Chapter 6

Common solution to system of unrelated generalized mixed vector equilibrium problems and fixed point problems in Banach space

6.1 Introduction

Throughout the chapter unless otherwise stated, let $E$ be a real Banach space with its dual space $E^*$ and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $E$ and $E^*$ and $\|\cdot\|$ denote the norm of $E$ as well as of $E^*$. Let $C$ be nonempty closed and convex subset of $E$. Let $Y$ be an ordered Banach space and let $P$ be a pointed, proper, closed and convex cone of $Y$ with $\text{int}P \neq \emptyset$.

We consider the following new system of unrelated generalized mixed vector equilibrium problems (in short, SUGMVEP): For each $i = 1, 2, 3, \ldots, N$, let $K_i$ be a nonempty closed and convex set in $E$ with $K = \bigcap_{i=1}^{N} K_i \neq \emptyset$; let $F_i : K_i \times K_i \rightarrow Y$ be a bimapping, $\psi_i : K_i \times K_i \rightarrow \mathbb{R}$ be a bifunction and $A_i : K_i \rightarrow B(E,Y)$ be a nonlinear mapping. Then SUGMVEP is to find $x^* \in \bigcap_{i=1}^{N} K_i$ such that

$$ F_i(x^*, y_i) + \langle y_i - x^*, A_i x^* \rangle + \psi_i(y_i, x^*) - \psi_i(x^*, x^*) \in P, \quad \forall y_i \in K_i, \quad i = 1, 2, 3, \ldots, N. \tag{6.1.1} $$

Then the set of solutions of SUGMVEP(6.1.1) is given by $\bigcap_{i=1}^{N} \text{Sol}(\text{GMVEP}(F_i, A_i, \psi_i, K_i))$, where $\text{Sol}(\text{GMVEP}(F_i, A_i, \psi_i, K_i))$ denotes the set of solutions of GMVEP(6.1.1) corresponding to the mappings $F_i, A_i, \psi_i$ and $K_i$.

When $i = 1$, then SUGMVEP(6.1.1) reduces to the generalized mixed vector equilibrium
problem (in short, GMVEP): Find \( x^* \in C \) such that

\[
F_1(x^*, y_1) + \langle y_1 - x^*, A x^* \rangle + \psi_1(y_1, x^*) - \psi_1(x^*, x^*) \in P, \quad \forall y_1 \in K_1.
\] (6.1.2)

The solution set of GMVEP(6.1.2) is denoted by \( \text{Sol}(\text{GMVEP}(6.1.2)) \).

If \( Y = \mathbb{R} \), then SUGMVEP(6.1.1) reduces to the system of unrelated generalized mixed equilibrium problem (in short, SUGMEP) of finding \( x^* \in \bigcap_{i=1}^{N} K_i \) such that

\[
F_i(x^*, y_i) + \langle y_i - x^*, A_i x^* \rangle + \psi_i(y_i, x^*) - \psi_i(x^*, x^*) \geq 0, \quad \forall y_i \in K_i, \quad i = 1, 2, 3, ..., N.
\] (6.1.3)

When \( i = 1 \), then SUGMEP(6.1.3) reduces to the generalized mixed equilibrium problem (in short, GMEP): Find \( x^* \in C \) such that

\[
F_1(x^*, y_1) + \langle y_1 - x^*, A_1 x^* \rangle + \psi_1(y_1, x^*) - \psi_1(x^*, x^*) \geq 0, \quad \forall y_1 \in K_1.
\] (6.1.4)

The solution set of GMEP(6.1.4) is denoted by \( \text{Sol}(\text{GMEP}(6.1.4)) \). The GMEP(6.1.4) with \( \psi(x, x^*) = \psi(x), \quad \forall x, x^* \in C \) has been studied by Ceng and Yao [27] in Hilbert space.

If \( Y = \mathbb{R} \) and \( F = 0 \), then \( P = [0, +\infty) \) and hence GMVEP(6.1.2) reduces to the following generalized variational inequality problem (in short, GVIP): Find \( x^* \in C \) such that

\[
\langle y_1 - x^*, A_1 x^* \rangle + \psi_1(y_1, x^*) - \psi_1(x^*, x^*) \geq 0, \quad \forall y_1 \in K_1.
\] (6.1.5)

If \( E = H \), Hilbert space, \( \psi_i = 0 \), and \( Y = \mathbb{R} \) for all \( i \), then SUGMVEP(6.1.1) is reduced to the system of unrelated mixed equilibrium problems introduced and studied by Djafari-Rouhani, Kazmi and Rizvi [51].

Recently, Plubtieng and Ungchittrakool [140] improved iterative method (1.3.42) by considering a pair of relatively nonexpansive mappings, and considered the following hybrid projection method:
\[
\begin{align*}
  x_0 &= x \in C, \\
  y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\
  z_n &= J^{-1}(\beta_{n,1} Jx_n + \beta_{n,2} JTx_n + \beta_{n,3} JSx_n), \\
  H_n &= \{ z \in C : \phi(z, y_n) \leq \phi(z, x_n) \}, \\
  W_n &= \{ z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0 \}, \\
  x_{n+1} &= \prod_{H_n \cap W_n} x, \; \forall n \geq 0,
\end{align*}
\]

where \( T \) and \( S \) are relatively nonexpansive mappings and \( \{\alpha_n\}, \{\beta_{n,1}\}, \{\beta_{n,2}\} \) and \( \{\beta_{n,3}\} \) are control sequences in \([0, 1]\). They obtained a strong convergence under some restrictions imposed on the control sequences in uniformly convex and uniformly smooth Banach space. Further related work see Su et al. [157].

In 2010, Qin and Agarwal [141] considered the shrinking projection method which was first introduced by Takahashi, Takeuchi and Kubota [162] in Hilbert spaces, for a pair of asymptotically quasi \( \phi \)-nonexpansive mappings \( T, S \) defined on a nonempty closed and convex subset \( C \) of uniformly smooth and strictly convex Banach space \( E \). They proved the strong convergence for the sequences generated by the following iterative method. Let \( \{x_n\} \) be a sequence generated by the iterative schemes:

\[
\begin{align*}
  x_0 &\in E, \; \text{chosen arbitrary}, \\
  C_1 &\equiv C, \\
  x_1 &= \prod_{C_1} x_0, \\
  z_n &= J^{-1}(\beta_{n,1} Jx_n + \beta_{n,2} JTx_n + \beta_{n,3} JSx_n), \\
  y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\
  C_{n+1} &= \{ z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + (g_n - 1) M_n \}, \\
  x_{n+1} &= \prod_{C_{n+1}} x_0, \; \forall n \geq 0,
\end{align*}
\]

where \( \{\alpha_n\}, \{\beta_{n,1}\}, \{\beta_{n,2}\} \) and \( \{\beta_{n,3}\} \) are sequences in \([0, 1]\); \( g_n = \max\{k_n, f_n\} \) for each \( n \geq 1 \); \( \{k_n\} \subset [1, \infty) \); \( \{f_n\} \subset [1, \infty) \) and \( M_n = \sup\{\phi(z, x_n) : z \in F(T) \cap F(S)\} \) for each \( n \geq 1 \).

Recently, Qin et al. [142] extended the work of Qin and Agrawal [141] for a family of
generalized asymptotically quasi $\phi$-nonexpansive mappings. Very recently, Song and Chen [154] proved a strong convergence theorem for common solution to MEP(1.3.12) and FPP(1.2.1) for a generalized asymptotically quasi $\phi$-nonexpansive mapping in Banach space.

Motivated by the work of Qin et al. [141, 142], Song and Chen [154], Shan and Huang [148], and by the ongoing research in this direction, we study the existence and properties of solution of GMVEP(6.1.2) in Banach space. Further, we introduce an iterative method based on hybrid method and convex approximation method for finding a common element to the set of solutions of SUGMVEP(6.1.1) and the set of solutions of fixed point problems for the two families of two generalized asymptotically quasi $\phi$-nonexpansive mappings in Banach space. Furthermore, we obtain a strong convergence theorem for the sequences generated by the proposed iterative scheme. Finally, we derive some consequences from our main result. The results presented in this paper extended and unify many of the previously known results in this area, see instance Song and Chen [154].

6.2 Existence of solution

Let $F, \psi : C \times C \to Y$ be two mappings and $A : C \to B(E,Y)$ be a nonlinear mapping. For any $z \in E$, define a mapping $G_z : C \times C \to Y$ as follows:

$$G_z(x, y) = F(x, y) + \langle y - x, Ax \rangle + \psi(y, x) - \psi(x, x) + \frac{e}{r}(y - x, Jx - Jz),$$  \hspace{1cm} (6.2.1)

where $r$ is a positive real number and $e \in \text{int}P$.

