. A point \(x_0\) is called a fixed point of \(T\), if \(Tx_0 = x_0\), i.e., a point which remains invariant under the transformation \(T\).

The fixed point problem (in short, FPP) for a mapping \(T : C \to C\) is to find \(x \in C\) such that

\[ x = Tx. \tag{1.2.1} \]

We denote \(\text{Fix}(T)\), the set of solutions of FPP(1.2.1).

**Remark 1.2.1.** It is well known that every nonexpansive operator \(T : H \to H\) satisfies, for all \((x, y) \in H \times H\), the inequality

\[ \langle (x - Tx) - (y - Ty), Ty - Tx \rangle \leq \frac{1}{2} \| (Tx - x) - (Ty - y) \|^2, \tag{1.2.2} \]

and therefore, we get, for all \((x, y) \in H \times \text{Fix}(T)\),

\[ \langle x - Tx, y - Tx \rangle \leq \frac{1}{2} \| Tx - x \|^2, \tag{1.2.3} \]
see, e.g., [48], Theorem 2.1.

**Definition 1.2.3.** [168] A mapping \( T : C \to C \) is said to be weakly contractive if

\[
\|T x - T y\| \leq \|x - y\| - \psi(\|x - y\|), \quad \forall x, y \in C,
\]

where \( \psi : [0, +\infty) \to [0, +\infty) \) is a continuous and strictly increasing function such that \( \psi \) is positive on \((0, +\infty)\) and \( \psi(0) = 0 \). If \( \psi(t) = (1 - k)t \) then \( T \) is \( k \)-Lipschitz continuous.

**Definition 1.2.4.** [8] Let \( T : H \to H \) be a nonlinear mapping. Then \( T \) is called:

(i) **monotone**, if

\[
\langle T x - T y, x - y \rangle \geq 0, \quad \forall x, y \in H;
\]

(ii) **\( \alpha \)-strongly monotone**, if there exists a constant \( \alpha > 0 \) such that

\[
\langle T x - T y, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in H;
\]

(iii) **\( \beta \)-inverse strongly monotone** (in short, \( \beta \)-ism), if there exists a constant \( \beta > 0 \) such that

\[
\langle T x - T y, x - y \rangle \geq \beta \|T x - T y\|^2, \quad \forall x, y \in H;
\]

(iv) **firmly nonexpansive**, if it is \( \beta \)-inverse strongly monotone with \( \beta = 1 \).

It is easy to observe that every \( \beta \)-inverse strongly monotone mapping \( T \) is monotone and \( \frac{1}{\beta} \)-Lipschitz continuous.

**Definition 1.2.5.** [8] For every point \( x \in H \), there exists a unique nearest point in \( C \) denoted by \( P_C x \) such that

\[
\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C,
\]  

(1.2.4)

where \( P_C \) is called the **metric projection of \( H \) onto \( C \)**.
Remark 1.2.2. [8] It is well known that $P_C$ is nonexpansive mapping and satisfies

$$\langle x - y, P_Cx - P_Cy \rangle \geq \|P_Cx - P_Cy\|^2, \quad \forall x, y \in H. \quad (1.2.5)$$

Moreover, $P_Cx$ is characterized by the fact $P_Cx \in C$ and

$$\langle x - P_Cx, y - P_Cx \rangle \leq 0, \quad (1.2.6)$$

and

$$\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2, \quad \forall x \in H, \ y \in C. \quad (1.2.7)$$

Definition 1.2.6. [8] A multi-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$, $u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v \rangle \geq 0.$$

Definition 1.2.7. [8] A multi-valued monotone mapping $M : H \to 2^H$ is maximal if the Graph$(M)$, the graph of $M$, is not properly contained in the graph of any other monotone mapping.

Remark 1.2.3. It is known that a multi-valued monotone mapping $M$ is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v \rangle \geq 0$, for every $(y, v) \in \text{Graph}(M)$ implies that $u \in Mx$.

Lemma 1.2.1. [17] (Demiclosed principle) Let $H$ be a real Hilbert space, $C$ be a closed and convex subset of $H$ and let $T : C \to H$ be a nonexpansive mapping. Then $I - T$ (I is the identity operator on $H$) is demiclosed at $y \in H$, i.e., for any sequence $\{x_n\}$ in $C$ such that $x_n \to \bar{x} \in C$ and $(I - T)x_n \to y$, we have $(I - T)\bar{x} = y$.

Definition 1.2.8. [8] A mapping $T : H \to H$ is said to be averaged if and only if (in short, iff) it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$T = (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive and $I$ is the identity operator on $H$. 
We note that the firmly nonexpansive mappings (in particular, projections on nonempty closed and convex subsets and resolvent operators of maximal monotone operators) are averaged. Obviously, averaged mapping is a nonexpansive mapping.

The following are some key properties of averaged mappings.

**Lemma 1.2.2.** [123]

(i) If \( T = (1 - \alpha)xS + \alpha V \), where \( S : H \to H \) is averaged, \( V : H \to H \) is nonexpansive and \( \alpha \in (0, 1) \), then \( T \) is averaged;

(ii) The composite of finitely many averaged mappings is averaged;

(iii) If the mappings \( \{T_i\} \) are averaged and have a common fixed point, then

\[
\bigcap_{i=1}^{N} \text{Fix}(T_i) = \text{Fix}(T_1T_2...T_N);
\]

(iv) If \( T \) is \( \tau \)-ism, then for \( \gamma > 0 \), \( \gamma T \) is \( \frac{x}{\gamma} \)-ism;

(v) \( T \) is averaged if and only if, its complement \( I - T \) is \( \tau \)-ism for some \( \tau > \frac{1}{2} \).

**Definition 1.2.9.** [149] A family \( \mathcal{S} := \{T(s) : 0 \leq s < \infty\} \) of mappings from \( C \) into itself is called nonexpansive semigroup on \( C \) if it satisfies the following conditions:

(i) \( T(0)x = x \) for all \( x \in C \);

(ii) \( T(s)T(t) = T(s + t) \) for all \( s, t \geq 0 \);

(iii) \( \|T(s)x - T(s)y\| \leq \|x - y\| \) for all \( x, y \in C \) and \( s \geq 0 \);

(iv) for all \( x \in C \), \( s \mapsto T(s)x \) is continuous.

