INTEGRAL SOLUTIONS OF \[ x^3 + y^3 + 8k(x+y) = (2k+1)z^3 \]

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ABSTRACT

The ternary cubic diophantine equation \[ x^3 + y^3 + 8k(x+y) = (2k+1)z^3 \] is analysed for its non-zero distinct integral solutions. A few relations among the solutions are presented.

Key words: Ternary cubic diophantine equation, integral solutions.

INTRODUCTION

Integral solutions for the cubic homogeneous or non-homogeneous diophantine equations is an interesting concept, as it can be seen from [1,2,3]. In [4,5,6], integral solutions of binary cubic non-homogeneous diophantine equations have been studied. In [3], two double parameter solutions for the equation \[ x^3 + y^3 = z^3 \] are given. In [7,8,9,10,11,12,13] a few special cases of ternary cubic diophantine equations are studied.

In this communication, we present the integral solutions of yet another ternary cubic equation \[ x^3 + y^3 + 8k(x+y) = (2k+1)z^3 \]. A few interesting relations between the solutions are obtained.

METHOD OF ANALYSIS

The diophantine equation to be solved is

\[ x^3 + y^3 + 8k(x+y) = (2k+1)z^3 \] (1)

Taking the linear transformations

\[ x = u+v, \quad y = u-v, \quad \text{and} \quad z = 2u \] (2)

in (1), it reduces to

\[ (8k+3)u^3 - 3v^2 = 8k \] (3)

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The assumptions, \( k = 1, \ u = X + 3T, \ v = X + 11T \)
in (3) lead to

\[
X^2 = 33T^2 + 1
\]  

(5)

The general solutions of the Pellian (5), is represented by

\[
X_n + \sqrt{33} T_n = (23 + 4\sqrt{33})^{n+1}, \ n = 0, 1, 2, ...
\]

(6)

since irrational roots occurs in pairs, we have

\[
X_n - \sqrt{33} T_n = (23 - 4\sqrt{33})^{n+1}, \ n = 0, 1, 2, ...
\]

(7)

From (6) and (7), we get

\[
X_n = \frac{f}{2}, \ T_n = \frac{g}{2\sqrt{33}}
\]

Where

\[
f = (23 + 4\sqrt{33})^{n+1} + (23 - 4\sqrt{33})^{n+1}
\]

and

\[
g = (23 + 4\sqrt{33})^{n+1} - (23 - 4\sqrt{33})^{n+1}
\]

The integral solutions of (1) are

\[
x_n = f + \frac{7g}{\sqrt{33}}
\]

\[
y_n = -\frac{4g}{\sqrt{33}}
\]

\[
z_n = f + \frac{3g}{\sqrt{33}}, \ n = 0, 1, 2, ...
\]

The recurrence relations satisfied by the solutions of \( x_n, y_n \) and \( z_n \) are respectively given by

\[
x_{n+2} - 46x_{n+1} + x_n = 0
\]

\[
y_{n+2} - 46y_{n+1} + y_n = 0
\]

\[
z_{n+2} - 46z_{n+1} + z_n = 0
\]

A few observations are presented below:

1. \((3y_n + 4z_n)^2 - 33y_n(z_n - x_n) = 64\)
2. \((3y_n + 4z_n)^2 = 33(x_n - z_n)^2 + 64\).

3. \(x_n^3 + y_n^3 - z_n^3 = -3x_n y_n z_n\)

4. Each of the following expressions is a Nasty number.
   a) \(6(3y_{2n+1} + 4z_{2n+1} + 8)\)
   b) \(198y_n (z_n - x_n)\)
   c) \(6((3y_n + 4z_n)^2 - 33y_n^2)\)
   d) \(6((x_n + y_n)z_n)\)
   e) \(6(4x_{2n+1} + 7y_{2n+1})\)

5. \(33(x_n - y_n - z_n)^2 = 132y_n (z_n - x_n)\).

6. \(24(4x_{2n+1} + 7y_{2n+1}) = (4x_n + 7y_n)^2\)

7. Each of the following expressions is a Cubical integer.
   a) \(16(4x_{3n+2} + 7y_{3n+2} + 12x_n + 21y_n)\)
   b) \((x_n + 7y_n + 7z_n)^2 - 132y_n^2\)

8. \(16(16x_n x_{3n+2} + 28x_n y_{3n+2} + 28x_n y_{3n+2} + 49y_{3n+2} y_n + 168x_n y_n + 48x_n^2 + 147y_n^2)\)
   is a quartic integer.

9. \(1089(x_n - z_n)^4 + 128(4x_n + 7y_n)^2 - (4x_n + 7y_n)^4\) is a perfect square.

10. a) \(\Delta^2 x_n \equiv 0 \pmod{4}\)
    b) \(\Delta^2 z_n \equiv 0 \pmod{4}\), when \(k = 1\)
    where \(\Delta\) is the forward difference operator.

Remark:
Suppose, instead of the transformations (4), we have

\(k = 1, \quad u = X - 3T, \quad v = X - 11T\)

Then, the integral solutions to (1) are given by

\[ x_n = f - \frac{7g}{\sqrt{33}} \]
\[ y_n = \frac{4g}{\sqrt{33}} \]
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\[ z_n = f - \frac{3g}{\sqrt{33}}, \quad n = 0,1,2,... \]

One may search for other patterns of solutions for the cubic equation under consideration and search for the properties.

REFERENCES


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INTEGRAL SOLUTIONS OF $x^3 + y^3 + 8k(x+y) = (2k+1)z^3$


A SPECIAL PYTHAGOREAN TRIANGLE

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Patterns of Pythagorean Triangles for each of which the Hypotenuse added with ratio \((\text{Area}/\text{Perimeter})\) is a Perfect square.

KEYWORDS: Pythagorean Triangles, Nasty Numbers.

INTRODUCTION

It is well known that Pythagorean triangle is a treasure house which contains many interesting results. For an extensive review of the literature one may refer [1-7]. In [8], Pythagorean triangles where, in each of which, the sum of the legs is represented by \(kz^2\) \((k > 0)\), \(z\) arbitrary are obtained. In [9], patterns of Pythagorean triangles for each of which the perimeter is represented by a Nasty number are obtained. In this communication, we obtain different patterns of Pythagorean triangles for each of which the Hypotenuse added with ratio \((\text{Area}/\text{Perimeter})\) is a Perfect square. A few interesting properties are given.

METHOD OF ANALYSIS

The general solution for the Pythagorean equation,
\[ x^2 + y^2 = z^2 \]
is given by
\[ x = 2pq, \quad y = p^2 - q^2, \quad z = p^2 + q^2, \quad \text{where} \quad p > q > 0 \]

The assumption that the Hypotenuse added with ratio \((\text{Area}/\text{Perimeter})\) is a Perfect square, namely \(\alpha^2\) yields,
\[ (4p + q)^2 + 7q^2 = (4\alpha)^2 \]
which is rewritten as,
\[ y^2 = 7q^2 + Z^2 \]
where
\[ Z = 4p + q, \quad Y = 4\alpha \]
The solution of (2) are given by
\[ Y = 7r^2 + s^2, \]
\[ Z = 7r^2 - s^2 \]

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From (3) and (4), we get
\[ p = 7r^2 - s^2 - 2rs/4, \quad q = 2rs, \quad \alpha = 7r^2 + s^2/4 \]

As our interest centers on finding integral solutions, the values of \( p \) and \( \alpha \) are integers for the following choices of \( r \) and \( s \).

(i) \( r = 2kS, \quad s = 2S \) \((k \neq 1)\)

(ii) \( r = 2R + 1, s = 2S + 1, R \neq S \)

For the choice (i), the sides of the Pythagorean triangle are:
\[
x = 16S^2k(7k^2 - 2k - 1)
y = 34(7R^2 + 6k - 1)(7k^2 - 10k - 1)
z = 34((7k^2 - 2k - 1)^2 + 64k^2)
\]

For the choice (ii), the sides of the Pythagorean triangle are:
\[
x = 2(7R^2 - S^2 + 6R - 2RS - 2S + 1)(8RS + 4R + 4S + 2)
y = (7R^2 - S^2 + 6R + 10R + 2S + 3)(7R^2 - S^2 - 10RS + 2R - 6S - 1)
z = (7R^2 - S^2 + 6R - 2RS - 2S + 1)^2 + (8RS + 4R + 4S + 2)^2
\]

with the condition,
\[ R > S, \quad 7R^2 + 6R + 1 > S^2 + 2RS + 2S \]

It is to be noted that another pattern of Pythagorean triangle is obtained as follows.

Equation (1) is written as
\[
(q + p)/2(\alpha - p) = (\alpha + p)/q = A/B \quad (B \neq 0)
\]

The above equation is equivalent to the double equations,
\[
qB + p(B - 2A) - 2A\alpha = 0
\]
\[
-\alpha q + pB + \alpha B = 0
\]

Applying the method of cross-multiplication the values of \( p \) and \( q \) are given by
\[
P = 2A^2 - B^2, \quad q = B^2
\]

Thus, the sides of the Pythagorean triangle are given by
\[
x = 2B^2(2A^2 - B^2)
y = 4A^2(A^2 - B^2)
z = 4A^4 - 4A^2B^2 + 2B^4
\]

where \( A \) and \( B \) should satisfy the conditions,
\[ A > B, \quad 2A^2 - B^2 > 0 \]

A few examples are given below:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>p</th>
<th>q</th>
<th>x</th>
<th>y</th>
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<td>8642</td>
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</table>
Properties:

1. $3(z \pm y)$ is a Nasty number.
2. $x - y + z$ is a perfect square.
3. $x - y + z = (A^2 + B^2)^2 - (A^2 - B^2)^2$, difference of two squares.
4. For the choice, $A = 4R^2 + 2S^2$, $B = 8RS$, $(R > S)$, $x$ is a perfect square.
5. When the parameters $A, B$ take the values of Hypotenuse and a leg of a.
6. Pythagorean triangle respectively, the value of $y$ is a perfect square.
7. $y = 0 \pmod{24}$, when $A > 2, B = 1$.
8. When $A > 1, B = 1, z \equiv y \pmod{2}$.
9. $y$ is a Nasty number.

In addition to the above three patterns of solutions, we have also the following patterns.

**Pattern 1:**

$$
x = 8AB^3 - 2B^4 - 16A^3 B + 4A^2 B^2,
\quad y = 8AB^3 - 20A^4 B^2 + 4A^4,
\quad z = 2B^4 + 4A^4 + 12A^2 B^2 - 8AB^3,
$$

with the condition

$$B^2 > A^2 + 2AB, \quad 4AB > B^3 > 2A^2.$$

A few examples are given below:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>p</th>
<th>q</th>
<th>x</th>
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<td>25000</td>
<td>36250</td>
</tr>
</tbody>
</table>

Properties:

1. $y \equiv 0 \pmod{4}$, $x \equiv 0 \pmod{2}$, $z \equiv 0 \pmod{2}$.
2. All the values of $x, y, z$ are even.
3. $3(z \pm y)$ is a Nasty number.
4. $x^2/2(z - y)$ is a perfect square.
5. $x + z$ is a perfect square.