First, we have the following assumption:

**Assumption 6.2.1.** Let $G_z, F, \psi$ satisfy the following conditions:

(i) For all $x \in C$, $F(x, x) \in P$;

(ii) $F$ is $P$-monotone, i.e., $F(x, y) + F(y, x) \in -P$, $\forall x, y \in C$;

(iii) $F(., y)$ is continuous, $\forall y \in C$;
(iv) $F(x, \cdot)$ is weakly continuous and $P$-convex, i.e.,
\[
tF(x, y_1) + (1 - t)F(x, y_2) \in F(x, ty_1 + (1 - t)y_2) + P, \quad \forall x, y_1, y_2 \in C, \quad \forall t \in [0, 1];
\]

(v) $G_z(\cdot, y)$ is lower $P$-continuous, $\forall y \in C$ and $z \in E$;

(vi) $\psi(\cdot, y)$ is $P$-convex and weakly continuous;

(vii) $G_z(x, \cdot)$ is proper $P$-quasiconvex, $\forall x \in C$ and $z \in E$;

(viii) $\psi$ is $P$-skew symmetric, i.e.,
\[
\psi(x, x) - \psi(x, y) - \psi(y, x) + \psi(y, y) \in P, \quad \forall x, y \in C.
\]

Remark 6.2.1. $P$-skew-symmetric bifunctions are natural extensions of skew-symmetric bifunctions. The skew-symmetric bifunctions have the properties that can be considered analogous to the monotonicity of the gradient and the non-negativity of the second derivative for convex functions. For the properties and applications of the skew-symmetric bifunctions, we refer the reader to [5].

For any $r > 0$, define a mapping $T_r : E \to C$ as follows:
\[
T_r(z) = \{ x \in C : F(x, y) + \langle y - x, Ax \rangle + \psi(y, x) - \psi(x, x) + \frac{e}{r} \langle y - x, Jx - Jz \rangle \in P, \quad \forall y \in C \},
\]
where $e \in \text{int} P$.

Now, we study some properties of set of solutions of GMVEP(6.1.2) and the mapping $T_r$ defined by (6.2.2).

Theorem 6.2.1. Let $E$ be a uniformly smooth and strictly convex Banach space and let $C$ be a nonempty compact and convex subset of $E$. Assume that $P$ is a pointed, proper, closed and convex cone of a real order Banach space $Y$ with $\text{int} P \neq \emptyset$. Let $G_z : C \times C \to Y$ be defined by (6.2.1). Let $F, \psi : C \times C \to Y$, and $G_z$ satisfy Assumption 6.2.1 and $A : C \to B(E, Y)$ be a continuous and $P$-monotone mapping. Let $T_r : E \to C$ defined by (6.2.2). Then the following conclusions hold:
(i) $T_r z \neq \emptyset$, $\forall z \in E$;

(ii) $T_r$ is single-valued;

(iii) $T_r$ is firmly nonexpansive type mapping, i.e., for all $z_1, z_2 \in E$,

$$
\langle T_r z_1 - T_r z_2, J T_r z_1 - J T_r z_2 \rangle \leq \langle T_r z_1 - T_r z_2, J z_1 - J z_2 \rangle;
$$

(iv) $\text{Fix}(T_r) = \text{Sol}(\text{GMVEP}(6.1.2))$;

(v) $\text{Sol}(\text{GMVEP}(6.1.2))$ is closed and convex.

**Proof.** (i) Let $g(x,y) = G_z(x,y)$ and $\Phi(y) = 0$ for all $x,y \in C$ and $z \in E$. It is easy to observe that $g(x,y)$ and $\Phi(y)$ satisfy all the conditions of Lemma 1.2.17. Then there exists a point $x \in C$ such that

$$
G_z(x,y) + \Phi(y) - \Phi(x) \in P, \quad \forall y \in C,
$$

and thus $T_r z \neq \emptyset$, $\forall z \in E$.

(ii) For each $z \in E$, $T_r z \neq \emptyset$, let $x_1, x_2 \in T_r z$. Then

$$
F(x_1, y) + \langle y - x_1, A x_1 \rangle + \psi(y, x_1) - \psi(x_1, x_1) + \frac{e}{\rho} (y - x_1, J x_1 - J z) \in P, \quad \forall y \in C, \quad (6.2.3)
$$

and

$$
F(x_2, y) + \langle y - x_2, A x_2 \rangle + \psi(y, x_2) - \psi(x_2, x_2) + \frac{e}{\rho} (y - x_2, J x_2 - J z) \in P, \quad \forall y \in C. \quad (6.2.4)
$$

Letting $y = x_2$ in (6.2.3) and $y = x_1$ in (6.2.4), and then adding, we have

$$
F(x_1, x_2) + F(x_2, x_1) + \langle x_2 - x_1, A x_1 - A x_2 \rangle + \psi(x_2, x_1) - \psi(x_1, x_1)
+ \psi(x_1, x_2) - \psi(x_2, x_2) + \frac{e}{\rho} (x_2 - x_1, J x_1 - J x_2) \in P.
$$

Since $F$ is $P$-monotone, $A$ is $P$-monotone, i.e., $\langle x_2 - x_1, A x_1 - A x_2 \rangle \in -P$ and $\psi$ is
$P$-skew symmetric, then we have

$$\frac{e}{r} \langle x_2 - x_1, Jx_1 - Jx_2 \rangle \in P.$$ 

Since $e \in \mathop{\text{int}} P$, $r > 0$ and $P$ is closed and convex cone, we have

$$\frac{1}{r} \langle x_2 - x_1, Jx_1 - Jx_2 \rangle \geq 0.$$ 

Since $E$ is strictly convex, preceding inequality implies $x_1 = x_2$. Hence $T_r$ is single-valued.

(iii) For any $z_1, z_2 \in E$, let $x_1 = T_r z_1$ and $x_2 = T_r z_2$. Then

$$F(x_1, y) + \langle y - x_1, Ax_1 \rangle + \psi(y, x_1) - \psi(x_1, x_1) + \frac{e}{r} \langle y - x_1, Jx_1 - Jz_1 \rangle \in P, \ \forall y \in C, \ (6.2.5)$$

and

$$F(x_2, y) + \langle y - x_2, Ax_2 \rangle + \psi(y, x_2) - \psi(x_2, x_2) + \frac{e}{r} \langle y - x_2, Jx_2 - Jz_2 \rangle \in P, \ \forall y \in C. \ (6.2.6)$$

Letting $y = x_2$ in (6.2.5) and $y = x_1$ in (6.2.6), and then adding, we have

$$F(x_1, x_2) + F(x_2, x_1) + \langle x_2 - x_1, Ax_1 - Ax_2 \rangle + \psi(x_2, x_1) - \psi(x_1, x_1)$$

$$+ \psi(x_1, x_2) - \psi(x_2, x_2) + \frac{e}{r} \langle x_2 - x_1, Jx_1 - Jx_2 - Jz_1 + Jz_2 \rangle \in P.$$ 

By using the monotonicity of $F, A$ and the properties of $\psi$ and $P$, we have

$$\frac{1}{r} \langle x_2 - x_1, Jx_1 - Jx_2 - Jz_1 + Jz_2 \rangle \geq 0.$$ 

Hence, we have

$$\langle x_2 - x_1, Jx_1 - Jx_2 \rangle + \langle x_2 - x_1, Jz_2 - Jz_1 \rangle \geq 0,$$

or,

$$\langle x_1 - x_2, Jx_1 - Jx_2 \rangle \leq \langle x_1 - x_2, Jz_1 - Jz_2 \rangle$$

i.e.,

$$\langle T_r z_1 - T_r z_2, JT_r z_1 - JT_r z_2 \rangle \leq \langle T_r z_1 - T_r z_2, Jz_1 - Jz_2 \rangle. \quad (6.2.7)$$
Thus $T_r$ is firmly nonexpansive-type mapping.