The set of all the common fixed points of a family \( \mathcal{S} \) is denoted by \( \text{Fix}(\mathcal{S}) \), i.e.,

\[
\text{Fix}(\mathcal{S}) := \{x \in C : T(s)x = x, 0 \leq s < \infty\} = \bigcap_{0 \leq s < \infty} \text{Fix}(T(s)),
\]  

(1.2.8)

where \( \text{Fix}(T(s)) \) is the set of fixed points of \( T(s) \). It is well known that \( \text{Fix}(\mathcal{S}) \) is closed and convex.
Lemma 1.2.3. [149] Let $C$ be a nonempty, bounded, closed and convex subset of a Hilbert space $H$ and let $\mathcal{T} := \{ T(s) : 0 \leq s < \infty \}$ be a nonexpansive semigroup on $C$. Then, for $t > 0$ and for every $0 \leq h < \infty$,\[ \lim_{t \to \infty} \sup_{x \in C} \| \frac{1}{t} \int_0^t T(s)x ds - T(h) \left( \frac{1}{t} \int_0^t T(s)x ds \right) \| = 0. \]

Definition 1.2.10. [115] An operator $B : H \to H$ is said to be strongly positive, if there exists a constant $\bar{\gamma} > 0$ such that\[ \langle Bx, x \rangle \geq \bar{\gamma} \| x \|^2, \quad \forall x \in H. \]

Lemma 1.2.4. [115] Assume that $B$ is a strongly positive self-adjoint bounded linear operator on a Hilbert space $H$ with constant $\bar{\gamma} > 0$ and $0 < \rho \leq \| B \|^{-1}$. Then $\| I - \rho B \| \leq 1 - \rho \bar{\gamma}$.

Definition 1.2.11. [8] Let $C$ be a nonempty subset of a Hilbert space $H$ and let $\{ x_n \}$ be a sequence in $H$. Then $\{ x_n \}$ is Fejer monotone with respect to $C$ if\[ \| x_{n+1} - x \| \leq \| x_n - x \|, \quad \forall x \in C. \]

Lemma 1.2.5. [65,133] In real Hilbert space $H$, the following hold:

(i) \[ \| x + y \|^2 \leq \| x \|^2 + 2 \langle y, x + y \rangle, \quad \forall x, y \in H; \quad (1.2.9) \]

(ii) \[ \| \lambda x + (1 - \lambda) y \|^2 = \lambda \| x \|^2 + (1 - \lambda) \| y \|^2 - \lambda(1 - \lambda) \| x - y \|^2, \quad (1.2.10) \]

for all $x, y \in H$ and $\lambda \in (0, 1)$;

(iii) (Opial’s condition) For any sequence $\{ x_n \}$ with $x_n \rightharpoonup x$ the inequality\[ \lim_{n \to \infty} \inf \| x_n - x \| < \lim_{n \to \infty} \inf \| x_n - y \|, \quad (1.2.11) \]

holds for every $y \in H$ with $y \neq x$;
(iv) (Kadec-Klee property) If \( \{x_n\} \) be a sequence in \( H \) which satisfies \( x_n \rightharpoonup x \) and \( \|x_n\| \to \|x\| \) as \( n \to \infty \), then \( \|x_n - x\| \to 0 \) as \( n \to \infty \).

**Lemma 1.2.6.** [17] Let \( \{a_n\} \) be a sequence of nonnegative real numbers such that

\[
a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0,
\]

where \( \{\alpha_n\} \) is a sequence in \( (0, 1) \) and \( \{\delta_n\} \) is a sequence in \( \mathbb{R} \) such that

(i) \( \sum_{n=1}^{\infty} \alpha_n = \infty \);

(ii) \( \limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \).

Then \( \lim_{n \to \infty} a_n = 0 \).

**Lemma 1.2.7.** [25] Let \( \{\lambda_n\} \) and \( \{\beta_n\} \) be two non negative real number sequences and let \( \{\alpha_n\} \) be a positive real number sequence satisfying the conditions \( \sum_{n=0}^{\infty} \alpha_n = \infty \) such that either \( \limsup_{n \to \infty} \frac{\beta_n}{\alpha_n} = 0 \) or \( \sum_{n=1}^{\infty} \beta_n < \infty \). Let the recursive inequality

\[
\lambda_{n+1} \leq \lambda_n - \alpha_n \psi(\lambda_n) + \beta_n, \quad n = 0, 1, 2, 3, \ldots,
\]

be given, where \( \psi(\lambda) \) is a continuous and strict increasing function for all \( \lambda \geq 0 \) with \( \psi(0) = 0 \). Then \( \lambda_n \) converges to zero as \( n \to \infty \).

Let \( E \) be a real Banach space with its dual space \( E^* \) and let \( \langle ., . \rangle \) denote the duality pairing between \( E \) and \( E^* \).

**Definition 1.2.12.** [35] The normalized duality mapping \( J : E \to 2^{E^*} \) is defined by

\[
J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad (1.2.12)
\]

for every \( x \in E \).

**Definition 1.2.13.** [35] A Banach space \( E \) is said to be:

(i) strictly convex if \( \frac{\|x + y\|}{2} < 1 \) for \( x, y \in E \) with \( \|x\| = \|y\| = 1 \) and \( x \neq y \);
(ii) uniformly convex if for each \( \epsilon \in (0, 2] \), there exists \( \delta > 0 \) such that \( \frac{\|x + y\|}{2} \leq 1 - \delta \) for \( x, y \in E \) with \( \|x\| = \|y\| = 1 \) and \( \|x - y\| \geq \epsilon \);

(iii) smooth if the limit \( \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \) exists for all \( x, y \in M(E) = \{ z \in E : \|z\| = 1 \} \);

(iv) uniformly smooth if the limit given in (iii) exists uniformly in \( x, y \in M(E) \).

Remark 1.2.4. [35]

(i) It follows from Hahn-Banach theorem that \( J(x) \) is nonempty.

(ii) If \( E \) is smooth, strictly convex and reflexive, then the normalized duality mapping \( J \) is single-valued, one-to-one and onto. The normalized duality mapping \( J \) is said to be weakly sequentially continuous if \( x_n \rightharpoonup x \) implies that \( Jx_n \rightharpoonup Jx \).

(iii) It is well known that if \( E \) is uniformly smooth, then \( J \) is uniformly norm-to-norm continuous on each bounded subset of \( E \). It is also well known that \( E \) is uniformly smooth if and only if \( E^* \) is uniformly convex. It is well known that if \( E \) is a uniformly convex Banach space then \( E \) enjoys the Kadec-Klee property.

Definition 1.2.14. [3] Let \( E \) be a smooth Banach space. The Lyapunov functional \( \phi : E \times E \to \mathbb{R}_+ \) is defined by

\[
\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.
\]

Observe that in a Hilbert space \( H \), equality is reduced to \( \phi(x, y) = \|x - y\|^2 \), \( \forall x, y \in H \). As we all known, if \( C \) is a nonempty closed convex subset of a Hilbert space \( H \) and \( P_C : H \to C \) is the metric projection of \( H \) onto \( C \), then \( P_C \) is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [3] recently introduced a generalized projection operator \( \prod_C \) in a Banach space \( E \) which is an analogue of the metric projection \( P_C \) in Hilbert spaces.
Definition 1.2.15. [3] The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x).$$

Remark 1.2.5. [3]

(i) The existence and uniqueness of the operator $\Pi_C$ follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping $J$ (see, for example, [3]). In Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of $\phi$, that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2(x - z, Jz - Jy),$$

and

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \quad (1.2.13)$$

(ii) If $E$ is a reflexive, strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$.