**Pattern 2:**

$$
x = 16A^3 B - 8A^4 - 8AB^2 + 4A^2 B^2
\quad y = 16A^3 B - 20A^4 B^2 + B^4
\quad z = 6A^4 + B^4 - 4A^2 B^2 + 4A^4 B^4 - 16A^3 B
$$

with the conditions, $4AB > 2A^2 > B^2, 4A^2 > B^2 + 4AB$
A few examples are given below:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>p</th>
<th>q</th>
<th>x</th>
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</table>

**Conclusion**

One may search for other patterns of Pythagorean triangles.

**References**

ON THE TRANSCENDENTAL EQUATION

\[ x + g \sqrt{x} + y + h \sqrt{y} = z + g \sqrt{z} \]

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Abstract: The transcendental equation with 3 unknowns given by \( x + g \sqrt{x} + y + h \sqrt{y} = z + g \sqrt{z} \) is analysed for its non-zero integral solutions. A few interesting relations between the solutions, special polygonal and pyramidal numbers are presented.

Keywords: Transcendental equation, Integral solutions, Special polygonal numbers, pyramidal numbers, Nanogonal numbers and Automorphic numbers.

1. INTRODUCTION

The theory of Diophantine equations plays a significant role in higher arithmetic and has a marvelous effect on credulous people and always occupies a remarkable position due to unquestioned historical importance. Most of the diophantine equations are algebraic equations [1-5].

It seems much work has not been done to obtain integral solutions of transcendental equation involving surds. In this context, one may refer [6]. In this communication, the transcendental equation represented by \( x + g \sqrt{x} + y + h \sqrt{y} = z + g \sqrt{z} \) is analysed for different patterns of non-trivial integral solutions. A few interesting relations between the solutions, special polygonal numbers and pyramidal numbers are presented.

2. NOTATIONS

<table>
<thead>
<tr>
<th>Triangular number of rank ( r )</th>
<th>( T_r = r(r + 1)/2. )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heptagonal number of rank ( r )</td>
<td>( HP_r = r(5r - 3)/2. )</td>
</tr>
<tr>
<td>Decagonal number of rank ( r )</td>
<td>( D_r = 4r^2 - 3r. )</td>
</tr>
<tr>
<td>Gnomonic number of rank ( r )</td>
<td>( G_r = (2r - 1). )</td>
</tr>
<tr>
<td>Centered decagonal number of rank ( r )</td>
<td>( CD_r = 5r^2 - 5r + 1. )</td>
</tr>
<tr>
<td>Heteromecic number of rank ( r )</td>
<td>( HM_r = r(r + 1). )</td>
</tr>
<tr>
<td>Dodecagonal number of rank ( r )</td>
<td>( DD_r = 5r^2 - 4r. )</td>
</tr>
</tbody>
</table>
Nanogonal number of rank \( r \)

\[ N_r = \frac{7r^2 - 5r}{2}. \]

Centered nanogonal number of rank \( r \)

\[ CN_r = \frac{(3r - 1)(3r - 2)}{2}. \]

3. METHOD OF ANALYSIS

The equation under consideration is

\[ x + g \sqrt{x} + y + h \sqrt{y} = z + g \sqrt{z} \]  \hspace{1cm} (1)

The substitutions

\[ x = \alpha^2, \quad y = \beta^2 \quad \text{and} \quad z = \gamma^2 \]  \hspace{1cm} (2)

in (1) give,

\[ \alpha^2 + g \alpha + \beta^2 + h \beta = \gamma^2 + g \gamma \]  \hspace{1cm} (3)

Multiplying (3) by 4 and performing some simplifications, we get

\[ (2 \alpha + g)^2 + (2 \beta + h)^2 = (2 \gamma + g)^2 + 1 \]  \hspace{1cm} (4)

We present below different patterns of non-zero integral solutions to (4).

Pattern 1: Equation (4) is written in the factorizable form as

\[ ((2 \alpha + g) + i(2 \beta + h))((2 \alpha + g) - i(2 \beta + h)) \]

\[ = ((2 \gamma + g) + i \gamma)(i \gamma)((2 \gamma + g) + i \gamma) \left( \frac{3 + 4i}{5} \right) \left( \frac{3 - 4i}{5} \right) \]

Define,

\[ ((2 \alpha + g) + i(2 \beta + h)) = ((2 \gamma + g) + i \gamma) \left( \frac{3 + 4i}{5} \right) \]  \hspace{1cm} (5)

Equating the real and imaginary parts in (5), we have

\[ \alpha = \frac{3 \gamma - g - 2h}{5} \]  \hspace{1cm} (6)

\[ \beta = \frac{4 \gamma - 2g - h}{5} \]  \hspace{1cm} (7)

Note that, when

\[ g = 2u - 1, \quad h = r, \quad p = u \quad \text{and} \quad \gamma = 5n - r - p + 3 \]  \hspace{1cm} (8)
we get the integral solutions of (1), as follows:

\[ x = x(n, r, p) = (3n - r - p + 2)^2 \]  \hfill (9)
\[ y = y(n, r) = (4n - r + 2) \]  \hfill (10)
\[ z = z(n, r, p) = (5n - r - p + 3)^2 \]  \hfill (11)

The following are the observations

1. \( y(n, 2n + 1) = 8T_n + 1 \).
2. \( x(n, 1, 2) - D_n - 2HP_n - 1 = 0 \).
3. \( z(n, n + 3, 1) - 4D_n - 1 \equiv 0 \pmod{4} \).
4. \( x(n, 2, 1) - 2CN_n + 1 \equiv 0 \pmod{3} \).
5. \( y(n, 3) = z(n, n + 3, 1) \).
6. \( z(n^3, 3n, 3) = 4HP_n^2 \).
7. \( y(n, 2n + 3) = G_n^2 \).
8. \( z(n^2, 5n, 2) = CD_n^2 \).
9. \( y(n^2, -4n + 2) = 16HM_n^2 \).
10. \( z(n, r, p) \) is an Automorphic number for particular values of \( n, r \) and \( p \). For illustration, \( z(1, 2, 1) = 5^2 = 25 \), \( z(15, 1, 1) = 76^2 = 5776 \).
11. \( z(n^2, 4n, 3) = DD_n^2 \).
12. \( y(n, 1) - x(n, 1, 1) - N_n - 1 \equiv 0 \pmod{13} \).
13. \( z(n, 1, 1) - 10HP_n - 1 \equiv 0 \pmod{25} \).
14. \( x(n, 2, 1) - y(n, 1) - N_n - 2 \equiv 0 \pmod{19} \).
15. \( x(n^2, -3n, 2) = 9HM_n^2 \).
16. Each of the following expressions is a Nasty Number.
   (a) \( 6(y(n, 2)) \)
   (b) \( 6(z(n, 1, 1) - x(n, 1, 1) - 10n + 1) \)
   (c) \( 6(x(n, 1, 1) + y(n, 2)) \)

**Pattern 2:** In equation (4), taking \( (2\gamma + g) = k(2\alpha + g) \), it is rewritten as

\[ Y^2 = DX^2 + h^2 \]  \hfill (12)

where \( Y = 2\beta + h, \ X = 2\alpha + g, \ D = k^2 - 1 \).
Employing the most cited solutions of (12), the non-zero integral solutions of (1) are found to be

\[ x = x(r, s, n) = r^2 s^2 - 2rsn + n^2 \]
\[ y = y(s) = s^4 \]
\[ z = z(k, r, s, n) = k^2 r^2 s^2 - 2krsn + n^2 \]

The observations are as follows:

1. \( y(s) \) is an automorphic number when \( s \) takes the values 1, 5, 6, 25, … in turn.
2. The difference of \( z(2, r, 1, 1) \) and \( x(r, 1, 1) \) is an octagonal number.
3. \( 6(x(r, 2, n) - 4HM_{n-1}) \) is a Nasty number.
4. \( x(r^2, 2, r^2) = Dr^2 \).
5. \( 6(x(r, 1, 1) - G_r) \) is a Nasty number.
6. \( HM_{r-1} - x(r, 1, 2) + 4 \equiv 0 \pmod{3} \).
7. \( z(2, r, 1, 1) - HM_{r-1} - 1 = 0 \).
8. \( z(3, r, 1, 1) - 1 = N_r + HE_r \).
9. \( z(2, 2, 1, 1) - 1 = 8 HE_r \).
10. \( 36(48 PN_r - 12(HY_r + HM_r) - 2(D_r + x(r, r, r + 1)) - 3 G_r + 1) \) is a cubic integer.

**Pattern 3:** In equation (4), assume \( (2\gamma + g) = k(2\beta + h) \)

Following the analysis similar to pattern 2 and performing a few calculations, we have

\[ Z^2 = DW^2 + h^2 \]

where \( Z = 2\alpha + g, \quad W = 2\beta + h, \quad D = k^2 - 1 \).

Using the standard solutions of the above equation, the values of \( \alpha, \beta \) and \( \gamma \) are given by

\[ \alpha = \frac{(k^2 - 1)r^2 + s^2 - g}{2} \]
\[ \beta = \frac{r^2 + s^2 + 2rs - k^2r^2}{2} \]
\[ \gamma = \frac{2krs - g}{2} \]

As our interest centers on finding integer solutions, in what follows, we present for simplicity, the different choices of integral solutions of (1).
Case 1:

\[ g = 2n, \quad k = 2A, \quad r = 2B, \quad s = 2C \]

The corresponding solutions are

\[ x = x(A, B, C, n) = ((4A^2 - 1)2B^2 + 2C^2 - n)^2 \]
\[ y = y(A, B, C) = ((2B^2 + 2C^2 + 4BC - 8A^2 B^2)^2 \]
\[ z = z(A, B, C, n) = (8ABC - n)^2 \]

Case 2:

\[ g = 2n, \quad k = 2A, \quad r = 2B + 1, \quad s = 2C + 1 \]

The corresponding solutions are

\[ x = x(A, B, C, n) = \left( \frac{(4A^2 - 1)(2B+1)^2 + (2C+1)^2 - 2n}{2} \right)^2 \]
\[ y = y(A, B, C) = \left( \frac{(2B-1)^2 + (2C+1)^2 + 2(2B+1)(2C+1) - 4A^2(2B+1)^2}{2} \right)^2 \]
\[ z = z(A, B, C, n) = \left( \frac{4A(2B+1)(2C+1) - 2n}{2} \right)^2 \]

Case 3:

\[ g = 2n, \quad k = 2A + 1, \quad r = 2B, \quad s = 2C \]

The corresponding solutions are

\[ x = x(A, B, C, n) = ((2A + 1)^2 - 1)2B^2 + 2C^2 - n)^2 \]
\[ y = y(A, B, C) = ((2B^2 + 2C^2 + 4BC - (2A + 1)^2 2B^2)^2 \]
\[ z = z(A, B, C, n) = (4(2A + 1)BC - n)^2 \]

Case 4:

\[ g = 2n, \quad k = 2A + 1, \quad r = 2B + 1, \quad s = 2C \]

The corresponding solutions are

\[ x = x(A, B, C, n) = \left( \frac{(2A + 1)^2 - 1)(2B+1)^2 + 4C^2 - 2n}{2} \right)^2 \]
\[ y = y(A, B, C) = \left( \frac{(2B+1)^2 + 4C^2 + 4(2B+1)C - (2A + 1)^2 (2B+1)^2}{2} \right)^2 \]
\[ z = z(A, B, C, n) = ((2A + 1)(2B + 1)2C - n)^2 \]
4. CONCLUSION

One may search for other patterns of solutions and their corresponding properties.