(iv) Let $x \in \text{Fix}(T_r)$. Then

$$F(x, y) + \langle y - x, Ax \rangle + \psi(y, x) - \psi(x, x) + \frac{e}{r} \langle y - x, Jx - Jx \rangle \in P, \quad \forall y \in C,$$

and so,

$$F(x, y) + \langle y - x, Ax \rangle + \psi(y, x) - \psi(x, x) \in P, \quad \forall y \in C.$$

Thus $x \in \text{Sol}(\text{GMVEP}(6.1.2))$. Let $x \in \text{Sol}(\text{GMVEP}(6.1.2))$. Then

$$F(x, y) + \langle y - x, Ax \rangle + \psi(y, x) - \psi(x, x) \in P, \quad \forall y \in C,$$

and so,

$$F(x, y) + \langle y - x, Ax \rangle + \psi(y, x) - \psi(x, x) + \frac{e}{r} \langle y - x, Jx - Jx \rangle \in P, \quad \forall y \in C.$$

Hence $x \in \text{Fix}(T_r)$. Thus $\text{Fix}(T_r) = \text{Sol}(\text{GMVEP}(6.1.2))$.

(v) As in the proof of Theorem 3.2.1(v), we have that, for $z_1, z_2 \in C$,

$$\phi(T_r z_1, T_r z_2) + \phi(T_r z_2, T_r z_1) \leq \phi(T_r z_1, z_2) + \phi(T_r z_2, z_1).$$

Taking $z_2 = u \in \text{Fix}(T_r)$, we have

$$\phi(u, T_r z_1) \leq \phi(u, z_1).$$

Next, we show that $\text{F} \text{I} \text{x}(T_r) = \text{Sol}(\text{GMVEP}(6.1.2))$. Indeed, let $p \in \text{F} \text{I} \text{x}(T_r)$. Then there exists $\{z_n\} \subset E$ such that $z_n \to p$ and $\lim_{n \to \infty} \|z_n - T_r z_n\| = 0$. Moreover, we get $T_r z_n \to p$. Hence, we have $p \in C$. Since $J$ is uniformly continuous on bounded sets, we have

$$\lim_{n \to \infty} \frac{\|Jz_n - JT_r z_n\|}{r} = 0, \quad r > 0. \quad (6.2.8)$$

From the definition of $T_r$, we have, $\forall y \in C$,

$$F(T_r z_n, y) + \langle y - T_r z_n, AT_r z_n \rangle + \psi(y, T_r z_n)$$

$$- \psi(T_r z_n, T_r z_n) + \frac{e}{r} \langle y - T_r z_n, JT_r z_n - Jz_n \rangle \in P.$$
Let \( y_t = (1 - t)p + ty \), \( \forall t \in (0, 1] \). Since \( y \in C \) and \( p \in C \), we get \( y_t \in C \) and hence

\[
0 \in F(y, T_r z_n) - \langle y - T_r z_n, AT_r z_n \rangle - \psi(y, T_r z_n) + \psi(T_r z_n, T_r z_n) - \frac{e}{r} \langle y - T_r z_n, JT_r z_n - J z_n \rangle + P.
\]

By using (6.2.9), we have

\[
0 \in F(y, T_r z_n) - \langle y - T_r z_n, AT_r z_n \rangle - \psi(y, T_r z_n) - \psi(T_r z_n, T_r z_n) - \frac{e}{r} \langle y - T_r z_n, JT_r z_n - J z_n \rangle + P.
\]

It follows from Assumption 6.2.1 (i), (iv) and (vi) that

\[
tF(y_t, y) + (1 - t)F(y_t, p) + t\psi(y, p) + (1 - t)\psi(p, p) - \psi(y_t, p) \in F(y_t, y_t) + \psi(y_t, p) - \psi(y_t, p) + P
\]

By using (6.2.9), we have

\[
- t[F(y_t, y) + \psi(y, p) - \psi(y_t, p) - \langle y_t - p, Ap \rangle] - \langle 1 - t \rangle [F(y_t, p) + \psi(p, p) - \psi(y_t, p) - \langle y_t - p, Ap \rangle] \in - P + \langle y_t - p, Ap \rangle
\]

\[
- t[F(y_t, y) + \psi(y, p) - \psi(y_t, p) - \langle y_t - p, Ap \rangle] - \langle 1 - t \rangle P \in - P + \langle y_t - p, Ap \rangle
\]

\[
- t[F(y_t, y) + \psi(y, p) - \psi(y_t, p) - \langle y_t - p, Ap \rangle] - \langle 1 - t \rangle P \in - P + \langle y_t - p, Ap \rangle
\]

\[
- t[F(y_t, y) + \psi(y, p) - \psi(y_t, p) - \langle y_t - p, Ap \rangle] - \langle 1 - t \rangle P \in - P + \langle y_t - p, Ap \rangle
\]

\[
- t[F(y_t, y) + \psi(y, p) - \psi(y_t, p) - \langle y_t - p, Ap \rangle] - \langle 1 - t \rangle P \in - P + \langle y_t - p, Ap \rangle
\]

\[
F(y_t, y) + \psi(y, p) - \psi(y_t, p) - \langle y_t - p, Ap \rangle \in P - \langle y - p, Ap \rangle.
\]
Letting $t \to 0_+$, we have

$$F(p, y) + \langle y - p, Ap \rangle + \psi(y, p) - \psi(p, p) \in P.$$ 

Thus $p \in \text{Sol}(GMVEP(6.1.2))$. So, we get $\text{Fix}(T_r) = \text{Sol}(GMVEP(6.1.2)) = \widehat{\text{Fix}}(T_r)$. Therefore $T_r$ is a relatively nonexpansive mapping. Further, it follows from Lemma 1.2.9 that $\text{Sol}(GMVEP(6.1.2)) = \text{Fix}(T_r)$ is closed and convex. This completes the proof. \(\square\)

Next, we have the following consequence of Theorem 6.2.1.

**Lemma 6.2.1.** Let $E$, $C$, $F$, $\psi$, $G_z$ be same as in Theorem 6.2.1 and let $r > 0$. Then, for $x \in E$ and $q \in \text{Fix}(T_r)$, we have

$$\phi(q, T_rx) + \phi(T_rx, x) \leq \phi(q, x).$$

### 6.3 Hybrid iterative method

We prove a strong convergence theorem for finding a common element to the set of solutions of $SUGMVEP(6.1.1)$ and set of fixed points of fixed point problems of two families of generalized asymptotically quasi-\(\phi\)-nonexpansive mappings in Banach space.

**Theorem 6.3.1.** Let $E$ be a uniformly smooth and strictly convex Banach space such that $E$ has Kadec-Klee property. For each $i \in I := \{1, 2, 3, ..., N\}$, let $K_i$ be a nonempty compact and convex subset of $E$ such that $K = \bigcap_{i=1}^{N} K_i \neq \emptyset$. Assume that $P$ is a pointed, proper, closed and convex cone of a real ordered Banach space $Y$ with $\text{int}P \neq \emptyset$.

Let for each $i$, the mappings $F_i, \psi_i : K_i \times K_i \to Y$ satisfy Assumption 6.2.1 and $A_i : K_i \to \text{B}(E, Y)$ be continuous and $P$-monotone mapping. For each fixed $i$, let $S_i, T_i : K_i \to E$ be closed, asymptotically regular and generalized asymptotically quasi-\(\phi\)-nonexpansive mappings with the sequences $\{\eta_{n,i}\}$, $\{\alpha_{n,i}\}$ and $\{\zeta_{n,i}\}$, $\{\xi_{n,i}\}$ such that $\Gamma := \left( \bigcap_{i=1}^{N} \text{Fix}(S_i) \right) \bigcap \left( \bigcap_{i=1}^{N} \text{Fix}(T_i) \right) \bigcap \left( \bigcap_{i=1}^{N} \text{Sol}(GMVEP(F_i, A_i, \psi_i, K_i)) \right) \neq \emptyset$. Assume that, for each fixed $i$, the sequences $\{r_{n,i}\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_{n,i}\} \subset (0, 1)$ and $\{\beta_{j,n,i}\} \subset (0, 1)$ $(j = 1, 2, 3)$ be such that

(i) $\beta_{1,n,i} + \beta_{2,n,i} + \beta_{3,n,i} = 1; \quad$
(ii) \( \liminf_{n \to \infty} \beta_{n,i}^1 \beta_{n,i}^2 > 0 \) and \( \liminf_{n \to \infty} \beta_{n,i}^1 \beta_{n,i}^3 > 0 \);

(iii) \( \limsup_{n \to \infty} \alpha_{n,i} < 1 \);

(iv) \( \liminf_{n \to \infty} r_{n,i} > 0 \).