Let $E$ be a smooth, strictly convex and reflexive Banach space. Let $C$ be a nonempty subset of $E$.

Definition 1.2.16. A mapping $T : C \to C$ is said to be:

(i) asymptotically regular on $C$ if for any bounded subset $K$ of $C$,

$$\lim_{n \to \infty} \sup \{ \|T^{n+1}x - T^nx\| : x \in K \} = 0;$$

(ii) closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n \to \infty} x_n = x_0$ and $\lim_{n \to \infty} Tx_n = y_0$,

then $Tx_0 = y_0$.

Definition 1.2.17. [163] Let $T : C \to C$ be a mapping. A point $p \in C$ is said to be an asymptotic fixed point of $T$ iff $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ so that $\lim_{n \to \infty} \|x_n - T x_n\| = 0$. 

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The set of asymptotic fixed points of $T$ is denoted by $\hat{\text{Fix}}(T)$.

**Definition 1.2.18.** [141,163,176] A mapping $T : C \to C$ is said to be:

(i) relatively nonexpansive iff

\[
\hat{\text{Fix}}(T) = \text{Fix}(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in \text{Fix}(T);
\]

(ii) relatively asymptotically nonexpansive iff

\[
\hat{\text{Fix}}(T) = \text{Fix}(T) \neq \emptyset, \quad \phi(p, T^nx) \leq (1 + \mu_n)\phi(p, x), \quad \forall x \in C, \forall p \in \text{Fix}(T), \forall n \geq 1,
\]

where $\{\mu_n\} \subset [0, \infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$;

(iii) quasi $\phi$-nonexpansive iff

\[
\text{Fix}(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in \text{Fix}(T);
\]

(iv) asymptotically quasi $\phi$-nonexpansive iff there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

\[
\text{Fix}(T) \neq \emptyset, \quad \phi(p, T^nx) \leq (1 + \mu_n)\phi(p, x), \quad \forall x \in C, \forall p \in \text{Fix}(T), \forall n \geq 1;
\]

(v) generalized asymptotically quasi $\phi$-nonexpansive if $\text{Fix}(T) \neq \emptyset$ and there exist two nonnegative sequences $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\{\xi_n\} \subset [0, \infty)$ with $\xi_n \to 0$ as $n \to \infty$ such that

\[
\phi(p, T^nx) \leq (1 + \mu_n)\phi(p, x) + \xi_n, \quad \forall x \in C, \forall p \in \text{Fix}(T), \forall n \geq 1.
\]

**Lemma 1.2.8.** [3] Let $E$ be a smooth, strictly convex and reflexive Banach space, and $C$ be a nonempty closed and convex subset of $E$. Then, the following conclusions hold:

(i) $\phi(x, \prod_C y) + \phi(\prod_C y, y) \leq \phi(x, y)$, $\forall x \in C, y \in E$;
(ii) Let \( x \in E \) and \( z \in C \) then \( z = \prod_{C} x \iff \langle z - y, Jx - Jz \rangle \geq 0, \forall y \in C \).

**Lemma 1.2.9.** [117] Let \( C \) be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space \( E \) and let \( T \) be a relatively nonexpansive mapping from \( C \) into itself. Then \( \text{Fix}(T) \) is closed and convex.

**Lemma 1.2.10.** [77] Let \( E \) be a smooth and uniformly convex Banach space and let \( \{x_n\} \) and \( \{y_n\} \) be sequences in \( E \) such that either \( \{x_n\} \) or \( \{y_n\} \) is bounded. If \( \lim_{n \to \infty} \phi(x_n, y_n) = 0 \) then \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).

**Lemma 1.2.11.** [169] Let \( E \) be a uniformly convex Banach space and let \( r > 0 \). Then there exists a strictly increasing, continuous and convex function \( g : [0, 2r] \to \mathbb{R} \) such that \( g(0) = 0 \) and

\[
\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|),
\]

for all \( x, y \in B_r \) and \( t \in [0, 1] \), where \( B_r = \{z \in E : \|z\| \leq r\} \).

**Lemma 1.2.12.** [38] Let \( E \) be a uniformly convex Banach space and let \( r > 0 \). Then there exists a strictly increasing, continuous and convex function \( g : [0, \infty) \to [0, \infty) \) such that \( g(0) = 0 \) and

\[
\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \mu\gamma g(\|y - z\|),
\]

for all \( x, y, z \in B_r(0) \) and \( \lambda, \mu, \gamma \in [0, 1] \), where \( B_r(0) = \{z \in E : \|z\| \leq r\} \).

**Lemma 1.2.13.** [77] Let \( E \) be a smooth and uniformly convex Banach space and let \( r > 0 \). Then there exists a strictly increasing, continuous and convex function \( g : [0, 2r] \to [0, 2r] \) such that \( g(0) = 0 \) and

\[
g(\|x - y\|) \leq \phi(x, y), \forall x, y \in B_r.
\]

**Lemma 1.2.14.** [65] Let \( C \) be a nonempty closed and convex subset of a strictly convex Banach space \( E \) and let \( T \) be a nonexpansive mapping from \( C \) into itself with \( \text{Fix}(T) \neq \emptyset \). Then \( \text{Fix}(T) \) is closed and convex.
Lemma 1.2.15. [156] (Suzuki Lemma) Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( E \) and \( \{\beta_n\} \) be a sequence in \( [0, 1] \) with \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \). Suppose \( x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n \), for all integers \( n \geq 0 \) and \( \limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then \( \lim_{n \to \infty} \|y_n - x_n\| = 0 \).

Definition 1.2.19. A mapping \( T : E \to E^* \) is demicontinuous at \( x_0 \) if for any sequence \( \{x_n\} \) converging to \( x_0 \), the sequence \( \{Tx_n\} \) converges weakly to \( Tx_0 \), i.e., \( x_n \to x_0 \Rightarrow Tx_n \rightharpoonup Tx_0 \).

Definition 1.2.20. [151] A mapping \( T : E \to E^* \) is hemicontinuous at \( x_0 \) if for any sequence \( \{x_n\} \) converging to \( x_0 \) along a line, the sequence \( \{Tx_n\} \) converges weakly to \( Tx_0 \), i.e., \( Tx_n = T(x_0 + t_n x) \rightharpoonup Tx_0 \) as \( t_n \to 0 \) and \( n \to \infty \) for all \( x \in E \).

Definition 1.2.21. [5] A bifunction \( \phi : E \times E \to \mathbb{R} \) is said to be skew-symmetric if

\[
\phi(x, x) - \phi(x, y) - \phi(y, x) + \phi(y, y) \geq 0, \quad \forall x, y \in H.
\]

The skew-symmetric bifunctions have the properties which can be considered an analog of monotonicity of gradient and nonnegativity of second derivative for the convex function. For properties and applications of the skew-symmetric bifunction, we refer to see [5].