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QUARTIC EQUATION IN 5 Unknowns

\[ x^4 - y^4 = 2(z^2 - w^2)p^2 \]

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ABSTRACT

The Quartic equation with 5 unknowns given by \( x^4 - y^4 = 2(z^2 - w^2)p^2 \) is analysed for its non-zero integral solutions. A few interesting relations between the solutions, special polygonal numbers, and pyramidal numbers are presented.

Key words: Quartic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers.

INTRODUCTION

Biquadratic Diophantine equations, homogeneous and non-homogeneous, have aroused the interest of numerous Mathematicians since ambiguity as can be seen from [1,4,11,14,15,16,18]. In the context one may refer [2,3,5,7,8,9,10,12,13,17] for varieties of problem on the diophantine equations with three and four variables. This communication concerns with the problem of determining non-zero integral solutions of the Quartic equation in 5 unknowns represented by \( x^4 - y^4 = 2(z^2 - w^2)p^2 \). Also a few interesting relations between the solutions and special polygonal numbers are presented.

NOTATIONS

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<td>Kynea number of rank ( n )</td>
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<td>polygonal number of rank ( n ) with sides ( m )</td>
<td>( t_{m,n} )</td>
<td>( n(1 + \frac{(n-1)(n-2)}{2}) )</td>
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<td>Pyramidal number of rank ( n ) with sides ( r )</td>
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METHOD OF ANALYSIS

The equation under consideration is
\[ x^4 - y^4 = 2(z^2 - w^2)p^2 \]  
(1)

The substitutions
\[ x = u + v , y = u - v , z = u + (k^2 + 1), w = u - (k^2 + 1) \]
and \[ p = a^2 + b^2 \]  
(2)
in (1) give,
\[ (u + v)^4 - (u - v)^4 = 8uv(k^2 + 1)(a^2 + b^2)^2 \]  
(3)
Performing some simplifications in (3), we get
\[ u^2 + v^2 = (k^2 + 1)(a^2 + b^2)^2 \]  
(4)
Employing the method of unique factorization, (4) is written as
\[ u + iv = (k + i)(a + ib)^2 \]  
(5)
Equating the real and imaginary parts we get,
\[ u = k(a^2 - b^2) - 2ab \]  
(6)
\[ v = a^2 - b^2 + 2 kab \]  
(7)
By substituting various values to \( k \), we have different patterns of non zero integral solutions to (5) and using (2), the corresponding integral solutions to (1) are obtained. In what follows, we exhibit patterns of integral solutions to (1) when \( k = 1, 2 \) and 3.

Pattern 1:
Let \( k = 1 \), Equations (6) and (7) reduce to
\[ u = a^2 - b^2 - 2ab , \quad v = a^2 - b^2 + 2ab \]
The integral solutions of (1) are as follows,
\[ x = x(a, b) = 2(a^2 - b^2) \]
\[ y = y(a, b) = -4ab \]
\[ z = z(a, b) = a^4 - 6a^2b^2 + b^4 + 2 \]
\[ w = w(a, b) = a^4 - 6a^2b^2 + b^4 - 2 \]
The following are the few observations:
1. \( x(a, a+1) - y(a,1) + 2 = 0 \).
2. \( t_{s,a} - 2 - (x(a,1) + y(a,1)) \) is a multiple of 3.
3. Each of the following expressions is a Nasty number.

\[ \left( \frac{2t_{s,a} + 3z(a,1)}{17} \right) \]
QUARTIC EQUATION IN 5 UNKNOWNS \( x^4 - y^4 = 2(z^2 - w^2)p^2 \)

b) \( \frac{2t_{i,a} - 5w(1, b) - 5}{27} \)
c) \( \frac{2w(a, 1) - t_{6,a} + 2}{11} \)

4. Each of the following expressions is a Perfect square.
a) \( \frac{x(a, 1) - y(a^2, 1) + 2}{2} \)
b) \( t_{8,a} - 3 - 3w(1, b) \)

5. \( y(a, a + 1) + 8t_{4,a} = 0 \)

6. \( x(2^n, 1) - y(2^n, 1) = 2KN_a \).

7. \( w(a, 1) + 4t_{1,a} + G_{ij} + 2 \) is a quartic integer.

8. \( x(1, 2^n) + y(1, 2^n) + 2KN_n = 0. \)

9. \( 2(1 - t_{3,a}) - (x(1, b) + y(1, b)) = 0(\text{mod} \ 2). \)

Pattern 2:
Taking \( k = 2 \), Equations (6) and (7) reduce to
\( u = 2a^2 - 2b^2 - 2ab, \ v = a^2 - b^2 + 4ab \)
The integral solutions of (1) are as follows,
\( x = x(a, b) = 3a^2 - 3b^2 + 2ab \)
\( y = y(a, b) = a^2 - b^2 - 6ab \)
\( z = z(a, b) = 2a^3 + 6a^3b - 6ab^3 - 12a^2b^2 + 2ab^4 + 5 \)
\( w = w(a, b) = 2a^4 + 6a^3b - 6ab^3 - 12a^2b^2 + 2ab^4 - 5 \)
The following are the few observations:
1. \( x(a, 1) - t_{8,a} + 3 = 0(\text{mod} \ 4). \)
2. \( x(a, 1) - 2t_{i,a} + t_{6,a} + 3 = 0(\text{mod} \ 4). \)
3. \( S_a = 6y(a, 1) - 2 = 0(\text{mod} \ 30). \)
4. \( x(a, 1) + y(a, 1) - 8t_{3,a} + 4 = 0 \)
5. \( x(2^n, 3) + 24 = 3KN_n. \)
6. \( x(2^n, 1) + y(2^n, 1) + 3(2^{n+2}) = 4KN_n. \)

Pattern 3:
Assuming \( k = 3 \), Equations (6) and (7) reduce to
\( u = 3a^2 - 3b^2 - 2ab, \ v = a^2 - b^2 + 6ab \)
The integral solutions of (1) are as follows
x = x(a, b) = 4a^2 - 4b^2 + 4ab
y = y(a, b) = 2a^2 - 2b^2 - 8ab
z = z(a, b) = 3a^4 + 16a^3b - 16ab^3 - 18a^2b^2 + 3b^4 + 10
w = w(a, b) = 3a^4 + 16a^3b - 16ab^3 - 18a^2b^2 + 3b^4 - 10

The following are the few observations:

1. \( x(2^n, 2) + 12 = 4KN_a \).

2. \( \frac{2x(a, b) + y(a, b)}{10} \) is the difference of squares.

3. \( \frac{2ct_{10,a} - 5y(a^4, 1) - 12}{30} \) is a quartic integer.

4. \( 9y(a, 1) - 4ct_{9,a} + 22 \equiv 0 \pmod{90} \).

5. \( z(a, 1) - 96t_5^3 + 16t_{10,a} - t_{9,a} - 13 = 0 \).

6. Each of the following expressions is a Nasty number.
   
a) \( \frac{6(x(a, b) - 2y(a, b))}{20} \)
   
b) \( \frac{6(x(a, b))^2 + (y(a, b))^2}{20} \)

Remark:
Suppose the choices of \( z \) and \( w \) in (2) are taken as
\[ z = u + (k^2 + 1)v \quad w = u - (k^2 + 1)v \]
then, the corresponding integral solutions to (1) when \( k = 1, 2 \) and 3 are respectively obtained as follows.

Case 1: \( k = 1 \). The integral solutions of (1) are as follows,
\[ x = x(a, b) = 2a^2 - b^2 \]
\[ y = y(a, b) = -4ab \]
\[ z = z(a, b) = 3a^2 - 3b^2 + 2ab \]
\[ w = w(a, b) = -(a^2 + 3b^2 + 6ab) \]
QUARTIC EQUATION IN 5 UNKNOWNS \( x^4 - y^4 = 2(z^2 - w^2)p^2 \)

Case 2: \( k = 2 \). The integral solutions of (1) are as follows,
\[
\begin{align*}
  x &= x(a,b) = 3a^2 - 3b^2 + 2ab \\
  y &= y(a,b) = a^2 - b^2 - 6ab \\
  z &= z(a,b) = 7a^2 - 7b^2 + 18ab \\
  w &= w(a,b) = -3a^2 + 3b^2 - 22ab
\end{align*}
\]

Case 3: \( k = 3 \). The integral solutions of (1) are as follows,
\[
\begin{align*}
  x &= x(a,b) = 4a^2 - 4b^2 + 4ab \\
  y &= y(a,b) = 2a^2 - 2b^2 - 8ab \\
  z &= z(a,b) = 13a^2 - 13b^2 + 58ab \\
  w &= w(a,b) = -7a^2 + 7b^2 - 62ab
\end{align*}
\]

It is worth to observe that there are other choices of transformations for solving (1), in addition to the transformations (2). For simplicity and brevity, we present below, two types of transformations for solving (1) and their corresponding integral solutions to (1).