Let \( \{x_n\} \) be a sequence generated by the iterative scheme:

\[
x_0 \in E,
C_{1,i} := K_i, \quad C_1 = \bigcap_{i=1}^N C_{1,i} = K,

x_1 = \prod C_1 x_0,

y_{n,i} = J^{-1}(\alpha_{n,i} Jx_n + (1 - \alpha_{n,i}) Jz_{n,i}),

z_{n,i} = J^{-1}(\beta_{n,i}^1 Jx_n + \beta_{n,i}^2 JT_i^n x_n + \beta_{n,i}^3 JS_i^n x_n),

u_{n,i} = T_{r_{n,i}}(y_{n,i})

C_{n+1,i} = \{v \in C_{n,i} : \phi(v, u_{n,i}) \leq \phi(v, x_n) + \delta_{n,i} M_n + \mu_{n,i}\},

C_{n+1} = \bigcap_{i \in I} C_{n+1,i},

x_{n+1} = \prod_{C_{n+1}} x_0, \quad \text{for every } n \in \mathbb{N} \cup \{0\},
\]

where \( M_n = \sup \{\phi(p, x_n) : p \in \Gamma; e \in \text{int} P; J : E \to E^* \text{ is the normalized duality mapping with its inverse } J^{-1}; \} \)

\[\delta_{n,i} = \beta_{n,i}^2 \zeta_{n,i} + \beta_{n,i}^3 \eta_{n,i} \text{ and } \mu_{n,i} = \xi_{n,i} \beta_{n,i}^2 + \varsigma_{n,i} \beta_{n,i}^3.\]

Then the sequence \( \{x_n\} \) converges strongly to \( \prod \Gamma x_0 \).

**Proof.** First, we show that \( C_n \) is closed and convex for every \( n \geq 1 \). It suffices to show that for each \( i \in I, C_{n,i} \) is closed and convex for every \( n \geq 1 \). This can be proved by induction on \( n \). In fact, for \( n = 1, C_{1,i} = K_i \) is closed and convex for each \( i \in I \). Assume that \( C_{n,i} \) is closed and convex for some \( n \geq 1 \) and for each \( i \in I \). For \( v \in C_{n+1,i}, \)

\[
\phi(v, u_{n,i}) \leq \phi(v, x_n) + \delta_{n,i} M_n + \mu_{n,i}
\]

which is equivalent to

\[
2\langle v, Jx_n - Ju_{n,i} \rangle \leq \|x_n\|^2 - \|u_{n,i}\|^2 + \delta_{n,i} M_n + \mu_{n,i}.
\]

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Further, it follows from Theorem 6.2.1 that $u$ are relatively nonexpansive. Therefore,

$$
\phi(w, z_{k,i}) = \phi(w, T_{r_{k,i}, y_{k,i}})
$$

Next, we prove $\Gamma \subset C_n$ for all $n \geq 1$. It suffices to show that for each $i \in I$, $\Gamma \subset C_{n,i}$. Indeed, $\Gamma \subset C_{1,i} = K_i$ is obvious. Suppose $\Gamma \subset C_{k,i}$ for some $k \geq 1$. Then, for all $w \in \Gamma \subset C_{k,i}$, we have

$$
\phi(w, z_{k,i}) = \phi(w, J^{-1}(\beta_{k,i}^1 Jx_k + \beta_{k,i}^2 J^T x_k + \beta_{k,i}^3 JS^k x_k))
\leq \|w\|^2 - 2\langle w, \beta_{k,i}^1 Jx_k + \beta_{k,i}^2 J^T x_k + \beta_{k,i}^3 JS^k x_k \rangle
+ \|\beta_{k,i}^1 Jx_k + \beta_{k,i}^2 J^T x_k + \beta_{k,i}^3 JS^k x_k\|^2
\leq \|w\|^2 - 2\beta_{k,i}^1 \langle w, Jx_k \rangle - 2\beta_{k,i}^2 \langle w, J^T x_k \rangle - 2\beta_{k,i}^3 \langle w, JS^k x_k \rangle
+ \beta_{k,i}^1 \|x_k\|^2 + \beta_{k,i}^2 \|T^k x_k\|^2 + \beta_{k,i}^3 \|S^k x_k\|^2
= \beta_{k,i}^1 \phi(w, x_k) + \beta_{k,i}^2 \phi(w, T^k x_k) + \beta_{k,i}^3 \phi(w, S^k x_k)
\leq \beta_{k,i}^1 \phi(w, x_k^k) + \beta_{k,i}^2 (1 + \xi_{k,i}) \phi(w, x_k) + \xi_{k,i} \beta_{k,i}^3
+ \beta_{k,i}^3 (1 + \eta_{k,i}) \phi(w, x_k) + \varsigma_{k,i} \beta_{k,i}^3
\leq \phi(w, x_k) + \delta_{k,i} \phi(w, x_k) + \mu_{k,i}. \quad (6.3.1)
$$

Further, it follows from Theorem 6.2.1 that $u_{n,i} = T_{r_{n,i}, y_{n,i}}$ for all $n \in N \cup \{0\}$, and $T_{r_{n,i}}$ are relatively nonexpansive. Therefore,

$$
\phi(w, u_{k,i}) = \phi(w, T_{r_{k,i}, y_{k,i}})
\leq \phi(w, y_{k,i})
= \phi(w, J^{-1}(\alpha_{k,i} Jx_k + (1 - \alpha_{k,i}) Jz_{k,i}))
= \|w\|^2 - 2\langle w, \alpha_{k,i} Jx_k + (1 - \alpha_{k,i}) Jz_{k,i} \rangle + \|\alpha_{k,i} Jx_k + (1 - \alpha_{k,i}) Jz_{k,i}\|^2
= \|w\|^2 - 2\alpha_{k,i} \langle w, Jx_k \rangle - 2(1 - \alpha_{k,i}) \langle w, Jz_{k,i} \rangle + \alpha_{k,i} \|x_k\|^2 + (1 - \alpha_{k,i}) \|z_{k,i}\|^2
= \alpha_{k,i} \phi(w, x_k) + (1 - \alpha_{k,i}) \phi(w, z_{k,i})
\leq \alpha_{k,i} \phi(w, x_k) + (1 - \alpha_{k,i}) \phi(w, x_k) - (1 - \alpha_{k,i})(\beta_{k,i}^2 \phi(w, x_k) + \beta_{k,i}^3 \phi(w, x_k))
+ (1 - \alpha_{k,i})(\xi_{k,i} \beta_{k,i}^3 + \varsigma_{k,i} \beta_{k,i}^3)
\[
\leq \phi(w, x_k) + (1 - \alpha_{k,i})[(\beta_{k,i}^2 \zeta_{k,i} + \beta_{k,i}^3 \eta_{k,i})\phi(w, x_k) + (\xi_{k,i}\beta_{k,i}^2 + \zeta_{k,i}\beta_{k,i}^3)] \\
\leq \phi(w, x_k) + (\beta_{k,i}^2 \zeta_{k,i} + \beta_{k,i}^3 \eta_{k,i})\phi(w, x_k) + (\xi_{k,i}\beta_{k,i}^2 + \zeta_{k,i}\beta_{k,i}^3) \\
\leq \phi(w, x_k) + \delta_{k,i} M_k + \mu_{k,i}. 
\] 

(6.3.2)

This shows that \( w \in C_{k+1} \). That is, \( \Gamma \subset C_{n,i} \), for all \( n \geq 1 \) and each \( i \in I \). Therefore, \( w \in C_n = \bigcap_{i \in I} C_{n,i} \) for all \( n \geq 1 \).

From Lemma 1.2.8, we have

\[
\phi(x_n, x_0) = \phi(\prod_{C_n} x_0, x_0) \\
\leq \phi(w, x_0) - \phi(w, x_n) \\
\leq \phi(w, x_0), \quad \text{for each } w \in \Gamma \subset C_n \text{ and for each } n \geq 1.
\]

Therefore, the sequence \( \{\phi(x_n, x_0)\} \) is bounded. It follows from (1.2.13) that the sequence \( \{x_n\} \) is also bounded. Since \( E \) is reflexive, without loss of generality, we may assume that \( x_n \to p \) as \( n \to \infty \). Since \( C_j \subset C_n \) for \( j \geq n \), we have \( x_j \in C_n \) for \( j \geq n \). Since \( C_n \) is closed and convex, \( p \in C_n \) for all \( n \geq 1 \). Hence \( p \in \bigcap_{n=1}^\infty C_n \). Since \( \phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \leq \phi(p, x_0) \), we have

\[
\phi(p, x_0) \leq \lim \inf_{n \to \infty} \phi(x_n, x_0) \leq \lim \sup_{n \to \infty} \phi(x_n, x_0) \leq \phi(p, x_0),
\]

which implies that \( \phi(x_n, x_0) \to \phi(p, x_0) \) as \( n \to \infty \). Hence \( \|x_n\| \to \|p\| \). By the Kadec-Klee property of \( E \), we have \( x_n \to p \in C \) as \( n \to \infty \).