Definition 1.2.22. [151] Let \( C \) be a nonempty convex subset of a Hausdorff topological vector space \( X \). A functional \( f : C \to \mathbb{R} \) is said to be

(i) convex, if, for any \( x, y \in C \) and \( 0 \leq \alpha \leq 1 \),

\[
f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y);
\]

(ii) lower semicontinuous on \( C \), if, for every \( \alpha \in \mathbb{R} \), the set

\[
\{x \in C : f(x) \leq \alpha\}
\]

is closed in \( C \);

(iii) concave, if \( -f \) is convex;
(iv) upper semicontinuous on $C$, if $-f$ is lower semicontinuous on $C$.

**Definition 1.2.23.** [54] Let $C$ be a nonempty subset of a Hausdorff topological vector space $X$ and $\text{conv}(C)$ denote the convex hull of $C$. Then a multi-valued mapping $G : C \to 2^X$ is said to be a KKM map if, for every finite subset $\{x_1, x_2, x_3, ..., x_n\} \subseteq C$, $\text{conv}(x_1, x_2, x_3, ..., x_n) \subseteq \bigcup_{i=1}^n G(x_i)$.

**Lemma 1.2.16.** [54] Let $C$ be a nonempty subset of a Hausdorff topological vector space $X$ and let $G : C \to 2^X$ be a KKM map. If $G(x)$ is closed for all $x \in C$ and is compact for at least one $x \in C$, then $\bigcap_{x \in C} G(x) \neq \emptyset$.

**Definition 1.2.24.** [111] Let $X$ and $Y$ be two Hausdorff topological spaces and let $D$ be a nonempty convex subset of $X$ and $P$ be a pointed, proper, closed and convex cone of $Y$ with interior of $P$, $\text{int}P \neq \emptyset$. Let 0 be the zero point of $Y$, $\mathbb{U}(0)$ be the neighborhood set of 0, $\mathbb{U}(x_0)$ be the neighborhood set of $x_0$ and $f : D \to Y$ be a mapping.

(i) If, for any $V \in \mathbb{U}(0)$ in $Y$, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V + P, \quad \forall x \in U \cap D,$$

then $f$ is called upper $P$-continuous on $x_0$. If $f$ is upper $P$-continuous for all $x \in D$, then $f$ is called upper $P$-continue on $D$;

(ii) If, for any $V \in \mathbb{U}(0)$ in $Y$, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V - P, \quad \forall x \in U \cap D,$$

then $f$ is called lower $P$-continuous on $x_0$. If $f$ is lower $P$-continuous for all $x \in D$, then $f$ is called lower $P$-continuous on $D$;

(iii) If, for any $x, y \in D$ and $t \in [0,1]$, the mapping $f$ satisfies

$$tf(x) + (1-t)f(y) \in f(tx + (1-t)y) + P,$$

then $f$ is called $P$-convex;
(iv) If, for any \(x, y \in D\) and \(t \in [0, 1]\), the mapping \(f\) satisfies
\[
f(x) \in f(tx + (1 - t)y) + P \quad \text{or} \quad f(y) \in f(tx + (1 - t)y) + P,
\]
then \(f\) is called proper \(P\)-quasiconvex.

**Lemma 1.2.17.** [66] Let \(X\) and \(Y\) be two real Hausdorff topological spaces, \(D\) is a nonempty compact and convex subset of \(X\) and \(P\) is a pointed, proper, closed and convex cone of \(Y\) with \(\text{int}P \neq \emptyset\). Assume that \(g : D \times D \to Y\) and \(\Phi : D \to Y\) are two nonlinear mappings. Suppose that \(g\) and \(\Phi\) satisfy

(i) \(g(x, x) \in P, \ \forall x \in D\);  

(ii) \(\Phi\) is upper \(P\)-continuous on \(D\);  

(iii) \(g(., y)\) is lower \(P\)-continuous, \(\forall x \in D\);  

(iv) \(g(x, .) + \Phi(.)\) is proper \(P\)-quasiconvex, \(\forall x \in D\).

Then there exists a point \(x \in D\) satisfies
\[
G(x, y) \in P \setminus \{0\}, \ \forall y \in D,
\]
where
\[
G(x, y) = g(x, y) + \Phi(y) - \Phi(x), \ \forall x, y \in D.
\]

### 1.3 Variational inequalities, equilibrium problems and iterative methods

In this section, we give brief survey of some classes of variational inequalities and equilibrium problems. Further, we give brief survey of some iterative methods for solving fixed point problems, variational inequalities and equilibrium problems.

#### 1.3.1 Variational inequalities

Let \(a(\cdot, \cdot) : H \times H \to \mathbb{R}\) be a bilinear form.
Problem 1.3.1. For given $f \in H^*$, find $x \in C$ such that

$$a(x, y - x) \geq \langle f, y - x \rangle, \quad \forall y \in C.$$  \hspace{1cm} (1.3.1)

The inequality (1.3.1) is termed as variational inequality which characterizes the classical Signorini problem of elasto-statistics, that is, the analysis of a linear elastic body in contact with a rigid frictionless foundation. This problem was investigated and studied by Lions and Stampacchia [109] by using the projection technique.

If the bilinear form is continuous, then by Riesz-FrÉchet theorem, we have

$$a(x, y) = \langle Ax, y \rangle, \quad \forall x, y \in H,$$

where $A : H \to H^*$ is a continuous linear operator. Then Problem 1.3.1 is equivalent to the following problem:

Problem 1.3.2. Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq \langle f, y - x \rangle, \quad \forall y \in C.$$  \hspace{1cm} (1.3.2)

If $f \equiv 0 \in H^*$, then (1.3.2) reduces to the following classical variational inequality problem introduced by Hartmann and Stampacchia [70].

Problem 1.3.3. Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C.$$  \hspace{1cm} (1.3.3)

The solution set of variational inequality problem (in short,VIP) (1.3.3) is denoted by $\text{Sol}(\text{VIP (1.3.3)})$. Since then various generalizations of the above mentioned variational inequality have been introduced and studied by a number of authors. Some of them are given below:
Problem 1.3.4. Find $x \in C$ such that
\[
\langle Tx, y - x \rangle + \psi(y) - \psi(x) \geq \langle Ax, y - x \rangle, \quad \forall y \in C,
\] (1.3.4)
where $T, A : H \to H$ and $\psi : H \to \mathbb{R}$.

Problem 1.3.4 has been studied by Siddiqi et al. [150] in the setting of Banach space.

Problem 1.3.5. Find $x \in C$ such that
\[
\langle Tx, y - x \rangle + \psi(x, y) - \psi(x, x) \geq 0, \quad \forall y \in C,
\] (1.3.5)
$\psi : H \times H \to \mathbb{R}$.

Problem 1.3.5 has been studied by Kikuchi and Oden [102], and Noor [129].

An important generalization of variational inequality problem is a vector variational inequality problem introduced and studied by Giannessi [59] in finite dimensional Euclidean spaces. Since then various generalizations of vector variational inequality in infinite dimensional spaces have been studied by many authors, see [60, 81, 83, 92, 93], and references therein.