**Pattern 4:**
Consider the transformations,
\[
  x = u + v, \quad y = u - v, \quad z = uv + 2^{2a}, \quad w = uv - 2^{2a}
\]

Following the procedure as given above, the non zero integral solutions to (1) are given by
\[
\begin{align*}
  x &= x(a, r, s) = 2^a(r^2 - s^2 + 2rs) \\
  y &= y(a, r, s) = 2^a(r^2 - s^2 - 2rs) \\
  z &= z(a, r, s) = 2^{2a}(2r^3s - 2rs^3 + 1) \\
  w &= w(a, r, s) = 2^{2a}(2r^3s - 2rs^3 - 1)
\end{align*}
\]

The following are the few observations:
1. \( \left( x(1, r, s) + y(1, r, s) \right) / 4 \) is the difference of squares.
2. \( x(1, 2^n, 1) = 2KN_n \),
3. \( \left( x(1, 2^n, 1)(y(1, r, 1) - t_{o,r} + 1) / 2KN_n \right) \equiv 0 \pmod{3} \),
4. \( 2^{16} + 2^{16} \equiv 1 \pmod{KN_n} \), where \( X - Y = x(1, r, 2) - y(1, r, 2) \)

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5. Each of the following expressions is a Nasty number.
   a) $3(x(1, r, s) - y(1, r, s))$, $rs = \alpha^2$
   b) $8OH_2 = z(1, r^2, 1)$
   c) $6(x(1, r, 2) + y(1, r, 2) + 16)$

Pattern 5:
The transformations considered are
   $x = u + v, y = u - v, z = u + 2^{2a} v, w = u - 2^{2a} v$

The corresponding nontrivial integral solutions to (1) are obtained as
   $x = x(\alpha, r, s) = 2^n (r^2 - s^2 + 2rs)$
   $y = y(\alpha, r, s) = 2^n (r^2 - s^2 - 2rs)$
   $z = z(\alpha, r, s) = 2^n (r^2 - s^2) + 2^{2a+1} rs$
   $w = w(\alpha, r, s) = 2^n (r^2 - s^2) - 2^{2a+1} rs$

The following are the few observations:
1. $x(1, r, s) + y(1, r, s) \text{ is } 4 \text{ times the difference of squares.}$
2. $x(1, 2^n, 1) = 2KN_n$
3. $x(1, r, 2) + y(1, r, 2) \text{ is a Perfect square.}$

One may search for other patterns of integral solutions to (1) and their corresponding properties.

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QUARTIC EQUATION IN 5 UNKNOWNS \[ x^4 - y^4 = 2(z^2 - w^2)p^2 \]


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OBSERVATION ON THE DIOPHANTINE EQUATION
\[ Y^2 = DX^2 + Z^2 \]

M.A.Gopalan* and J.Kaliga Rani**

ABSTRACT

Special patterns of pythagorean triangles are generated through the solutions of the ternary quadratic diophantine equation \[ Y^2 = DX^2 + Z^2 \].

INTRODUCTION

The ternary quadratic diophantine equations are rich in variety. For an extensive review of various problems one may refer [1-9].

In this communication, we consider yet another interesting ternary quadratic diophantine equation \[ Y^2 = DX^2 + Z^2 \], where D is a square free integer. Employing suitable forms of D and Y, infinitely many Pythagorean triangles where in each of which Hypotenuse + 4α²(Area / Perimeter) = Z², α non zero, are obtained. A few numerical illustrations with some properties are presented.

METHOD OF ANALYSIS

The most cited non-trivial integral solutions of the ternary quadratic diophantine equation \[ Y^2 = DX^2 + Z^2 \], where D is a square free integer are given by

\[ X = 2rs, \quad Z = Dr^2 - s^2, \quad Y = Dr^2 + s^2 \]  \hspace{1cm} (1)

Define

\[ D = \alpha^4 + 2\alpha^2 - 1 \]  \hspace{1cm} (2)
\[ Y = P + \alpha^2 X \]  \hfill (3)

where \( P, \alpha \) are non-zero integers.

From (1), (2) and (3) we have

\[ P = (\alpha^4 + 2 \alpha^2 - 1) r^2 + s^2 - 2 \alpha^2 rs \]

Considering \( P \) and \( X \) to be the generators of a pythagorean triangle \((u, v, w)\) the corresponding sides are given by

\[ u = 2PX = 4\alpha^4 r^3 s + 8 \alpha^2 r^3 s - 4r^3 s + 4r s^3 - 8\alpha^2 r^2 s^2 \]

\[ v = P^2 - X^2 = \alpha^8 r^4 + 4 \alpha^6 r^4 - 4\alpha^6 r^3 s + 2\alpha^4 r^4 + 6\alpha^4 r^2 s^2 - 8 \alpha^4 r^3 s - 4\alpha^2 r^4 + 4\alpha^2 r^2 s^2 + 2 \alpha^2 r^3 s - 6r^2 s^2 + r^4 + s^4 \]

\[ w = P^2 + X^2 = \alpha^8 r^4 + 4 \alpha^6 r^4 - 4\alpha^6 r^3 s + 2\alpha^4 r^4 + 2\alpha^4 r^2 s^2 - 8 \alpha^4 r^3 s - 4\alpha^2 r^4 + 8\alpha^2 r^2 s^2 + 4 \alpha^2 r^3 s - 4\alpha^2 r s^3 + 2r^2 s^2 + r^4 + s^4 \]

For different choices of \( \alpha, r, s \) in the above equations, we generate pythagorean triangles where in each of which the relation

\[ \text{Hypotenuse} + 4\alpha^2(\text{Area} / \text{Perimeter}) = Z^2 \]

holds good.

In particular for the choices \( \alpha = 2, 3 \) various relations among the sides of the corresponding Pythagorean triangle are exhibited below.

**Case(1):** The generators of the pythagorean triangle for the choice \( \alpha = 2 \) are

\[ P = 23r^2 + s^2 - 8rs \quad , \quad X = 2rs \]

For obtaining various relations among the sides, one has to go in for particular values of \( r \) and \( s \).

(I): Taking \( r = s \), are obtain the sides of the triangle to be
\[ u = 64r^4, \quad v = 252r^4, \quad w = 260r^4, \]

**PROPERTIES**

1. \(4u\) is a quartic integer.
2. \(2(w - v)\) is a quartic integer.
3. \(4u - v = 4, \quad w - 4u = 4.\)
4. \(8(w - v) - u = 0.\)
5. \(512(w^3 - v^3) - u^3 = 192uvw.\)
6. \(u \equiv 0 \pmod{(w - v)}.\)
7. \(u + w, \quad u + v + w\) are a perfect square.
8. \(3(w \pm v)\) is a nasty number.
9. The triple \((u/4 \ r^4, \ v/4 \ r^4, \ w/4 \ r^4)\) is primitive.

(II) The substitution of \(s = 2r\) gives

\[ u = 88r^4, \quad v = 105r^4, \quad w = 137r^4 \]

**PROPERTIES**

1. \((u/4 \ r^4, \ v/4 \ r^4, \ w/4 \ r^4)\) is primitive.
2. \((w \pm v)/2\) is a perfect square.
3. \(2v - w - 9\) is a perfect square.
4. \(2v - w - 8\) is a perfect square.
5. Area = \(4620 \ r^8\)

From the area, the following Second order Ramanuja numbers are obtained.

3365, 5780, 8468, 17300, 29588, 55700, 80660, 151700, 154388, 1345172, 1387412, ...
6. $21(u+v-w)$ is a Nasty number.
7. $u+v-w$ is expressed as the difference of two cubes.

Case(2): The generators of the Pythagorean triangles for the choice $\alpha = 3$ are

$$P = 98r^2 + s^2 - 18rs, \quad X = 2rs$$

(III) On substituting $s = r$, we have,

$$u = 324r^4, \quad v = 6557r^4, \quad w = 6565r^4$$

**PROPERTIES**

1. $20u+10v-10w$ is a perfect square.
2. $2(w-v)$ is a quartic integer.
3. $u - 40(w-v) = 4$
4. $4u$ is a quartic integer.
5. $u$ is a perfect square.
6. $(u/\ r^4, \ v/\ r^4, \ w/\ r^4)$ is primitive.

(IV) Setting $s = 4r$, we get

$$u = 672r^4, \quad v = 1700r^4, \quad w = 1828r^4$$

**PROPERTIES**

1. $7(u+v+w)$ is a Nasty number.
2. $(u/\ 4r^4, \ v/\ 4r^4, \ w/\ 4r^4)$ is primitive.
3. $u+w$ is a perfect square.
4. $4(u+v+w) = 0 \pmod{7}$
5. $4(u+v+w) = 7$ times a Nasty number.
6. $u - 5(w-v) + 4$ is a perfect square.
CONCLUSION

One may search varieties of patterns of Pythagorean triangles for other values of $\alpha$, $r$ and $s$.

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On the Transcendental Equation \( x + y - \sqrt{xy} = (k^2 + 3)z^2 \)

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Abstract—The transcendental equation with three unknowns given by \( x + y - \sqrt{xy} = (k^2 + 3)z^2 \) is analysed for its non-zero integral solutions and two sets of solutions are obtained. A few interesting relations between the solutions, Special polygonal numbers and pyramidal numbers are presented.

Keywords—Transcendental equation, Integral solutions, Special polygonal numbers, pyramidal numbers.

2000 Mathematics Subject Classification—11D99

1. INTRODUCTION

The theory of Diophantine equations plays a significant role in higher arithmetic and has a marvelous effect on credulous people and always occupies a remarkable position due to unquestioned historical importance. In [1-12], the Diophantine equations for which we require integral solutions are algebraic equations with integer coefficients. It seems much work has not been done with regard to integral solutions for transcendental equation involving surds.

A special Transcendental equation is studied for its non-trivial integral solutions in [13]. In this communication, the transcendental equation represented by \( x + y - \sqrt{xy} = (k^2 + 3)z^2 \) is analyzed for its non-trivial integral solutions. A few interesting relations between the solutions, Special polygonal numbers and pyramidal numbers are presented.

2. METHOD OF ANALYSIS

The equation under consideration is \( x + y - \sqrt{xy} = (k^2 + 3)z^2 \)  \( (1) \)

The substitution \( x = \alpha^2, \ y = \beta^2 \)

\( in \ (1), \ gives \ \alpha^2 + \beta^2 - \alpha\beta = (k^2 + 3)z^2 \) \( (2) \)

Using the transformations, \( \alpha = u + v, \ \beta = u - v, \ z = a^2 + 3b^2 \) \( (4) \)

in (3), it is written in the factorizable form as,

\( (u + i\sqrt{3}v)(u - i\sqrt{3}v) = (k + i\sqrt{3})(k - i\sqrt{3})(a + i\sqrt{3}b)^2(a - i\sqrt{3}b)^2 \) \( (5) \)

in which \( u, v \) are non-zero distinct integers and \( a, b \in z \{-0\} \)

Define \( (u + i\sqrt{3}v) = (k + i\sqrt{3})(a + i\sqrt{3}b)^2 \) \( (6) \)

Equating the real and imaginary parts in (6), we get

\( u = k(a^2 - 3b^2) - 6ab \) \( (7) \)

\( v = 2kab + a^2 - 3b^2 \) \( (8) \)

By using (7), (8) in (4), we get the values of \( \alpha \) and \( \beta \) given by

\( \alpha = \alpha(k, a, b) = k(a^2 - 3b^2 + 2ab) + (a^2 - 3b^2 - 6ab) \)
\[ \beta = \beta(k, a, b) = k(a^2 - 3b^2 - 2ab) - (a^2 - 3b^2 + 6ab) \]

Thus, the values of x, y and z are obtained as

\[ x = x(k, a, b) = (k(a^2 - 3b^2 + 2ab) + (a^2 - 3b^2 - 6ab))^2 \]

\[ y = y(k, a, b) = (k(a^2 - 3b^2 - 2ab) - (a^2 - 3b^2 + 6ab))^2 \]

\[ z = z(a, b) = a^2 + 3b^2 \]

It is to be noted that, choosing suitably the values of k, a and b one obtains distinct integer solutions.