Since \( x_n \to p \) and \( J : E \to E^* \) is demicontinuous, we have \( Jx_n \rightharpoonup Jp \in E^* \). Note that

\[
\|Jx_n - Jp\| = \|x_n - p\| \leq \|x_n - p\|.
\]

This implies that \( \|Jx_n\| \to ||Jp|| \). Since \( E \) is uniformly smooth, \( E^* \) is uniformly convex Banach space and hence it enjoys the Kadec-Klee property, we see that

\[
\lim_{n \to \infty} \|Jx_n - Jp\| = 0.
\]

(6.3.3)

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On the other hand, since \( x_n = \prod_{C_n} x_0 \) and \( x_{n+1} = \prod_{C_{n+1}} x_0 \in C_{n+1} \subseteq C_n \), we have

\[
\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \text{for all } n \geq 1.
\]

Therefore, \( \{\phi(x_n, x_0)\} \) is nondecreasing. Further, it follows that the limit of \( \{\phi(x_n, x_0)\} \) exists. By the construction of \( C_n \), one has that \( C_m \subseteq C_n \) and \( x_m = \prod_{C_m} x_0 \in C_n \) for any positive integer \( m \geq n \). It follows that

\[
\phi(x_m, x_n) = \phi(x_m, \prod_{C_n} x_0) \\
\leq \phi(x_m, x_0) - \phi(\prod_{C_n} x_0, x_0) \\
= \phi(x_m, x_0) - \phi(x_n, x_0).
\]

Letting \( m, n \to \infty \) in (6.3.4), we have \( \phi(x_m, x_n) \to 0 \).

Next, we show that \( p \in (\bigcap_{i=1}^N \text{Fix}(S_i)) \bigcap (\bigcap_{i=1}^N \text{Fix}(T_i)) \). By taking \( m = n + 1 \) in (6.3.4), we have

\[
\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0. \tag{6.3.5}
\]

Noticing that \( x_{n+1} \in C_{n+1} \), from the definition of \( C_n \), for every \( i \in I \), we have

\[
\phi(x_{n+1}, u_{n,i}) \leq \phi(x_{n+1}, x_n) + \delta_{n,i} M_n + \mu_{n,i}. \tag{6.3.6}
\]

It follows from (6.3.5) and (6.3.6) that

\[
\lim_{n \to \infty} \phi(x_{n+1}, u_{n,i}) = 0. \tag{6.3.7}
\]

It follows from (6.3.7) and inequality \( 0 \leq (\|x_{n+1}\| - \|u_{n,i}\|)^2 \leq \phi(x_{n+1}, u_{n,i}) \) that \( \|u_{n,i}\| \to \|p\| \) and consequently, we have \( \|Ju_{n,i}\| \to \|Jp\| \). This implies that \( \{J(u_{n,i})\} \) is bounded. Since \( E \) is reflexive, \( E^* \) is also reflexive. So, we may assume that \( J(u_{n,i}) \rightharpoonup f_{0,i} \in E^* \). On the other hand, in view of the reflexivity of \( E \), we have \( J(E) = E^* \), which means that for \( f_{0,i} \in E^* \), there exists \( e_i \in E \), such that \( Je_i = f_{0,i} \). Using weakly lower semi-continuity of \( \|\cdot\|^2 \), we have
\[
\lim \inf_{n \to \infty} \phi(x_{n+1}, u_{n,i}) = \lim \inf_{n \to \infty} \left( \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_{n,i} \rangle + \|u_{n,i}\|^2 \right)
= \lim \inf_{n \to \infty} \left( \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_{n,i} \rangle + \|Ju_{n,i}\|^2 \right)
\geq \|p\|^2 - 2\langle p, f_{0,i} \rangle + \|f_{0,i}\|^2
= \|p\|^2 - 2\langle p, Je_i \rangle + \|Je_i\|^2
= \phi(p, e_i).
\]

It follows from (6.3.7) that \(\phi(p, e_i) = 0\). Hence \(p = e_i\), which implies that \(f_{0,i} = Jp\).

Hence \(Ju_{n,i} \rightharpoonup Jp \in E^*\). Since \(\|Ju_{n,i}\| \to \|Jp\|\) and by the Kadec-Klee property of \(E^*\), we have

\[
\|Ju_{n,i} - Jp\| \to 0, \quad \forall i \in I. \tag{6.3.8}
\]

Since \(J^{-1} : E^* \to E\) is demicontinuous, therefore \(u_{n,i} \rightharpoonup p\). Since \(\|u_{n,i}\| \to \|p\|\) and using the Kadec-Klee property of \(E^*\), we have

\[
u_{n,i} \to p, \quad \text{as} \ n \to \infty, \quad \forall i \in I. \tag{6.3.9}
\]

Hence,

\[
\lim_{n \to \infty} \|x_n - u_{n,i}\| = 0, \quad \forall i \in I. \tag{6.3.10}
\]

Since \(J\) is uniformly norm-to-norm continuous on bounded sets, we have

\[
\lim_{n \to \infty} \|Jx_n - Ju_{n,i}\| = 0, \quad \forall i \in I. \tag{6.3.11}
\]

Now,

\[
\phi(w, x_n) - \phi(w, u_{n,i}) = \|x_n\|^2 - \|u_{n,i}\|^2 - 2\langle w, Jx_n - Ju_{n,i} \rangle
\leq \|x_n - u_{n,i}\| (\|x_n\| + \|u_{n,i}\|) + 2\|w\| \|Jx_n - Ju_{n,i}\|.
\]
It follows from (6.3.10) and (6.3.11), we have

\[ \phi(w, x_n) - \phi(w, u_{n,i}) \to 0, \text{ as } n \to \infty. \quad (6.3.12) \]

From \( u_{n,i} = T_{r_{n,i}}y_{n,i} \) and Lemma 6.2.1, we have

\[
\begin{align*}
\phi(u_{n,i}, y_{n,i}) &= \phi(T_{r_{n,i}}y_{n,i}, y_{n,i}) \\
&\leq \phi(w, y_{n,i}) - \phi(w, u_{n,i}) \\
&\leq \alpha_{n,i}\phi(w, x_n) + (1 - \alpha_{n,i})\phi(w, z_{n,i}) - \phi(w, u_{n,i}) \\
&\leq \alpha_{n,i}\phi(w, x_n) + (1 - \alpha_{n,i})[\phi(w, x_n) + \delta_{n,i}M_n + \mu_{n,i}] \\
&\quad - \beta_{n,i}^3\|Jx_n - J\tilde{x}_n\| - \phi(w, u_{n,i}) \\
&\leq \phi(w, x_n) - \phi(w, u_{n,i}) + (1 - \alpha_{n,i})[\delta_{n,i}M_n + \mu_{n,i}].
\end{align*}
\]

(6.3.13)

Using (6.3.12) and the restrictions on the sequences in (6.3.13), we have

\[ \phi(u_{n,i}, y_{n,i}) \to 0, \text{ as } n \to \infty. \quad (6.3.14) \]

Since \( 0 \leq (\|u_{n,i}\| - \|y_{n,i}\|)^2 \leq \phi(u_{n,i}, y_{n,i}) \), it follows from (6.3.9) that \( \|y_{n,i}\| \to \|p\| \) and consequently \( \|Jy_{n,i}\| \to \|Jp\| \). This implies that \( \{Jy_{n,i}\} \) is bounded. Since \( E \) is reflexive, \( E^* \) is also reflexive, so we may assume that \( J(y_{n,i}) \to h_{0,i} \in E^* \). On the other hand, in view of the reflexivity of \( E \), we have \( J(E) = E^* \), which means that for \( h_{0,i} \in E^* \), there exists \( d_i \in E \), such that \( Jd_i = h_{0,i} \). Using lower semi-continuity of \( \|\cdot\|^2 \), we have

\[
\liminf_{n \to \infty} \phi(u_{n,i}, y_{n,i}) = \liminf_{n \to \infty}(\|u_{n,i}\|^2 - 2\langle u_{n,i}, Jy_{n,i} \rangle + \|y_{n,i}\|^2) \\
= \liminf_{n \to \infty}(\|u_{n,i}\|^2 - 2\langle u_{n,i}, Jy_{n,i} \rangle + \|y_{n,i}\|^2) \\
\geq \|p\|^2 - 2\langle p, h_{0,i} \rangle + \|h_{0,i}\|^2 \\
= \|p\|^2 - 2\langle p, Jd_i \rangle + \|d_i\|^2 \\
= \phi(p, d_i).
\]