Throughout the rest of the chapter, let $X$ and $Y$ be topological vector spaces; let $C \subset X$ be a nonempty closed and convex set, and let $P \subset Y$ be a closed, convex and pointed cone with $\text{int}P \neq \emptyset$. Let $L(X, Y)$ denote the space of all continuous linear mappings from $X$ into $Y$, and $T : C \to L(X, Y)$ a given mapping then the weak vector variational inequality problem (in short, WVVIP) is to:

Problem 1.3.6. Find $x \in C$ such that
\[
\langle Tx, y - x \rangle \notin -\text{int}P, \quad \forall y \in C,
\] (1.3.6)
which is considered and studied by Chen [34].

Fang and Huang [56] consider the strong vector variational inequality problem (in short, SVIP) is to:
Problem 1.3.7. Find $x \in C$ such that

$$\langle Tx, y - x \rangle \notin -P \setminus \{0\}, \; \forall y \in C.$$  \hspace{1cm} (1.3.7)

The system of variational inequalities is another important generalization of variational inequality. In 1971, Caffarelli [21] studied the system of variational inequalities arising in membrane problem. Later the Nash equilibrium problem [128] for differentiable functions can be formulated in the form of a variational inequality problem over product of sets [6]. A number of problems arising in operation research, economics, game theory, mathematical physics and other areas can also be uniformly modelled as a variational inequality problem over product of sets. In 1985, Pang [134] decomposed the original variational inequality problem defined on the product of sets into a system of variational inequalities, which is easy to solve, and to establish some solution methods for variational inequality problem over product of sets. Later, it was found that these two problems are equivalent. Since then a number of researchers studied the existence and iterative approximations of solutions of various systems of abstract variational inequalities, see for example [4, 37, 79, 88, 121, 164].

Here, we give some classes of systems of variational inequalities.

Ceng et al. [26] considered and studied the following system of variational inequalities (in short, SVIP):

Problem 1.3.8. Find $(x, y) \in C \times C$ such that

$$\begin{cases} 
\langle \rho_1 B_1 y + x - y, z - x \rangle \geq 0, & \forall z \in C, \\
\langle \rho_2 B_2 x + y - x, z - y \rangle \geq 0, & \forall z \in C,
\end{cases}$$

where $B_i : C \to C$ is a nonlinear mapping and $\rho_i > 0$ for each $i = 1, 2$. The set of solutions of SVIP(1.3.8) is denoted by $\text{Sol}(\text{SVIP}(1.3.8))$.

For each $i = 1, 2$, let $K_i$ be a nonempty closed and convex subset of Hilbert space $H_i$ and $T_i : H_1 \times H_2 \to H_i$ be a nonlinear mapping.
**Problem 1.3.9.** Find \((x_1, x_2) \in K_1 \times K_2\) such that

\[
\begin{cases}
\langle T_1(x_1, x_2), y_1 - x_1 \rangle \geq 0, & \forall y_1 \in K_1, \\
\langle T_2(x_1, x_2), y_2 - x_2 \rangle \geq 0, & \forall y_2 \in K_2.
\end{cases}
\]  

(1.3.9)

Problem 1.3.9 is called the system of variational inequalities which has been introduced and studied by Kassay and Kolumban [79] for multi-valued mappings.

For each \(i = 1, 2, ..., N\), let \(T_i : H_i \to H_i\) be a nonlinear mapping.

**Problem 1.3.10.** Find \(x \in \bigcap_{i=1}^{N} K_i\) such that

\[
\langle T_i x, y_i - x \rangle \geq 0, \quad \forall y_i \in K_i,
\]  

(1.3.10)

for \(i = 1, 2, ..., N\). Problem 1.3.10 is called the system of unrelated variational inequalities which has been introduced and studied by Censor et al. [31] for multi-valued mappings.

We also observe that if \(T_i = 0\), for all \(i\), then Problem 1.3.10 is reduced to the problem of finding a point \(x \in \bigcap_{i=1}^{N} K_i\) which is well known convex feasibility problem. If the set \(K_i\) are fixed sets of family of operators \(S_i : H \to H\) then the convex feasibility problem is the the common fixed point problem (in short, CFPP).

### 1.3.2 Equilibrium problems

Equilibrium problems which were initially introduced by Zuhovickii et al. [177], Fan [55], Brezis et al. [16], perhaps motivated by minimax problems that appeared in economic equilibrium. But it was Blum and Oettli [14], who understand equilibrium problem. They introduced the following abstract equilibrium problem (in short, EP):

**Problem 1.3.11.** Find \(x \in C\) such that

\[
F(x, y) \geq 0, \quad \forall y \in C,
\]  

(1.3.11)

where \(F : C \times C \to \mathbb{R}\) be a bifunction.
The solution set of EP(1.3.11) is denoted by Sol(EP(1.3.11)). Since then various gener-
alizations of EP(1.3.11) have been introduced and studied by many authors.

In 1999, Moudafi and Thèra [124] introduced and studied the following mixed equilib-
rium problem (in short, MEP):

**Problem 1.3.12.** Find \( x \in C \) such that

\[
F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C,
\]

(1.3.12)

where \( A : C \to H \) be a nonlinear mapping. The solution set of MEP(1.3.12) is denoted
by Sol(MEP(1.3.12)).

Equilibrium problems have potential and useful applications in nonlinear analysis and
mathematical economics. For example, if we set \( F(x, y) = \sup_{\zeta \in Mx} \langle \zeta, y - x \rangle \) with \( M : C \to 2^C \) a multi-valued maximal monotone operator. Then MEP(1.3.12) reduces to the
following variational inclusion:

(1) **Variational inclusion.** Find \( x \in C \) such that

\[
0 \in A(x) + M(x), \quad \forall y \in C.
\]

(1.3.13)

For further related work, see [1, 80, 89].

Set \( F(x, y) = \psi(y) - \psi(x) \), where \( \psi : C \to \mathbb{R} \) is a real function then MEP(1.3.12)
reduces to the following problem.

(2) **Mixed variational inequality.** Find \( x \in C \) such that

\[
\langle Ax, y - x \rangle + \psi(y) - \psi(x) \geq 0, \quad \forall y \in C,
\]

(1.3.14)

which has been studied by Noor [130].

Set \( F(x, y) = \psi(y) - \psi(x), \quad \forall x, y \in C \), where \( \psi : C \to \mathbb{R} \) is a real function and
\( A = 0 \), then MEP(1.3.12) reduces to the following minimization problem subject
to the implicit constraints.
(3) Optimization. Find $\bar{x} \in C$ such that

$$
\psi(\bar{x}) \leq \psi(y), \quad \forall y \in C. \tag{1.3.15}
$$

(4) Saddle point problem. Let $\psi : C_1 \times C_2 \to \mathbb{R}$. Then $(\bar{x}_1, \bar{x}_2)$ is called saddle point of $\psi$ if and only if

$$(\bar{x}_1, \bar{x}_2) \in C_1 \times C_2, \quad \psi(\bar{x}_1, y_2) \leq \psi(y_1, \bar{x}_2), \quad \forall (y_1, y_2) \in C_1 \times C_2. \tag{1.3.16}$$

Set $C = C_1 \times C_2$ and define $F : C \times C \to \mathbb{R}$ by $f((x_1, x_2), (y_1, y_2)) = \psi(y_1, x_2) - \psi(x_1, y_2)$ then $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of (1.3.16) if and only if $(\bar{x}_1, \bar{x}_2)$ satisfies (1.3.11).