2.1 A Few Interesting Properties Noticed are Listed Below

1. \( x(1, a, 1) - H_a \equiv 0 \pmod{(3a + b)} \)
2. \( \sqrt{x(1, a, 1)} - HE_a + 6 \equiv 0 \pmod{3} \)
3. \( 2\sqrt{x(1, a, 1)} - D_a - 12 \equiv 0 \pmod{5} \)
4. \( \sqrt{x(1, a, 1)} - 4T_{a-1} + G_a + 7 = 0 \)
5. \( H_a - \sqrt{x(1, a, 2)} - T_{a-1} - 25 \equiv 0 \pmod{6} \)
6. \( 2CT_a - z(a, 1) - \sqrt{x(1, a, 2)} + 23 \equiv 0 \pmod{5} \)
7. \( x(1, 1, b) + 8O_b - 4O_b^2 = 4 \)
8. \( 2H_a - \sqrt{x(1, 2, b)} + 10 \equiv 0 \pmod{14} \)
9. \( D_b - \sqrt{x(1, 2, b)} - 4T_b + 8 \equiv 0 \pmod{9} \)
10. \( T_{a-1} - \sqrt{y(2, a, 1)} - 3G_a - 6 \equiv 0 \pmod{3} \)
11. \( \sqrt{x(2, a, 2)} - 2CT_a + a = -38 \)
12. Each of the following expressions is a perfect square.
   a) \( x(2, a, 1) + 18O_a - O_a^2 \)
   b) \( 48z^2(3, a, b) - 3x(3, a, b) \)
   c) \( y(1, a, 1) \)
   d) \( y(1, 1, b) \)
   e) \( x(3, 4b, b) \)
   f) \( \sqrt{x(1, a, 8a)} - \sqrt{y(3, a, 8a)} \)
   g) \( 6(\sqrt{x(1, 3b, b)} + \sqrt{y(1, 3b, b)}) \)
14. Each of the following expressions is a Nasty number.
   a) \( 6(x(2, a, b) + 9y(2, a, b) - 6\sqrt{x(2, a, b)} \sqrt{y(2, a, b)}) \)
   b) \( 6x(3, 4b, b) \)
   c) \( \sqrt{x(1, a, 8a)} - \sqrt{y(3, a, 8a)} \)
15. \( \sqrt{x(1,1,b)} + 12T_{n} - 2 \equiv 0 \pmod{2} \)

16. \( x(1,a,b) - y(1,a,b) - 96PN_{n} + 24(HY_{n} - TH_{n}) + 4RH_{n} + 248T_{n} - 32 \equiv 0 \pmod{124} \)

17. \( \sqrt{x(1,a,a+1)} - \sqrt{y(1,a,a+1)} = 8PR_{n} \)

However, another set of non-zero integral solutions of (1) are obtained by rewriting (5) as

\[(u + i\sqrt{3}v)(u - i\sqrt{3}v) = (k + i\sqrt{3})(k - i\sqrt{3})(a + i\sqrt{3}b)^2(a - i\sqrt{3}b)^2 \left( \frac{(1 + i\sqrt{3})}{2} \right) \frac{(1 - i\sqrt{3})}{2} \]

Now defining

\[(u + i\sqrt{3}v) = (k + i\sqrt{3})(a + i\sqrt{3}b)^2 \left( \frac{(1 + i\sqrt{3})}{2} \right) \]

and following the procedure presented above, the second set of non-zero integral solutions of (1) are seen to be

\[x = x(k,a,b) = (k(a^2 - 3b^2 - 2ab) - (a^2 - 3b^2 + 6ab))^2\]

\[y = y(k,a,b) = (k(-4ab) - 2a^2 + 6b^2)^2\]

\[z = z(a,b) = a^2 + 3b^2\]

2.2 A Few Results Observed Are

1. \( T_{n-1} - \sqrt{x(2,a,1)} - 3G_{n} - 6 \equiv 0 \pmod{3} \)

2. \( 2z(1,b) + \sqrt{y(2,1,b)} - 18T_{n-1} + G_{n} = 1 \)

3. \( \sqrt{x(1,a,-8a^2)} \) is a cubical integer.

4. Each of the following expressions is a perfect square.
   a) \( x(1,a,b) \)
   b) \( \sqrt{x(1,a,1)} - \sqrt{y(2,a,1)} + 12 \)
   c) \( x(1,a,1) \)
   d) \( x(1,1,b) \)

5. Each of the following expressions is a Nasty number
   a) \( 6x(1,a,b) \)
   b) \( (6(\sqrt{x(1,a,1)} - \sqrt{y(2,a,1}) + 12) \)
   c) \( 6x(1,a,1) \)
   d) \( 6x(1,1,b) \)

3. CONCLUSION

One may search for solutions patterns and other relations.

NOTATIONS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>HE_t</td>
<td>( r(2r - 1) )</td>
<td>Hexagonal number of rank ( r ).</td>
</tr>
<tr>
<td>O_r</td>
<td>( r(3r - 2) )</td>
<td>Octagonal number of rank ( r ).</td>
</tr>
<tr>
<td>G_r</td>
<td>( 2r - 1 )</td>
<td>Gnomonic number of rank ( r ).</td>
</tr>
<tr>
<td>D_r</td>
<td>( 4r^2 - 3r )</td>
<td>Decagonal number of rank ( r ).</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
<td>Name</td>
</tr>
<tr>
<td>--------</td>
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</tr>
<tr>
<td>$T_r$</td>
<td>$r(r+1)/2$</td>
<td>Triangular number of rank $r$.</td>
</tr>
<tr>
<td>$PN_r$</td>
<td>$(r+1)(r+2)(r+3)/24$</td>
<td>Pentatope number of rank $r$.</td>
</tr>
<tr>
<td>$HY_r$</td>
<td>$(r+1)(5r-2)/6$</td>
<td>Heptagonal pyramidal number of rank $r$.</td>
</tr>
<tr>
<td>$RD_r$</td>
<td>$(2r - 1)(2r^2 - 2r + 1)$</td>
<td>Rhombic dodecahedral number of rank $r$.</td>
</tr>
<tr>
<td>$PR_r$</td>
<td>$r(r+1)$</td>
<td>Pronic number of rank $r$.</td>
</tr>
<tr>
<td>$CT_r$</td>
<td>$(3r^2 - 3r + 2)/2$</td>
<td>Centered triangular number of rank $r$.</td>
</tr>
<tr>
<td>$TH_r$</td>
<td>$(r+1)(r+2)/6$</td>
<td>Tetrahedral number of rank $r$.</td>
</tr>
<tr>
<td>$H_r$</td>
<td>$(3r^2 - 3r + 2)$</td>
<td>Hex number of rank $r$.</td>
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</tbody>
</table>

**REFERENCE**


INTEGRAL SOLUTIONS OF $x^2 - y^2 = zw$

M. A. GOPALAN* & J. KALIGA RANI**

ABSTRACT

The quadratic diophantine equation with four unknowns given by $x^2 - y^2 = zw$ is analysed for its non-zero integral solutions. A few interesting relations between the solutions, Special polygonal numbers and Pyramidal numbers are presented.

KEYWORDS

Quadratic diophantine equation with four unknowns, Integral solutions, Special polygonal numbers, Special pyramidal numbers.

INTRODUCTION

The problem of finding non-trivial integral solutions for the quadratic diophantine equation, homogeneous or non-homogeneous with four unknowns has been an interest to various mathematicians since antiquity [1, 2, 3]. In particular, one may refer [4, 5, 6, 7].

In this communication, We consider yet another interesting quadratic diophantine equation with four unknowns given by $x^2 - y^2 = zw$ and obtain different patterns of solutions. A few interesting properties among the solutions are also given.

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NOTATIONS

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>DEFINITION</th>
<th>NAME</th>
</tr>
</thead>
<tbody>
<tr>
<td>TH_r</td>
<td>( (r(r+1)(r+2))/6 )</td>
<td>Tetrahedral number of rank r.</td>
</tr>
<tr>
<td>T_r</td>
<td>( (r(r+1))/2 )</td>
<td>Triangular number of rank r.</td>
</tr>
<tr>
<td>G_r</td>
<td>( 2r - 1 )</td>
<td>Gnomonic number of rank r.</td>
</tr>
<tr>
<td>HE_r</td>
<td>( r(2r - 1) )</td>
<td>Hexagonal number of rank r.</td>
</tr>
<tr>
<td>D_r</td>
<td>( 4r^2 - 3r )</td>
<td>Decagonal number of rank r.</td>
</tr>
<tr>
<td>O_r</td>
<td>( r(3r - 2) )</td>
<td>Octagonal number of rank r.</td>
</tr>
<tr>
<td>PY_r</td>
<td>( (r^2(r+1))/2 )</td>
<td>Pentagonal pyramidal number of rank r.</td>
</tr>
<tr>
<td>SO_r</td>
<td>( r(2r^2 - 1) )</td>
<td>Stella octangula number of rank r.</td>
</tr>
<tr>
<td>CS_r</td>
<td>( r^2 + (r - 1)^2 )</td>
<td>Centered square number of rank r.</td>
</tr>
<tr>
<td>CT_r</td>
<td>( (3r^2 - 3r + 2)/2 )</td>
<td>Centered triangular number of rank r.</td>
</tr>
</tbody>
</table>

METHOD OF ANALYSIS

The equation under consideration is

\[
x^2 - y^2 = zw,
\]

Substituting the linear transformation

\[
z = p + q, \quad w = p - q, \quad p \neq q
\]
INTEGRAL SOLUTIONS OF $x^2 - y^2 = zw$

in (1), it is written in the factorizable form as

$$(x + iq)(x - iq) = (p + iy)(p - iy)((3 +4i)/5)((3 -4i)/5)$$

Define

$$(x + iq) = (p + iy)((3 +4i)/5) \quad (3)$$

Equating the real and imaginary parts in (3), we get

$$x = (5p - 4q)/3, \quad y = (5q - 4p)/3 \quad (4)$$

The values of $x$ and $y$ are integers when

$$p = 3n + 5(k - 1), \quad q = 3n + 4(k - 1) \quad (5)$$

Thus using (5) in (2) and (4), the non-trivial integer values of $x, y, z$ and $w$ are given by

$$x = x(n, k) = n + 3k -3$$

$$y = y(n) = n$$

$$z = z(n, k) = 6n + 9k -9$$

$$w = w(k) = k - 1$$

A few interesting properties observed among the solutions, are presented below.

1. $6y(n) + 9w(k) = z(n, k)$.  

2. $y(n) + 3w(k) = x(n, k)$.  

3. $6x(n, k) - z(n, k) = 9w(k)$.  