It follows from (6.3.14) that \( \phi(p, d_i) = 0 \). Hence \( p = d_i \), which implies that \( h_{0,i} = Jp \).
Hence $J_{y,n,i} \rightharpoonup Jp \in E^*$. Since $J_{y,n,i} \to Jp$ and Kadec-Klee property of $E^*$, we have $J_{y,n,i} - Jp \to 0$. Since $J^{-1} : E^* \to E$ is demicontinuous, we have $y_{n,i} \to p$. Further, since $\|y_{n,i}\| \to \|p\|$ and $E$ has the Kadec-Klee property, we have

$$\|y_{n,i} - p\| \to 0, \text{ as } n \to \infty. \quad (6.3.15)$$

It follows from (6.3.9) and (6.3.15) that

$$\|y_{n,i} - u_{n,i}\| \leq \|u_{n,i} - p\| + \|y_{n,i} - p\| \to 0, \text{ as } n \to \infty.$$

Since $J$ is uniformly norm-to-norm continuous on bounded sets, we have

$$\|J_{y,n,i} - Ju_{n,i}\| \to 0, \text{ as } n \to \infty. \quad (6.3.16)$$

It follows from (6.3.11) and (6.3.16) that

$$\|Jx_n - J_{y,n,i}\| \leq \|Jx_n - Ju_{n,i}\| + \|Ju_{n,i} - J_{y,n,i}\| \to 0, \text{ as } n \to \infty. \quad (6.3.17)$$

From iterative scheme, (6.3.17) and $\limsup_{n \to \infty} \alpha_{n,i} < 1$, it follows that

$$\|Jz_{n,i} - Jx_n\| \leq \frac{1}{1 - \alpha_{n,i}} \|Jx_n - J_{y,n,i}\| \to 0, \text{ as } n \to \infty. \quad (6.3.18)$$

Hence,

$$\|Jz_{n,i} - Jp\| \leq \|Jz_{n,i} - Jx_n\| + \|Jx_n - Jp\| \to 0, \text{ as } n \to \infty. \quad (6.3.19)$$

Since $J^{-1} : E^* \to E$ is demicontinuous, we have $z_{n,i} \to p$. From (6.3.19), we have that

$$\|z_{n,i} - \|p\| = \|Jz_{n,i} - \|p\|\| \leq \|Jz_{n,i} - Jp\| \to 0, \text{ as } n \to \infty.$$
This implies that \( \| z_{n,i} \| \to \| p \| \). Since \( E \) enjoys the Kadec-Klee property, we see that

\[
\lim_{n \to \infty} \| z_{n,i} - p \| = 0. \tag{6.3.20}
\]

Further,

\[
\| z_{n,i} - x_n \| \leq \| z_{n,i} - p \| + \| x_n - p \| \to 0, \quad \text{as } n \to \infty. \tag{6.3.21}
\]

Now,

\[
\phi(w, x_n) - \phi(w, z_{n,i}) = \| x_n \|^2 - \| z_{n,i} \|^2 - 2 \langle w, Jx_n - Jz_{n,i} \rangle
\]

\[
\leq \| x_n - z_{n,i} \| (\| x_n \| + \| z_{n,i} \|) + 2 \| w \| \| Jx_n - Jz_{n,i} \|.
\]

It follows from (6.3.18) and (6.3.21) that

\[
\phi(w, x_n) - \phi(w, z_{n,i}) \to 0, \quad \text{as } n \to \infty. \tag{6.3.22}
\]

Let \( r = \max \left\{ \sup_{n \geq 1} \{ \| x_n \| \}, \sup_{n \geq 1} \{ \| T^n_i x_n \| \}, \sup_{n \geq 1} \{ \| S^n_i x_n \| \} \right\} \). Since \( E \) is uniformly smooth, then \( E^* \) is uniformly convex. In the light of Lemma 1.2.12, we have, for any fixed \( w \in \text{Fix}(T_i) \cap \text{Fix}(S_i) \),

\[
\phi(w, z_{n,i}) = \phi(w, J^{-1}(\beta_{n,i}^1 Jx_n + \beta_{n,i}^2 JT^n_i x_n + \beta_{n,i}^3 JS^n_i x_n))
\]

\[
= \| w \|^2 - 2 \langle w, \beta_{n,i}^1 Jx_n + \beta_{n,i}^2 JT^n_i x_n + \beta_{n,i}^3 JS^n_i x_n \rangle
\]

\[
+ \| \beta_{n,i}^1 Jx_n + \beta_{n,i}^2 JT^n_i x_n + \beta_{n,i}^3 JS^n_i x_n \|^2
\]

\[
\leq \| w \|^2 - 2 \beta_{n,i}^1 \langle w, Jx_n \rangle - 2 \beta_{n,i}^2 \langle w, JT^n_i x_n \rangle - 2 \beta_{n,i}^3 \langle w, JS^n_i x_n \rangle
\]

\[
+ \beta_{n,i}^1 \| x_n \|^2 + \beta_{n,i}^2 \| T^n_i x_n \|^2 + \beta_{n,i}^3 \| S^n_i x_n \|^2 - \beta_{n,i}^1 \beta_{n,i}^3 g(\| Jx_n - JS^n_i x_n \|)
\]

\[
= \beta_{n,i}^1 \phi(w, x_n) + \beta_{n,i}^2 (1 - \zeta_{n,i}) \phi(w, x_n) + \beta_{n,i}^3 \xi_{n,i}
\]

\[
+ \beta_{n,i}^3 (1 + \eta_{n,i}) \phi(w, x_n) + \beta_{n,i}^3 \zeta_{n,i} - \beta_{n,i}^3 \beta_{n,i}^3 g(\| Jx_n - JS^n_i x_n \|)
\]

\[
= \phi(w, x_n) + (\beta_{n,i}^2 \zeta_{n,i} + \beta_{n,i}^3 \eta_{n,i}) \phi(w, x_n)
\]

\[
+ \beta_{n,i}^2 \xi_{n,i} + \beta_{n,i}^3 \zeta_{n,i} - \beta_{n,i}^3 \beta_{n,i}^3 g(\| Jx_n - JS^n_i x_n \|)
\]

\[
= \phi(w, x_n) + \delta_{n,i} \phi(w, x_n) + \mu_{n,i} - \beta_{n,i}^1 \beta_{n,i}^3 g(\| Jx_n - JS^n_i x_n \|)
\]

\[
\leq \phi(w, x_n) + \delta_{n,i} M_n + \mu_{n,i} - \beta_{n,i}^1 \beta_{n,i}^3 g(\| Jx_n - JS^n_i x_n \|).
\]
This implies that
\[
\beta_{n,i}^1 \beta_{n,i}^3 g(\|Jx_n - JS_i^n x_n\|) \leq \phi(w, x_n) - \phi(w, z_n) + \delta_{n,i} M_n + \mu_{n,i}. \tag{6.3.23}
\]

Taking limit on both sides of (6.3.23) and using \(\liminf_{n \to \infty} \beta_{n,i}^1 \beta_{n,i}^3 > 0\) and (6.3.22), we have
\[
g(\|Jx_n - JS_i^n x_n\|) \to 0, \quad \text{as } n \to \infty. \tag{6.3.24}
\]

Therefore, using Lemma 1.2.12, (6.3.24) implies that
\[
\|Jx_n - JS_i^n x_n\| \to 0, \quad \text{as } n \to \infty. \tag{6.3.25}
\]

Next,
\[
\|JS_i^n x_n - Jp\| \leq \|JS_i^n x_n - Jx_n\| + \|Jx_n - Jp\|.
\]

Using (6.3.25) and (6.3.3) in above inequality, we get
\[
\lim_{n \to \infty} \|JS_i^n x_n - Jp\| = 0. \tag{6.3.26}
\]

Since \(J^{-1}\) is demicontinuous, it follows that \(S_i^n x_n \rightharpoonup p\) for each fixed \(i\). Using \(\|S_i^n x_n\| - \|p\| = \|JS_i^n x_n - Jp\| \leq \|JS_i^n x_n - Jp\|\) with (6.3.26), we get \(S_i^n x_n \to \|p\|\) for each fixed \(i\) as \(n \to \infty\). Since \(E\) enjoys the Kadec-Klee property, we obtain
\[
\lim_{n \to \infty} \|S_i^n x_n - p\| = 0. \tag{6.3.27}
\]

Similarly, we obtain
\[
\lim_{n \to \infty} \|T_i^n x_n - p\| = 0. \tag{6.3.28}
\]

Notice that
\[
\|S_i^{n+1} x_n - p\| \leq \|S_i^{n+1} x_n - S_i^n x_n\| + \|S_i^n x_n - p\|. \tag{6.3.29}
\]

Since \(S_i^n\) is asymptotic regular, then from (6.3.27) and (6.3.29), we have
\[
\lim_{n \to \infty} \|S_i^{n+1} x_n - p\| = 0.
\]
that is, $S_i^n x_n - p \to 0$ as $n \to \infty$. It follows from the closedness of $S_i$ that $S_i p = p$, for each fixed $i$. Hence $p \in \bigcap_{i=1}^N \text{Fix}(S_i)$. In similar way, we can obtain $p \in \bigcap_{i=1}^N \text{Fix}(T_i)$.