(5) Nash equilibria. Let $I$ be the index set (the set of players). For every $i \in I$ let there be given a set $C_i$ (the strategy set of $i^{th}$ player). Let $C = \prod_{i \in I} C_i$. For every $i \in I$ let there be given a function $f_i : C \to \mathbb{R}$ (the loss function of the $i^{th}$ player depending on the strategies of all players). For $x = (x_i)_{i \in I} \in C$, we define $x^i := (x_j)_{j \in I, j \neq i}$. The point $\bar{x} = (\bar{x}_i)_{i \in I} \in C$ is called Nash equilibrium if and only if, for all $i \in I$, there holds

$$
f_i(\bar{x}_i) \leq f_i(\bar{x}^i, y_i), \quad \forall y_i \in C_i, \tag{1.3.17}
$$

(i.e., no player can reduce his loss by varying his strategy alone). Define $F : C \times C \to \mathbb{R}$ by

$$
F(x, y) = \sum_{i \in I} (f_i(x^i, y_i) - f_i(x)).
$$

Then $\bar{x} \in C$ is a Nash equilibrium if and only if $\bar{x}$ fulfills (1.3.11). Indeed if (1.3.17) holds for all $i \in I$ we choose $y \in C$ in such a way that $x^i = y^i$, then $F(\bar{x}, y) = f_i(\bar{x}^i, y_i) - f_i(\bar{x})$. Hence (1.3.11) implies (1.3.17) for all $i \in I$.

(6) Fixed point problem. Let $T : C \to C$ be a given mapping. The problem is to find
\( \bar{x} \in C \) such that

\[ \bar{x} = T\bar{x}. \quad (1.3.18) \]

Set \( F(x, y) = \langle x - Tx, y - x \rangle \). Then \( \bar{x} \) solves (1.3.11) if and only if \( \bar{x} \) is a solution of (1.3.18).

(7) **Variational inequality.** Let \( T : C \to H^* \) be a given mapping. The problem is to find \( \bar{x} \in H \) such that \( \bar{x} \in C \),

\[ \langle T\bar{x}, y - \bar{x} \rangle \geq 0 \qquad \forall \ y \in C. \quad (1.3.19) \]

We set \( F(x, y) = \langle Tx, y - x \rangle \). Clearly (1.3.11) \( \Rightarrow \) (1.3.19).

(8) **Complementarity problem.** This is special case of previous example. Let \( C \) be a closed and convex cone with \( C^* = \{ x^* \in H^* : \langle x^*, y \rangle \geq 0, \ \forall y \in C \} \) denotes the polar cone. Let \( T : C \to H^* \) be a given mapping. The problem is to find \( \bar{x} \in H \) such that \( T\bar{x} \in C^* \),

\[ \langle T\bar{x}, \bar{x} \rangle = 0. \quad (1.3.20) \]

It is easily seen that (1.3.20) is equivalent with (1.3.19).

EP(1.3.11) and MEP(1.3.12) have been generalized by many authors. Some generalizations of these problems are given below.

**Problem 1.3.13.** Find \( x \in C \) such that

\[ F(x, y) + \psi(y) - \psi(x) \geq 0, \quad \forall y \in C, \quad (1.3.21) \]

where \( \psi : H \to \mathbb{R} \cup \{+\infty\} \) is a nonlinear functional. Problem 1.3.13 has been studied by Ceng and Yao [27].

**Problem 1.3.14.** Find \( x \in C \) such that

\[ F(x, y) + \langle Ax, y - x \rangle + \psi(y) - \psi(x) \geq 0, \quad \forall y \in C, \quad (1.3.22) \]
where $A : C \to H$ is a nonlinear mapping. Problem 1.3.14 has been studied by Peng and Yao [135].

Let $F : C \times C \to \mathbb{R}$, $\psi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be nonlinear bifunctions. The generalized equilibrium problem (in short, GEP) is:

**Problem 1.3.15.** Find $x \in C$ such that

$$F(x, y) + \psi(y, x) - \psi(x, x) \geq 0, \quad \forall y \in C,$$

which has been studied by Noor [131].

One of the useful generalizations of EP(1.3.11) is vector equilibrium problem which has wide range of applications in multi objective optimizations. For the existence theory of various types of vector equilibrium problems, see for instance [60, 81, 82, 84].

Let $F : C \times C \to Y$ be a given mapping. The weak vector equilibrium problem (in short, WVEP) is to:

**Problem 1.3.16.** Find $x \in C$ such that

$$F(x, y) \notin -\text{int}P, \quad \forall y \in C.$$  

WVEP(1.3.24) is introduced and studied, independently, by Kazmi [84], Konnov [104] and Bianchi *et al.* [13].

The strong vector equilibrium problem (in short, SVEP) is to:

**Problem 1.3.17.** Find $x \in C$ such that

$$F(x, y) \notin -P \setminus \{0\}, \quad \forall y \in C.$$  

SVEP(1.3.25) is introduced and studied by Kazmi and Khan [92].

The generalized mixed vector equilibrium problem (in short, GMVEP) is to:
Problem 1.3.18. Find $x \in C$ such that

$$F(x, y) + \psi(y) - \psi(x) + e < Ax, y - x > \in P, \quad \forall y \in C,$$

where $X = H$, a Hilbert space, $e \in \text{int}P$, $\psi : X \to Y$ be nonlinear vector-valued mapping and $A : C \to X$. GMVEP(1.3.26) is introduced and studied by Shan and Huang [148].

1.3.3 Iterative methods

We give a brief survey of some iterative methods for solving fixed point problems, variational inequalities and equilibrium problems.

Picard iterative method. Let $T$ be a self-mapping defined on a nonempty closed subset $C$ of a complete metric space $X$, which posses at least one fixed point $p \in \text{Fix}(T)$. For a given $x_0 \in X$, the Picard iterative method generates the sequence $\{x_n\}$ given by

$$x_n = T(x_{n-1}) = T^n(x_0), \quad n = 1, 2, \ldots .$$

The sequence $\{x_n\}$ converges strongly to the unique fixed points of $T$ when $T$ is a contraction mapping.

When the contractive conditions are slightly weaker, e.g., nonexpansive, firmly nonexpansive, then the Picard iterations need not converge to a fixed point of the operator $T$, and some other iteration procedures must be considered.

Let $E$ be a real Banach space and $C$ be a nonempty closed and convex subset of $E$. Let $T : C \to C$ be a self-mapping, $x_0 \in C$ and $\lambda \in [0, 1]$.