4. $216y^3(n) + 729w^3(k) + 162y(n)w(k)z(n, k) = z^3(n, k)$.
5. \(y^3(n) + 27w^3(k) + 9 \ y(n) w(k) x(n, k) = x^3(n, k)\).

6. \(216x^3(n, k) + 162w(k)x(n, k)z(n, k) = z^3(n, k) + 729w^3(k)\).

7. \(x(n+1, 1) y(n) = 2T_n\).

8. \(y^2(n) w(n+2) = 2PY_n\).

9. \(x(n, k) + z(n, k) - 7y(n) \equiv 0 \pmod{12}\).

10. \(x(k^2 - 2k, k) w(k) - 6TH_{k-1} \equiv 0 \pmod{3}\).

11. \(x(n-1, 1) + y(n) = G_n\).

12. \(y(n) w(2n^2) = SO_n\).

13. \(y^2(n) + w^2(n) = CS_n\).

14. \(y^2(n) w(n+2) = 2PY_n\).

15. \(TH_\alpha - x(\alpha^3, \alpha^2 - \alpha) - 3 \equiv 0 \pmod{5}\).

16. \(6(x(n, O_n) + 5n - 3)\) is a Nasty number.

17. \(x(n, D_n) + 8n + 3 \equiv 0 \pmod{12}\).

18. \(x(2\alpha^3, 2T_\alpha) - 12TH_\alpha \equiv -3 \pmod{2}\).

19. \(z(n, HE_n) + 3n + 9 \equiv 0 \pmod{18}\).

However, we have other patterns of solutions which are exhibited below.

**Pattern 1:**
The solutions are

\[ x = 2rsm \]

\[ y = (m^2 - 1)r^2 + s^2 \]

\[ z = 2rsm - (m^2 - 1)r^2 - s^2. \]

\[ w = 2rsm + (m^2 - 1)r^2 + s^2. \]

where \( m, r, s \) are any non-zero integers.

**Properties:**

1. \( y \pm x \) is written as the difference of squares.
2. \((z, x, w)\) form an Arithmetic progression.
3. \( w - z = 2y \).
4. When \( r, s, m \) forms a geometric progression each of the following expressions forms a Nasty number.
   (i) \( 3x \)
   (ii) \( 3(w - y) \)
   (iii) \( 3(y + z) \)
   (iv) \( 6(z + w) \)

**Pattern 2:**

The solutions are

\[ x = 2rsm \]
\[ y = 2rs \]
\[ z = 2rsm - 2rs \]
\[ w = 2rsm + 2rs. \]

where \( m, r, s \) are any non-zero integers.

**Properties:**

1. \((z, x, w)\) forms an Arithmetic progression.
2. \((x - z)(x - w) + y^2 = 0.\)
3. \((z + y)(w - y) = x^2.\)
4. When \( m \) is a perfect square \((6 (w - y ))/ (x - z )\) is a Nasty number.

**Pattern 3:**

The solutions are
\[ x = \alpha^2 (\tilde{R}^2 + \tilde{S}^2) \]
\[ y = 2\alpha^2 \tilde{R} \tilde{S} \]
\[ z = \alpha^3 (\tilde{R}^2 - \tilde{S}^2) \]
\[ w = \alpha (\tilde{R}^2 - \tilde{S}^2) \]

where \( \alpha, \tilde{R}, \tilde{S} \) are any non-zero integers.

**Pattern 4:**

The solutions are
\[ x = u + v \]
\[ y = v - u \\
\[ z = 2u \\
\[ w = 2v. \\
where \( u \) and \( v \) are any non-zero integers.

Properties :

1. \( xy \) is written as difference of squares.
2. The difference of square of \( x \) and \( y \) is a perfect square, when \( u = v \).
3. when \( v = u + 1 \), \( x^2 - y^2 = 8 T_u \).

Pattern 5:

The solutions are

\[ x = u + v + 1 \\
\[ y = v - u \\
\[ z = 2u + 1 \\
\[ w = 2v + 1. \\
where \( u \) and \( v \) are any non-zero integers.

Properties :

1. \( x + y - w = 0 \)
2. \( x - y - z = 0 \)
3. \( x^2 - D_u - 1 = 0 \pmod{7} \).
4. \[ z^2 - y^2 - 2CT_u + 2 \equiv 0 \pmod{9}. \]

5. \[ y^2 + w - 2 \] is a perfect square.

**Pattern 6:**

The solutions are

\[
\begin{align*}
x &= (m^2 - 1)r^2 - s^2. \\
y &= (m^2 - 1)r^2 + s^2 \\
z &= -2s^2. \\
w &= 2(m^2 - 1)r^2.
\end{align*}
\]

where \( m, r, s \) are any non-zero integers.

**Properties:**

1. \( x + y = w \)
2. \( x - y = z. \)
3. The following expressions forms a Nasty number.
   
   (i) \( 3(w - 2x) \)
   
   (ii) \( 3(2y - w) \)
   
   (iii) \( 6(y - x - z) \)

**Pattern 7:**

The solutions are

\[
\begin{align*}
x &= (1 - m^2)r^2 + s^2. \\
y &= 2rs
\end{align*}
\]
\[ z = (1 - m^2)r^2 + s^2 + 2rs. \]

\[ w = (1 - m^2)r^2 + s^2 - 2rs. \]

where \( m, r, s \) are any non-zero integers.

**Properties:**

1. \((z, x, w)\) form an Arithmetic progression.
2. \(w - z = 2y.\)
3. The following expressions form a Nasty number.
   
   (i) \(6y^2\)
   
   (ii) \(6((w - z)y)\)
   
   (iii) \(6((x - z)y)\)

**REMARK**

It is to be noted that, if the values of \( z \) and \( w \) presented in the above patterns are taken as the sides of rectangles, then the area of each rectangle is expressed as the difference of two squares.

**REFERENCES**


INTEGRAL SOLUTIONS OF
\[ x^3 + y^3 + (x + y) w^2 = 4z^3 \]

M.A. GOPALAN & J. KALIGA RANI

**ABSTRACT:** Integral solutions of the cubic diophantine equation with 4 unknowns are obtained by employing linear transformation. A few interesting relations between the solutions and special polygonal numbers are presented.

1. **INTRODUCTION**

It is known that finding integral solutions for the ternary cubic homogeneous or non-homogeneous diophantine equations have been an interest to many mathematicians since antiquity as can be seen from [1, 2, 3]. In [4, 5] integral solutions of binary cubic non-homogeneous diophantine equations have been studied. In [3] two double parameter solutions for the equation \( x^2 + y^2 = z^3 \) are given.

In this communication, we look into the integral solutions of \( x^3 + y^3 + (x + y) w^2 = 4z^3 \) by using the linear transformations. A few interesting relations between the solutions and special polygonal numbers namely, Triangular number, Hex number, Octogonal number, Heptagonal number and Pentagonal number are obtained.

2. **METHOD OF ANALYSIS**

The equation under consideration to be solved is

\[ x^3 + y^3 + (x + y) w^2 = 4z^3 \]

which simplifies to

\[ u^2 = 3 v^2 + w^2 \]

by using the linear transformations,

\[ x = u + v, \quad y = u - v, \quad z = u \]

we present below three different patterns of non-trivial integral solutions of (2) and thus in view of (3), we obtain the corresponding integral solutions of (1).

**Pattern 1:** It is well known that (2) is satisfied by

\[ v = 2rs, \quad w = 3r^2 - s^2, \quad u = 3r^2 + s^2 \]

---

*Mathematical Subject Code: 11D25*

*Keywords: Cubic equation with 4 unknowns, Integral solution, Nasty number, Special polygonal numbers.*
and hence, using (3), the non-trivial solutions of (1) are represented by
\[ x = x(r, s) = 3r^2 + s^2 + 2rs \]
\[ y = y(r, s) = 3r^2 + s^2 - 2rs \]
\[ w = w(r, s) = 3r^2 - s^2 \]
\[ z = z(r, s) = 3r^2 + s^2 \]

A few interesting properties observed are given below.

(1) \( z(r, s) + w(r, s) = 6r^2 \) is a Nasty number.

(2) Each of the expressions \( 3(x^2(r, r) - y^2(r, r)) \) and \( 2(z^2(r, r) - w^2(r, r)) \) is a Nasty number.

(3) \( x(r, s) - y(r, s) = 4r^2 \) is a perfect square.

(4) \( z(r, r + 1) - D_r - 1 \equiv 0 \pmod{5} \).

(5) \( x(r, s) + y(r, s) = 2z(r, s) \), \( \therefore x, y, z \) forms an arithmetic progression.

(6) \( w(r + 1, s) + w(r - 1, s) = 2w(r, s) + 6 \).

(7) \( 3(z(r, s) - y(r, s))^2 = z^2(r, s) - w^2(r, s) \).

(8) \( y(r, 1) - H_r \equiv 0 \pmod{1} \).

(9) \( y(r, 1) - 1 = 0 \).

(10) \( x(r, r + 1) - y(r, r + 1) = 8T_r \).

**Pattern 2:** It is to be noted that the equation (2), is also satisfied by
\[ v = 2rs, \ w = r^2 - 3s^2, \ u = r^2 + 3s^2 \]

and the corresponding solutions of (1) are found to be
\[ x = x(r, s) = r^2 + 3s^2 + 2rs \]
\[ y = y(r, s) = r^2 + 3s^2 - 2rs \]
\[ w = w(r, s) = r^2 - 3s^2 \]
\[ z = z(r, s) = r^2 + 3s^2 \]

**Properties**

(1) \( z(r + 1, s) + z(r - 1, s) = 2(z(r, s) + 1) \).

(2) \( 6(x(r, s) - y(r, s))^2 = 8(z^2(r, s) - w^2(r, s)) \).

(3) \( w(r + 1, s) - w(r - 1, s) = x(r, 1) - y(r, 1) \).

(4) \( 2x(r, 1) - HE_r - 6 \equiv 0 \pmod{5} \).
Integral Solutions of \( x^3 + y^3 + (x + y) w^2 = 4z^3 \)

(5) \( 2y(r, 1) - HE_r - 6 \equiv 0 \pmod{5} \).
(6) \( 4z(r, 1) + x(r, 1) - 2 HP_r - 15 \equiv 0 \pmod{5} \).
(7) \( x(r, 1) + w(r, 1) = 2T_r \).
(8) \( x(r^2, r + 1) + w(r^2, r + 1) - 2r^4 = 4PY_r \).
(9) \( y(1, s) - 2P_s - 1 \equiv 0 \pmod{1} \).
(10) \( y(1, s) - 2C_s - 1 \equiv 0 \pmod{1} \).