Hence $p \in \left( \bigcap_{i=1}^N \text{Fix}(T_i) \right) \cap \left( \bigcap_{i=1}^N \text{Fix}(S_i) \right)$.

Next, we prove $p \in \bigcap_{i=1}^N \text{Sol}(\text{GMVEP}(F_i, A_i, \psi_i, K_i))$. It follows from (6.3.16) and $\lim_{n \to \infty} r_{n,i} > 0$ that

$$\lim_{n \to \infty} \frac{\|J_{n,i} y_n - J_{T_{n,i} y_{n,i}}\|}{r_{n,i}} = 0.$$ 

Since $u_{n,i} = T_{n,i} y_{n,i}$, we have

$$F(T_{n,i} y_{n,i}, y_i) + \langle y_i - T_{n,i} y_{n,i}, AT_{n,i} y_{n,i} \rangle + \psi(y_i, T_{n,i} y_{n,i}) - \psi(T_{n,i} y_{n,i}, T_{n,i} y_{n,i}) + \frac{e}{r_{n,i}} \langle y_i - T_{n,i} y_{n,i}, J_{T_{n,i} y_{n,i}} - J y_{n,i} \rangle + P, \quad \forall y_i \in K_i,$$

Let $y_{i,t} = (1-t)p + ty_i, \forall t \in (0,1]$. Since $y_i \in K_i$ and $p \in K_i$, we get $y_{i,t} \in K_i$ and hence

$$0 \in F(y_i, T_{n,i} y_{n,i}) - \langle y_{i,t} - T_{n,i} y_{n,i}, AT_{n,i} y_{n,i} \rangle - \psi(y_{i,t}, T_{n,i} y_{n,i}) + \psi(T_{n,i} y_{n,i}, T_{n,i} y_{n,i}) + \frac{e}{r_{n,i}} \langle y_i - T_{n,i} y_{n,i}, J_{T_{n,i} y_{n,i}} - J y_{n,i} \rangle + P, \quad \forall y_i \in K_i.$$

It follows from Assumption 6.2.1 (i), (iv) and (vi) that

$$tF(y_{i,t}, y_i) + (1-t)F(y_i, p) + t\psi(y_i, p) + (1-t)\psi(p, p) - \psi(y_{i,t}, p)$$

$$\in F(y_i, p) + \psi(y_{i,t}, p) - \psi(p, p) + P$$

$$\in P.$$
By using (6.2.9), we have to estimate

\[-t[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap \rangle] -
(1-t)[F(y_{i,t}, y_i) + \psi(p, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap \rangle] \in -P + \langle y_{i,t} - p, Ap \rangle
\]

\[-t[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap \rangle] - (1-t)P \in -P + \langle y_{i,t} - p, Ap \rangle
\]

\[-t[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap \rangle] \in -P + (1-t)P + \langle y_{i,t} - p, Ap \rangle
\]

\[\in -tP + t\langle y_i - (1-t)p - p, Ap \rangle
\]

\[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap \rangle \in P - \langle y_i - p, Ap \rangle.
\]

Letting \( t \to 0 \), we have

\[F(p, y_i) + \psi(y_i, p) - \psi(p, p) - \langle p - p, Ap \rangle + \langle y_i - p, Ap \rangle \in P
\]

\[F(p, y_i) + \langle y_i - p, Ap \rangle + \psi(y_i, p) - \psi(p, p) \in P.
\]

Thus \( p \in \text{Sol}(\text{GMVEP}(F_i, A_i, \psi_i, K_i)). \) Hence \( p \in \Gamma \). Finally, we prove that \( \hat{x} = \prod_{\Gamma} x \).

From \( x_n = \prod_{C^n} x_0 \), we have

\[\langle x_n - u, Jx_0 - Jx_n \rangle \geq 0, \quad \forall u \in C_n.\]

Since \( \Gamma \subset C^n \) for all \( n \geq 0 \), we have

\[\langle x_n - w, Jx_0 - Jx_n \rangle \geq 0, \quad \forall w \in \Gamma. \quad (6.3.30)\]

By taking the limit in (6.3.30), we have

\[\langle \hat{x} - z, Jx - J\hat{x} \rangle \geq 0, \quad \forall z \in \Gamma.
\]

Further, in view of Lemma 1.2.8, we see that \( \hat{x} = \prod_{\Gamma} x \). This completes the proof.
6.4 Consequences

We have the following consequences of Theorem 6.3.1.

**Corollary 6.4.1.** Let $E$ be a uniformly smooth and strictly convex Banach space such that $E$ has Kadec-Klee property. Let $K$ be a nonempty compact and convex subset of $E$. Assume that $P$ is a pointed, proper, closed and convex cone of a real ordered Banach space $Y$ with $\text{int}P \neq \emptyset$. Let the mappings $F, \psi : K \times K \to Y$ satisfy Assumption 6.2.1 and $A : K \to B(E,Y)$ be a continuous and $P$-monotone mapping. Let $S, T : K \to E$ be closed, asymptotically regular and generalized asymptotically quasi-\(\phi\)-nonexpansive mappings with the sequences \(\{\eta_n\}, \{\zeta_n\}, \{\xi_n\}\) such that \(\Gamma := \text{Fix}(S) \cap (\text{Fix}(T)) \cap (\text{Sol}(\text{GMVEP}(6.1.2))) \neq \emptyset\). Assume that, the sequences \(\{r_n\} \subset [a, \infty)\) for some $a > 0$, \(\{\alpha_n\} \subseteq (0, 1)\) and \(\{\beta^{(j)}_n\} \subseteq (0, 1)\) ($j = 1, 2, 3$) be such that

(i) \(\beta^1_n + \beta^2_n + \beta^3_n = 1\);

(ii) \(\lim \inf_{n \to \infty} \beta^1_n \beta^2_n > 0\) and \(\lim \sup_{n \to \infty} \beta^2_n \beta^3_n = 0\);

(iii) \(\lim \sup_{n \to \infty} \alpha_n < 1\);

(iv) \(\lim \inf_{n \to \infty} r_n > 0\).

Let \(\{x_n\}\) be a sequence generated by the iterative scheme:

\[x_0 \in E,\]
\[x_1 = \prod_{C_1} x_0,\]
\[y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n),\]
\[z_n = J^{-1}(\beta^1_n Jx_n + \beta^2_n JT^n x_n + \beta^3_n JS^n x_n),\]
\[u_n = Tr_n(y_n),\]
\[C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n) + \delta_n M_n + \mu_n\},\]
\[x_{n+1} = \prod_{C_{n+1}} x_0, \quad \text{for every } n \in N \cup \{0\},\]

where $M_n = \sup \{\phi(p, x_n) : p \in \Gamma\}; e \in \text{int}P; J : E \to E^*$ is the normalized duality mapping with its inverse $J^{-1}$; \(\delta_n = \beta^2_n \zeta_n + \beta^3_n \eta_n\) and \(\mu_n = \xi_n \beta^2_n + \zeta_n \beta^3_n\).

Then the sequence \(\{x_n\}\) converges strongly to $\prod_{\Gamma} x_0$. 