Krasnosel’skiĭ iteration method. The sequence $\{x_n\}$ given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n = 1, 2, \ldots ,$$

is called the Krasnosel’skiĭ iteration method, see Krasnosel’skiĭ [107].
It is easy to see that the Krasnosel’skii iteration \( \{x_n\}_{n=0}^{\infty} \) given by (1.3.28) is exactly the Picard iteration corresponding to the averaged operator

\[
T_{\lambda} = (1 - \lambda)I + \lambda T,
\]

(1.3.29)

where \( I \) is the identity operator. The study of iterative methods for approximating fixed points of a nonexpansive mapping \( T \) has yielded a host of works in the last decades. The most relevant progresses are mainly based on two types of iterative algorithms: Mann and Halpern iterative algorithms. Both algorithms have extensively been studied for decades.

**Mann iterative method.** Mann iterative algorithm, initially due to Mann [114], is essentially an averaged algorithm which generates a sequence recursively

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0,
\]

(1.3.30)

where the initial guess \( x_0 \in C \) and \( \{\alpha_n\} \) is a sequence in \( (0, 1) \). Later, Krasnosel’skii [107] studied the iterative algorithm (1.3.30) in the particular case when \( \alpha_n = \lambda \).

In 1974, Ishikawa [73] enlarged and improved Mann iterative algorithm to a new iterative algorithm which generates the sequence \( \{x_n\} \) defined by

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T[\beta_n x_n + (1 - \beta_n)Tx_n],
\]

(1.3.31)

where \( 0 \leq \alpha_n \leq \beta_n \leq 1, \quad \lim_{n \to \infty} \beta_n = 0, \quad \sum_{n \geq 1} \alpha_n \beta_n = \infty \). However, the iterative algorithms of both Mann and Ishikawa converge weakly in Banach space. As a matter of fact, Mann’s iterative algorithm may fail to converge while Ishikawa iterative algorithm can still converge for a Lipschitz pseudocontractive mapping in a Hilbert space.

**Halpern explicit iterative method.** Halpern [69] was the first in introducing the explicit iterative algorithm which generates a sequence via the recursive formula

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0,
\]

(1.3.32)
where the initial guess $x_0 \in C$ and anchor $u \in C$ are arbitrary (but fixed) and the sequence $\{\alpha_n\}$ is contained in $[0, 1]$, for finding a fixed point of a nonexpansive mapping $T : C \to C$ with $\text{Fix}(T) \neq \emptyset$, where $C$ is a nonempty, closed and convex subset of a Hilbert space $H$. This iterative method is now commonly known as Halpern iterative method although Halpern initially considered the case where $C$ is the unit closed ball and $u = 0$.

**Viscosity approximation method.** Given a nonexpansive self-mapping $T$ on a nonempty, closed and convex subset $C$, a real number $t \in (0, 1]$ and a contraction mapping $f$ on $C$, define the mapping $T_t : C \to C$ by

$$T_t x = tf(x) + (1 - t)Tx, \quad x \in C.$$  

It is easily seen that $T_t$ is a contraction and hence $T_t$ has a unique fixed point which is denoted by $x_t$. That is, $x_t$ is the unique solution to the fixed point equation

$$x_t = tf(x_t) + (1 - t)Tx_t, \quad t \in (0, 1]. \quad (1.3.33)$$

The explicit iterative discretization of (1.3.33) is

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0, \quad (1.3.34)$$

where $\{\alpha_n\} \subset [0, 1]$. Note that these two iterative processes (1.3.33) and (1.3.34) generalize the results of Browder [17] and Halpern [69] in another direction. The viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [118] in the framework of a Hilbert space. The convergence of the implicit (1.3.33) and explicit (1.3.34) algorithms has been the subject of many papers because under suitable conditions these iterations converge strongly to the unique solution $q \in \text{Fix}(T)$ of the variational inequality

$$\langle (I - f)q, x - q \rangle \geq 0, \quad \forall x \in \text{Fix}(T). \quad (1.3.35)$$

This fact allows us to apply this method to convex optimization, linear programming
and monotone inclusions. In 2004, Xu [171] extended the result of Moudafi to uniformly smooth Banach spaces and obtained strong convergence theorem. For related work, see [12, 35, 116, 172].

**Hybrid iterative method.** The hybrid iterative method is also known as outer-approximation method. This type of method was originally introduced by Haugazeau [71] in 1968 and was successfully generalized and extended by Combettes [44], Nakajo and Takahashi [127], Kikkawa and Takahashi [101].

In 2003, Nakajo and Takahashi [127] introduced and studied the following iterative method for a nonexpansive mapping $T$ over a Hilbert space:

\[
\begin{aligned}
x_0 &= x \in C \subseteq H, \\
y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\
C_n &= \{ z \in C : \|y_n - z\| \leq \|x_n - z\| \}, \\
Q_n &= \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \}, \\
x_{n+1} &= P_{C_n \cap Q_n} x.
\end{aligned}
\]  

They proved that the sequence $\{x_n\}$ generated by (1.3.36) converges strongly to $P_{\text{Fix}(T)} x_0$, where $P_{\text{Fix}(T)}$ denotes the metric projection from $H$ onto Fix$(T)$.

In 1967, Lions and Stampacchhia [109] proved the first general theorem for the existence and uniqueness of solution of variational inequality problem in Hilbert space using projection mapping. The variational inequality of finding $x \in C$ such that

\[
\langle Tx, y - x \rangle \geq 0, \quad \forall y \in C,
\]

where $T : C \to H$ be a nonlinear mapping, is equivalent to finding a fixed point $x$ of the equation

\[
x = P_{C}(x - \lambda Tx),
\]

where $\lambda > 0$. Using this fixed point formulation, one can have an iterative algorithm which generates the sequence $\{x_n\}$ given by
\[ x_{n+1} = P_C(x_n - \lambda T(x_n)), \]

where \( x_0 \in C \) is given and \( \lambda > 0 \), see Baiocchi and Capelo [7], Glowinski, Lions and Tremolières [64].

In 1976, Korpelevich [106] proposed an extragradient method with iterative scheme:

\[
\begin{aligned}
  x_1 &= x \in C \\
  y_n &= P_C(x_n - \lambda Fx_n), \\
  x_{n+1} &= P_C(x_n - \lambda Fy_n),
\end{aligned}
\]

for every \( n = 0, 1, 2, \ldots \), and \( \lambda > 0 \), where \( P_C \) is an orthogonal projection onto \( C \) in the finite dimensional Euclidean space. The idea of the extragradient iterative process introduced by Korpelevich was successfully generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see, e.g., He et al. [72], Iusem and Svaiter [74], Solodov and Svaiter [153], Wang et al. [166].