**Pattern 3:** Let \( a, b \) be two non-zero integers so that

\[
u = a^2 + 3b^2
\]  

Substituting (4) in (2) and employing the method of factorization, we have,

\[
(w + i\sqrt{3})v = (a + i\sqrt{3}b)^2 (a - i\sqrt{3}b)^2
= (a + i\sqrt{3}b)^2 (a - i\sqrt{3}b)^2 ((1 + i\sqrt{3})/2)((1 - i\sqrt{3})/2)
\]

(5)

Now, equation (5) is equivalent to the following system of double equations

\[
(w + i\sqrt{3})v = (a + i\sqrt{3}b)^2 ((1 + i\sqrt{3})/2)
\]

(6)

\[
(w - i\sqrt{3})v = (a - i\sqrt{3}b)^2 ((1 - i\sqrt{3})/2)
\]

(7)

Equating real and imaginary parts in either (6) or (7) we have

\[
w = w(a, b) = (a^2 - 3b^2 - 6ab)/2
\]

(8)

\[
v = v(a, b) = (a^2 - 3b^2 + 2ab)/2
\]

(9)

From (3), (4), (8) and (9) the values of \( x, y, z \) and \( w \) satisfying (1) are given by,

\[
x = x(a, b) = (3a^2 + 3b^2 + 2ab)/2
\]

\[
y = y(a, b) = (a^2 + 9b^2 - 2ab)/2
\]

\[
w = w(a, b) = (a^2 - 3b^2 - 6ab)/2
\]

\[
z = z(a, b) = a^2 + 3b^2
\]

As our interest centers on finding integral solutions of (1), it is seen that the above solutions will be in integers when \( a \) and \( b \) are of the same parity. Thus, we have the following two choices of solutions,

**Choice 1:** Let \( a = 2k, b = 2l, k, l \in z - \{0\} \)

The corresponding solutions are given by,

\[
x = x(k, l) = 6k^2 + 6l^2 + 4kl
\]
\[ y = y(k, l) = 2k^2 + 13l^2 - 4kl \]
\[ w = w(k, l) = 2k^2 - 6l^2 - 12kl \]
\[ z = z(k, l) = 4k^2 + 12l^2 \]

**Properties**

1. \( x(k, 1) - 2O_k - 6 \equiv 0 \pmod{8} \).
2. \( x(k, 1) - 4P_k - 6 \equiv 0 \pmod{6} \).
3. \( x(k, 1) - 4C_k + 4 \equiv 0 \pmod{10} \).
4. \( 2T_k - w(k, 1) - 6 \equiv 0 \pmod{14} \).
5. \( x(k, 1) - 2CS_k + 4G_k = 12 \).
6. \( z(l, l) = 16F \) is a perfect square.
7. \( x(l, l) + w(l, l) = 0 \).
8. \( 4w(k, l) + 2z(k, l) \) is expressed as difference of squares.
9. \( z(l, l) - 2w(l, l) \) is 8 times a Nasty number.
10. Each of the expression \( 6z(3r^2 - s^2, 2rs) \), \( 6z(r^2 - 3s^2, 2rs) \) and \( (2k + 2l)^2 - 2(z(k, 1) - y(k, 1)) \) is a Nasty number.

**Choice 2:** Let \( a = 2k - 1, b = 2l - 1 \)

For this choice, the corresponding solutions are obtained as

\[ x = x(k, l) = 6k^2 + 6l^2 - 8k - 8l + 4kl + 4 \]
\[ y = y(k, l) = 2k^2 + 13l^2 - 14kl - 16l + 4 \]
\[ w = w(k, l) = 2k^2 - 6l^2 + 4k + 12l - 12kl - 4 \]
\[ z = z(k, l) = 4k^2 + 12l^2 - 4k + 12l + 4 \]

**Properties**

1. \( x(k, 1) - 2H_k \equiv 0 \pmod{2} \).
2. \( x(k, 1) - 4C_k \equiv 0 \pmod{2} \).
3. \( x(k, 1) - 2P_k - 2 \equiv 0 \pmod{2} \).
4. \( z(1, l) = 4HE_l \).
5. \( x(1, l) - 2HE_l \equiv 0 \pmod{2} \).
6. \( 2z(k, l) - x(k, l) - 2(k - 1)^2 - 24T_l \equiv 0 \pmod{4} \).
7. \( 2z(k, l) - x(k, l) - y(k, l) - 10T_l \equiv 0 \pmod{23} \).
Integral Solutions of $x^3 + y^3 + (x + y)w^2 = 4z^4$

(8) Each of the expression $3(x(\alpha^2, 1) - 2H_{\alpha^2}), 60(z(k, l) - 8T_{k,l} + 4T_l - 4), 6(2z - x - 2(k - l)^2 - 24T_l - 4)$ is a Nasty number.

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INTEGRAL SOLUTIONS OF
\[ x^3 + y^3 + (x + y)xy = z^3 + w^3 + (z + w)zw \]

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ABSTRACT

Integral solutions of the cubic diophantine equation with four unknowns are obtained by employing linear transformation.

Key words: Cubic diophantine equation with 4 unknowns, Integral solution, Special polygonal numbers.

INTRODUCTION

It is known that finding integral solutions for the cubic homogeneous or non-homogeneous diophantine equations is an interesting concept to many mathematicians, as it can be seen from [1, 2, 5]. In [3, 4] integral solutions of binary cubic non-homogeneous diophantine equations have been studied. In [1] two double parameter solutions for the equation \( x^2 + y^2 = z^3 \) are given.

In this communication, we seek non-zero integral solutions of
\[ x^3 + y^3 + (x + y)xy = z^3 + w^3 + (z + w)zw \]
by using the linear transformations. A few interesting relations between the solutions and special polygonal numbers namely, Triangular number, Hex number, Octogonal number, Heptagonal number, Centered square number, Centered triangular number Gnomonic number, Stella octangula number, Octahedral number, Decogonal number and Pentagonal number are obtained.

Notations:
- \( D_r = 4r^2 - 3r \) = Decogonal number of rank \( r \).
- \( H_r = 3r^2 - 3r + 1 \) = Hex number of rank \( r \).
- \( O_r = r(3r - 2) \) = Octagonal number of rank \( r \).
- \( T_r = (r(r + 1))/2 \) = Triangular number of rank \( r \).
- \( H_r = r(2r - 1) \) = Hexagonal number of rank \( r \).
- \( H_r = (5r - 3)/2 \) = Heptagonal number of rank \( r \).
- \( S_{O_r} = r(2r^2 - 1) \) = Stella octangula number of rank \( r \).
- \( C_{T_r} = (3r^2 - 3r + 2)/2 \) = Centered triangular number of rank \( r \).

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\[ CS_r = r^2 + (r - 1)^2 \]  = Centered square number of rank r.
\[ PY_r = (r^2 (r + 1)) / 2 \]  = Pentagonal pyramidal number of rank r.
\[ G_r = 2r - 1 \]  = Gnomonic number of rank r.
\[ OC_r = (r (2r^2 + 1)) / 3 \]  = Octahedral number of rank r.
\[ C_r = r^3 \]  = Cubic number.

**METHOD OF ANALYSIS**

The equation under consideration to be solved is
\[ x^3 + y^3 + (x + y)xy = z^3 + w^3 + (z + w)zw \]  (1)
which simplifies to
\[ u(u^2 + v^2) = r(r^2 + s^2) \]  (2)
by using the linear transformations,
\[ x = u + v, \quad y = u - v, \quad z = r + s, \quad w = r - s \]  (3)
in which \( u, v, r, s \) are non-zero and distinct integers.

We present below the patterns of non-trivial integer solutions of (2) and thus in view of (3), we obtain the corresponding integral solutions of (1).

**Pattern 1:**

Assume \( u + iv = (A + iB)(r + is) \)  (4)
where \( A, B \) are both non-zero integers.

Equating the real and imaginary parts in (4), we get
\[ u = Ar - Bs \]  (5)
\[ v = Br + As \]  (6)

Using (5) and (6) in (2), we get, after some algebra,
\[ s = \lambda(A(A^2 + B^2) - 1) \]  (7)
\[ r = \lambda B(A^2 + B^2) \]  (8)
where \( \lambda \) is any non-zero integer.

From (5), (6), (7) and (8), the values of \( x, y, z \) and \( w \) satisfying (1) are given by
\[ x = x(A, B, \lambda) = \lambda B + \lambda (A^2 + B^2)^2 - A \]
\[ y = y(A, B, \lambda) = \lambda B - \lambda (A^2 + B^2)^2 + A \]
\[ z = z(A, B, \lambda) = \lambda (A^2 + B^2)(A + B) - \lambda \]
\[ w = w(A, B, \lambda) = \lambda (A^2 + B^2)(B - A) + \lambda \]

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\[ x^3 + y^3 + (x+y)xy = z^3 + w^3 + (z+w)zw \]

A few interesting properties observed are given below.
1. Each of the expressions is a Nasty number.
   a) \( x(A, B, 3\alpha^2) - y(A, B, 3\alpha^2) + 2A, \)
   b) \( x(A, 3\lambda, \lambda^2) + y(A, 3\lambda, \lambda) \)
   c) \( 6w(6, B, \alpha^2 + 1) - Bz(6, B, \alpha^2 + 1) - y(6, B, \alpha^2 + 1) \)
2. \( x(A, B, 2) - y(A, B, 2) + 2A \) is a perfect square.
3. \( z(\alpha, \beta, \lambda) + w(\alpha, \beta, \lambda) = (x(\alpha, B, \lambda) + y(\alpha, B, \lambda)) A^2 + B^2 \)
4. \( x(1, 1, \lambda) - D_\lambda + C_\lambda = 0 \mod 2. \)
5. \( z(1, 1, \lambda) - 2PY_A = 1. \)
6. \( w(A, 1, 1) + 2PY_A - HE_A = 2. \)
7. \( z(A, 1, 1) - 2PY_A - G_A = 2. \)
8. \( z(1, 1, 1) = C_B + T_B. \)
9. \( 2w(1, B, 1) = 30C_B - HE_B. \)
10. \( Aw(A, B, \lambda) - Bz(A, B, \lambda) - y(A, B, \lambda) = A(\lambda - 1). \)
11. \( 2w(-1, B, \lambda) - \lambda PY_B = 0 \mod 2. \)

Pattern 2:

Here we assume
\[(u + iv) / (r + is) = (A + iB) (3 + 4i)/5 \quad (9)\]
Equating the real and imaginary parts in (9), we get
\[u = (A(3r - 4s) - B(4r + 3s))/5 \quad (10)\]
\[v = (A(4r + 3s) + B(3r - 4s))/5 \quad (11)\]
There are three choices for obtaining \( u \) and \( v \) in integers.