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Proof. The proof follows by taking $i = 1$ in Theorem 6.3.1. \hfill \square

**Corollary 6.4.2.** Let $E$ be a uniformly smooth and strictly convex Banach space such that $E$ has Kadec-Klee property. For each $i \in I := \{1, 2, 3, \ldots, N\}$, let $K_i$ be a nonempty compact and convex subset of $E$ such that $K = \bigcap_{i=1}^{N} K_i \neq \emptyset$. Let for each $i$, the mappings $F_i, \psi_i : K_i \times K_i \rightarrow \mathbb{R}$ satisfy Assumption 6.2.1 and $A_i : K_i \rightarrow E^*$ be continuous and monotone mapping. For each fixed $i$, let $S_i, T_i : K_i \rightarrow E$ be closed, asymptotically regular and generalized asymptotically quasi $\phi$-nonexpansive mappings with the sequences $\{\eta_{n,i}\}, \{\varsigma_{n,i}\}, \{\zeta_{n,i}\}$ and $\{\xi_{n,i}\}$ such that $\Gamma := (\bigcap_{i=1}^{N} \text{Fix}(S_i)) \bigcap (\bigcap_{i=1}^{N} \text{Fix}(T_i)) \bigcap (\bigcap_{i=1}^{N} \text{Sol}(SUGMEP (6.1.3))) \neq \emptyset$. Assume that, for each fixed $i$, the sequences $\{r_{n,i}\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_{n,i}\} \subseteq (0, 1)$ and $\{\beta_{n,i}^{j}\} \subseteq (0, 1) \ (j = 1, 2, 3)$ be such that

(i) $\beta_{n,i}^{1} + \beta_{n,i}^{2} + \beta_{n,i}^{3} = 1$;

(ii) $\lim \inf_{n \rightarrow \infty} \beta_{n,i}^{1} \beta_{n,i}^{2} > 0$ and $\lim \inf_{n \rightarrow \infty} \beta_{n,i}^{1} \beta_{n,i}^{3} > 0$;

(iii) $\lim \sup_{n \rightarrow \infty} \alpha_{n,i} < 1$;

(iv) $\lim \inf_{n \rightarrow \infty} r_{n,i} > 0$.

Let $\{x_n\}$ be a sequence generated by the iterative scheme:

$x_0 \in E$,

$C_{1,i} := K_i, \ C_1 = \bigcap_{i=1}^{N} C_{1,i} = K$,

$x_1 = \Pi_{C_1} x_0$,

$y_{n,i} = J^{-1}(\alpha_{n,i} J x_n + (1 - \alpha_{n,i}) J z_{n,i})$,

$z_{n,i} = J^{-1}(\beta_{n,i}^{1} J x_n + \beta_{n,i}^{2} J T_i x_n + \beta_{n,i}^{3} J S_i x_n)$,

$u_{n,i} = T_{r_{n,i}}(y_{n,i})$,

$C_{n+1,i} = \{v \in C_{n,i} : \phi(v, u_{n,i}) \leq \phi(v, x_n) + \delta_{n,i} M_n + \mu_{n,i}\}$,

$C_{n+1} = \bigcap_{i \in I} C_{n+1,i}$,

$x_{n+1} = \Pi_{C_{n+1}} x_0, \ \text{for every} \ n \in \mathbb{N} \cup \{0\},$

where $M_n = \sup \{\phi(p, x_n) : p \in \Gamma\}; \ e \in \text{int}P; \ J : E \rightarrow E^*$ is the normalized duality mapping with its inverse $J^{-1}$; $\delta_{n,i} = \beta_{n,i}^{2} \zeta_{n,i} + \beta_{n,i}^{3} \eta_{n,i}$ and $\mu_{n,i} = \xi_{n,i} \beta_{n,i}^{2} + \varsigma_{n,i} \beta_{n,i}^{3}$.

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Then the sequence \( \{x_n\} \) converges strongly to \( \prod_\Gamma x_0 \).

**Proof.** The proof follows by taking \( Y = \mathbb{R}, \ P = [0, \infty) \) in Theorem 6.3.1.

**Corollary 6.4.3.** Let \( E \) be a uniformly smooth and strictly convex Banach space such that \( E \) has Kadec-Klee property. Let \( K \) be a nonempty compact and convex subset of \( E \). Let the mappings \( F, \psi : K \times K \to \mathbb{R} \) satisfy Assumption 6.2.1 and \( A : K \to E^* \) be continuous and monotone mapping. Let \( S, T : K \to E \) be closed, asymptotically regular and generalized asymptotically quasi \( \phi \)-nonexpansive mappings with the sequences \( \{\eta_n\}, \{\varsigma_n\}, \{\zeta_n\} \) such that \( \Gamma := \text{Fix}(S) \cap \text{Fix}(T) \cap \text{Sol}(\text{GMEP}(6.1.4)) \neq \emptyset \). Assume that, the sequences \( \{r_n\} \subset [a, \infty) \) for some \( a > 0 \), \( \{\alpha_n\} \subseteq (0, 1) \) and \( \{\beta_n^j\} \subseteq (0, 1) \ (j = 1, 2, 3) \) be such that

1. \( \beta_n^1 + \beta_n^2 + \beta_n^3 = 1; \)
2. \( \liminf_{n \to \infty} \beta_n^1 \beta_n^2 > 0 \) and \( \lim_{n \to \infty} \beta_n^1 \beta_n^3 = 0; \)
3. \( \limsup_{n \to \infty} \alpha_n < 1; \)
4. \( \liminf_{n \to \infty} r_n > 0. \)

Let \( \{x_n\} \) be a sequence generated by the iterative scheme:

\[
x_0 \in E, \\
x_1 = \prod_{C_1} x_0; \\
y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\
z_n = J^{-1}(\beta_n^1 Jx_n + \beta_n^2 JT^n x_n + \beta_n^3 JS^n x_n), \\
u_n = T_{r_n}(y_n), \\
C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n) + \delta_n M_n + \mu_n\}, \\
x_{n+1} = \prod_{C_{n+1}} x_0, \quad \text{for every} \ n \in N \cup \{0\},
\]

where \( M_n = \sup\{\phi(p, x_n) : p \in \Gamma\}; \ e \in \text{int} P; \ J : E \to E^* \) is the normalized duality mapping with its inverse \( J^{-1}; \ \delta_n = \beta_n^2 \zeta_n + \beta_n^3 \eta_n \) and \( \mu_n = \xi_n \beta_n^2 + \varsigma_n \beta_n^3. \)

Then the sequence \( \{x_n\} \) converges strongly to \( \prod_\Gamma x_0. \)
Proof. The proof follows by taking $Y = \mathbb{R}$, $P = [0, \infty)$, $i = 1$ in Theorem 6.3.1. The following Corollary is due to Qin and Agrawal [141].

**Corollary 6.4.4.** Let $E$ be a uniformly smooth and strictly convex Banach space such that $E$ has Kadec-Klee property. Let $K$ be a nonempty compact and convex subset of $E$. Let $S, T : K \to E$ be closed, asymptotically regular and generalized asymptotically quasi-$\phi$-nonexpansive mappings with the sequences $\{\eta_n\}$, $\{\varsigma_n\}$ and $\{\zeta_n\}$, $\{\xi_n\}$ such that $\Gamma := \text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$. Assume that, the sequences $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta^j_n\} \subseteq (0, 1)$ $(j = 1, 2, 3)$ be such that

(i) $\beta^1_n + \beta^2_n + \beta^3_n = 1$;

(ii) $\lim \inf_{n \to \infty} \beta^1_n \beta^2_n > 0$ and $\lim \beta^1_n \beta^3_n = 0$;

(iii) $\lim \sup_{n \to \infty} \alpha_n < 1$;

Let $\{x_n\}$ be a sequence generated by the iterative scheme:

\[
x_0 \in E,
\]

\[
x_1 = \prod_{C_1} x_0,
\]

\[
y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J z_n),
\]

\[
z_n = J^{-1}(\beta^1_n J x_n + \beta^2_n J T^n x_n + \beta^3_n J S^n x_n),
\]

\[
C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n) + \delta_n M_n + \mu_n\},
\]

\[
x_{n+1} = \prod_{C_{n+1}} x_0, \quad \text{for every } n \in N \cup \{0\},
\]

where $M_n = \sup \{\phi(p, x_n) : p \in \Gamma\}; e \in \text{int} P; J : E \to E^*$ is the normalized duality mapping with its inverse $J^{-1}$; $\delta_n = \beta^2_n \varsigma_n + \beta^3_n \eta_n$ and $\mu_n = \xi_n \beta^2_n + \zeta_n \beta^3_n$.

Then the sequence $\{x_n\}$ converges strongly to $\prod_{\Gamma} x_0$.

Proof. The proof follows by taking $Y = \mathbb{R}$, $P = [0, \infty)$, $i = 1$, and $F = 0$ in Theorem 6.3.1.