In 2006, by combining a hybrid iterative method with an extragradient method, Nadezhkina and Takahashi [126] introduced the following iterative method:

\[
\begin{aligned}
  x_0 &= x \in C, \\
  y_n &= P_C(x_n - \lambda_n Ax_n), \\
  z_n &= \beta_n x_n + (1 - \beta_n) TP_C(x_n - \lambda_n Ay_n), \\
  C_n &= \{ z \in C : \| z_n - z \|^2 \leq \| x_n - z \|^2 \}, \\
  Q_n &= \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \}, \\
  x_{n+1} &= P_{C_n \cap Q_n} x,
\end{aligned}
\]

for every \( n = 0, 1, 2, \ldots \). They proved that under certain appropriate conditions on \( \{ \beta_n \} \) and \( \{ \lambda_n \} \), the sequences \( \{ x_n \} \), \( \{ y_n \} \) and \( \{ z_n \} \) generated by (1.3.38) converge strongly to \( z \in \text{Fix}(T) \cap \text{Sol(VIP(1.3.3)))}. \) Ceng et al. [24] introduced the extragradient-like iterative method, an extension of method given by Nadezhkina and Takahashi [125,126], for approximating common solution of FPP(1.2.1) for a nonexpansive mapping \( T \) and
VIP(1.3.3) for a monotone, Lipschitz-continuous mapping.

In 2008, Ceng et al. [26] introduced and studied the following iterative method so called relaxed extragradient method for approximating a common solution of SVIP(1.3.8) with \(\alpha\)-inverse strongly monotone mappings and FPP(1.2.1) for a nonexpansive mapping \(T\):

\[
\begin{cases}
x_0 \in C, \\
y_n = P_C(x_n - \mu B_2x_n), \\
x_{n+1} = \alpha_n x_0 + \beta_n x_n + \gamma_n TP_C(y_n - \lambda B_1y_n),
\end{cases}
\]

(1.3.39)

where \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\) are sequences in \([0,1]\), and \(\lambda, \mu > 0\). For further related work, see Ceng et al. [23], Yao et al. [173].

On the other hand, Takahashi and Takahashi [160] in 2007, proposed an iterative method based on viscosity approximation method which improves the result of Moudafi [118] for approximating the common solution of EP(1.3.11) and FPP(1.2.1) for a nonexpansive mapping \(T\) in Hilbert space.

\[
\begin{cases}
F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0 \ \forall y \in C, \\
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n.
\end{cases}
\]

(1.3.40)

They proved that under certain appropriate conditions on \(\{\alpha_n\}, \{r_n\}\) and \(\{\lambda_n\}\), the sequences \(\{x_n\}\) and \(\{u_n\}\) generated by (1.3.40) converge strongly to \(z \in \text{Fix}(T) \cap \text{Sol}(\text{EP}(1.3.11))\), where \(z = P_{\text{Fix}(T) \cap \text{Sol}(\text{EP}(1.3.11))}f(z)\). For related work, see [42, 75, 113, 139, 144]. Further, Tada and Takahashi [158] introduced a hybrid method for approximating a common solution of EP(1.3.11) and FPP(1.2.1) for a nonexpansive mapping in a Hilbert space. Starting with an arbitrary \(x_1 \in H\), define sequences \(\{x_n\}\) and \(\{u_n\}\) by

\[
\begin{cases}
F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0 \ \forall y \in C, \\
w_n = (1 - \alpha_n)x_n + \alpha_n Tu_n, \\
C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n}x.
\end{cases}
\]

(1.3.41)
They proved that under certain appropriate conditions on \(\{\alpha_n\}\) and \(\{r_n\}\), the sequences \(\{x_n\}\) and \(\{u_n\}\) generated by (1.3.41) converge strongly to \(P_{\text{Fix}(T) \cap \text{Sol}(\text{EP}(1.3.11))}(x)\).

Using the idea of Takahashi and Takahashi [160], Plubtieng and Punpaeng [139] introduced the general iterative method for finding a common solution of \(\text{EP}(1.3.11), \text{VIP}(1.3.3)\) and \(\text{FPP}(1.2.1)\) for a nonexpansive mapping \(T\). For further related work, see for instance [33, 40, 78, 108, 110, 135, 147, 167] and the relevant references cited therein.

The idea to generalize the hybrid iterative method (1.3.36) of Nakajo and Takahashi [127] from Hilbert space to Banach space has been given by Matsushita and Takahashi [117] in 2005. They proved a strong convergence theorem with generalized projection for a relatively nonexpansive mapping \(T\) on nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space \(E\): The sequence \(\{x_n\}\) is generated by the iterative scheme

\[
\begin{aligned}
x_0 & = x \in C, \\
y_n & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JT x_n), \\
H_n & = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
W_n & = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} & = \prod_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \\
\end{aligned}
\]

where \(\alpha_n \in (0, 1)\) is a sequence and \(J\) is normalized mapping from \(E\) into \(E^*\) with its inverse \(J^{-1}\).

In 2007, Plubtieng and Ungchittrakool [140] proved the strong convergence theorem to obtain the common fixed points a pair of relatively nonexpansive mappings in Banach space, see [2] for related work.

In 2009, Takahashi and Zembayashi [163] generalized Tada and Takahashi [158] iterative method (1.3.41) for \(\text{EP}(1.3.11)\) and \(\text{FPP}(1.2.1)\) by considering relative nonexpansive mapping in Banach space: The sequence \(\{x_n\}\) is denoted by iterative scheme

\[
\begin{aligned}
x_0 & = x \in C, \\
y_n & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JT x_n), \\
H_n & = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
W_n & = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} & = \prod_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \\
\end{aligned}
\]

where \(\alpha_n \in (0, 1)\) is a sequence and \(J\) is normalized mapping from \(E\) into \(E^*\) with its inverse \(J^{-1}\).
\[
\begin{cases}
  x_0 = x \in C, \\
y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\
u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}(y - u_n, Ju_n - Jx_n) \geq 0 \ \forall y \in C, \\
H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, \ldots.
\end{cases}
\]

In 2009, Qin et al. [143] introduced a hybrid iterative method to find a common element of the set of solutions of EP(1.3.11) and the set of common fixed points of two quasi \(\phi\)-nonexpansive mappings in Banach space:

\[
\begin{cases}
x_0 = x \in C, \quad \text{chosen arbitrary}, \\
C_1 = C, \\
x_1 = \Pi_{C_1} x_0 \\
y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\
u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}(y - u_n, Ju_n - Jy_n) \geq 0 \ \forall y \in C, \\
C_{n+1} = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0.
\end{cases}
\]

They proved strong convergence theorems for hybrid iterative method for two families of quasi \(\phi\)-nonexpansive mappings in the framework of Banach space.

Zhang [175] proved a strong convergence theorem for the sequence generated by hybrid iterative method for EP(1.3.11) and FPP(1.2.1) for a finite family of quasi \(\phi\)-nonexpansive mappings. For related work, see Petrot et al. [136].

In 2010, Qin [141] investigated hybrid iterative methods for a pair of asymptotically quasi \(\phi\)-nonexpansive mappings in Banach space. For related work, see Zhou et al. [176], Qin et al. [142], Song and Chen [154].