Choice 1

Let \( A = 5A, B = 5B \)
Equations (10) and (11) reduces to
\[u = A(3r - 4s) - B(4r + 3s) \quad (12)\]
\[v = A(4r + 3s) + B(3r - 4s) \quad (13)\]
Substituting (12) and (13) in (3), the values of \( x, y, z \) and \( w \) are obtained as follows.
\[x = x(A, B, r, s) = u + v = A(7r - s) + B(-7r - s) \quad (14)\]
\[y = y(A, B, r, s) = u - v = A(-r - 7s) - B(7r - s) \quad (15)\]

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\[ z = z(r, s) = r + s \]
\[ w = w(r, s) = r - s \]

Properties:

1. \( 2 \left( z(r^2, s) + w(r^2, s) \right) \) is a perfect square.
2. \( z(r, r - 1) = G_r \).
3. \( z(r, s) w(r, s) \) is expressed as difference of squares.

4. \( 3z((r + 1)^2, 1) - 2 H_r - 5 \equiv 0 \pmod{9} \).
5. \( 3z((r + 1)^2, 1) - 2 C T_r - 3 \equiv 0 \pmod{9} \).
6. \( x(1, 1, r, s) + y(1, 1, r, s) + 8 z(r, s) = 6 w(r, s) \).
7. \( z(r^2, (r - 1)^2) = CS_r \).
8. Each of the expressions is a Nasty number.
   a) \( 3(z(r^2, s) + w(r^2, s)) \)
   b) \( x(1, 1, r^2, 1) - 8 \)

9. \( 2P_r + 4H P_r - x(2, 1, r^2, r) \equiv 0 \pmod{2} \).
10. \( z(r^3, (r + 1)^2) = 6 T H_r + G_r - T_r \).
11. \( 5O_r + x(1, 1, r^3, r) + 15r = 0 \)
12. \( 3O_r + x(1, 1, r^3, r) + 13r = 0 \).

Choice 2:
Assume \( s = 2k - 1, r = 5n + k - 3 \) \((14)\)

Substituting \((14)\) in \((10)\) and \((11)\), we get,
\[ u = A(3n - k - 1) - B(4n + 2k - 3) \]
\[ v = A(4n + 2k - 3) + B(3n - k - 1) \]

The values of \( x, y, z \) and \( w \) satisfying \((1)\) are given by
\[ x = x(A, B, n, k) = u + v = A(7n + k - 4) + B(-n - 3k + 2) \]
\[ y = y(A, B, n, k) = u - v = A(-n - 3k + 2) + B(-7n - k + 4) \]
\[ z = z(n, k) = r + s = 5n + 3k - 4 \]
\[ w = w(n, k) = r - s = 5n - k - 2 \]

Properties:

1. \( y(1, 1, n^2, n) + z(n^2, n) + 2 C T_n + 2G_n = 2 \).
2. \( w(k^2, k) + 1 = H_k + O_k + G_k \).
3. \( z(n, k) - w(n, k) = 2G_k \).
4. \( z(k^2, k) + w(k^2, k) - 4H_k + 6 \equiv 0 \pmod{8} \).
5. \( z(k^2, k) + w(k^2, k) - 4C T_k - 2H_k + 10 \equiv 0 \pmod{10} \).
6. \( z(k^2, k) + w(k^2, k) - 2C T_k - 2H_k + 8 \equiv 0 \pmod{11} \).
7. \( w(n^2, 3n) + 2 = 2 H P_n \).
8. \( w(n^2, 3n) + 2 = D_n + S_n \).

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\[ x^3 + y^3 + (x+y)xy = z^3 + w^3 + (z+w)zw \]

9. \( w(n^2, n^2) + 2 \) is a perfect square.
10. \( 6(w(n^2, n^2) + 2) \) is a Nasty number.

Choice 3:

Assume \( s = 2k, r = 5n + k - 5 \) \hspace{1cm} (15)

Employing (15) in (10) and (11), we get

\[
\begin{align*}
u &= A (3n - k - 3) - B (4n + 2k - 4)
v &= A (4n + 2k - 4) + B (3n - k - 3)
\end{align*}
\]

The values of \( x, y, z \) and \( w \) satisfying (1) are given by

\[
\begin{align*}
x &= x(A, B, n, k) = u + v = A (7n + k - 7) + B (-n - 3k + 1)
y &= y(A, B, n, k) = u - v = A (-n - 3k + 1) + B (-7n - k + 7)
z &= z(n, k) = r + s = 5n + 3k - 5
w &= w(n, k) = r - s = 5n - k - 5
\end{align*}
\]

Properties:

1. \( w(n^2, n) + 4 = 2HP_n + G_n \).
2. \( z(n, k^2) - w(n, k^2) = 4k^2 \) is a perfect square.
3. Each of the expressions is a Nasty number
   a) \( 6(z(n, k^2) - w(n, k^2)) \)
   b) \( 6(w(n^2, n^2) + 5) \)
4. \( x(1, 1, n^2, n) + 8 = 4CT_n + 2G_n \).
5. \( y(1, 1, n, k) + 4(G_n + G_k) = 0 \mod 4 \).
6. \( z(n^2, n) + 2 = HP_n + 3G_n \).
7. \( 2x(1, 1, n, k) - y(1, 1, n, k) + 20 = 0 \mod 20 \).

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INTEGRAL SOLUTIONS OF \( x^2 - ay^2 = (a-1)z^3 \)  
(\( a>1 \) and \( a \) is square free)

M.A. Gopalan* and J. Kaliga Rani**

ABSTRACT
The ternary cubic Diophantine equation \( x^2 - ay^2 = (a-1)z^3 \)  
(\( a>1 \) and \( a \) is square free) is analysed for its non-zero integral solutions. A few interesting relations between the solutions, Special polygonal numbers and pyramidal numbers are presented for \( a=2 \) and \( a=3 \).

KEYWORDS
Cubic Diophantine equation with three unknowns, Integral solutions, Special polygonal numbers, Special pyramidal numbers.

INTRODUCTION
Integral solutions for the cubic homogeneous or non-homogeneous diophantine equations is an interesting concept, as it can seen from [1,2,3]. In [4,5,6], integral solutions of binary cubic non-homogeneous Diophantine equations have been studied. In [3], two double parameter solutions for the equation \( x^2 + y^2 = z^2 \) are given. In [7,8,9], a few special cases of ternary cubic Diophantine equations are studied.

In this communication, we look into the integral solutions of yet another ternary cubic equation \( x^2 - ay^2 = (a-1)z^3 \), (\( a>1 \) and \( a \) is square free). A few interesting relations between the solutions and Special numbers namely Rhombic dodecahedral number, Tetrahedral number, Triangular number, Pronic number, Gnomonic number, Octahedral number, Square pyramidal number, Hex number, Decagonal number, Centered triangular number, Heptagonal number, Heptagonal pyramidal number, Pentagonal number are obtained.

NOTATIONS

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>DEFINITION</th>
<th>NAME</th>
</tr>
</thead>
<tbody>
<tr>
<td>( RD_r )</td>
<td>((2r-1)(2r^2-2r+1))</td>
<td>Rhombic dodecahedral number of rank ( r ).</td>
</tr>
<tr>
<td>( TH_r )</td>
<td>((r(r+1)(r+2))/2)</td>
<td>Tetrahedral number of rank ( r ).</td>
</tr>
<tr>
<td>Symbol</td>
<td>Formula</td>
<td>Description</td>
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<tr>
<td>$T_r$</td>
<td>$(r(r+1)/2)$</td>
<td>Triangular number of rank $r$.</td>
</tr>
<tr>
<td>$PR_r$</td>
<td>$(r(r+1))$</td>
<td>Pronic number of rank $r$.</td>
</tr>
<tr>
<td>$G_r$</td>
<td>$2r - 1$</td>
<td>Gnomonic number of rank $r$.</td>
</tr>
<tr>
<td>$OC_r$</td>
<td>$(r(2r^2+1)/3)$</td>
<td>Octahedral number of rank $r$.</td>
</tr>
<tr>
<td>$SP_r$</td>
<td>$(r(r+1)(2r+1)/6)$</td>
<td>Square pyramidal number of rank $r$.</td>
</tr>
<tr>
<td>$H_r$</td>
<td>$3r^2-3r+1$</td>
<td>Hex number of rank $r$.</td>
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<tr>
<td>$D_r$</td>
<td>$4r^2-3r$</td>
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</tr>
<tr>
<td>$C_r$</td>
<td>$(3r^2-3r+2)/2$</td>
<td>Centered triangular number of rank $r$.</td>
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<td>$HP_r$</td>
<td>$(r(5r-3)/2$</td>
<td>Heptagonal number of rank $r$.</td>
</tr>
<tr>
<td>$HY_r$</td>
<td>$(r(r+1)(5r-2)/6$</td>
<td>Heptagonal pyramidal number of rank $r$.</td>
</tr>
<tr>
<td>$P_r$</td>
<td>$(r(3r-1)$</td>
<td>Pentagonal number of rank $r$.</td>
</tr>
</tbody>
</table>

**METHOD OF ANALYSIS**

The equation under consideration is

$$x^2 - a^2 = (a-1)z^2, \quad (a>1 \text{ and } a \text{ is square free})$$

To start with, it is observed that equation (1) is satisfied by the values of $x, y$ and $z$ given by

$$x = (a-1)^2 \alpha^k, \quad y = 0, \quad z = (a-1) \alpha^{2k}, \text{ and}$$

$$x = a^2 \alpha^{3k}, \quad y = a\alpha^{3k}, \quad z = a\alpha^{2k},$$

where $k$ is any positive integer.

However, there are also other values of $x, y$ and $z$ satisfying (1), which we present below.

We take $z = z(p, q, a) = p^2 - aq^2, \quad (p, q \in Z^2)$

Then, equation (1), on factorization, is written as

$$(x+\sqrt{a} y)(x-\sqrt{a} y) = (\sqrt{a} + 1)(\sqrt{a} - 1)(p+\sqrt{a} q)^3(p-\sqrt{a} q)^3$$

And is satisfied by

$$x = x(p, q, a) = p^3 + 3p^2aq + a^2q^2 + 3paq^2$$

$$y = y(p, q, a) = p^3 + 3paq^2 + 3p^2q + aq^3$$

Thus, equations (2), (3) and (4) represent the non-trivial integral solutions of (1).

To obtain the nature of solutions of (1), one has to evaluate equations (2), (3) and (4) when $a,p$ and $q$ takes particular values.

For simplicity and clear understanding we present below the integral solutions and the corresponding properties of (1) when $a = 2, a = 3$.

**CHOICE 1**

Let $a = 2$. The corresponding equation to be solved is

$$x^2 - 2y^2 = z^2$$

If $(x_0, y_0, z_0)$ is any solution of (5), then $(3x_0 - 4y_0, 2x_0 - 3y_0, z_0)$ will also satisfy (5).

Substituting $a = 2$ in (3), (4) and (2) we get

$$x = x(p, q, 2) = p^3 + 6p^2q + 4q^2 + 6pq^2$$

$$y = y(p, q, 2) = p^3 + 6pq^2 + 3p^2q + 2q^3$$

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\[ z = z(p, q, 2) = p^2 - 2q^2 \]

**PROPERTIES**

1. \( y(1, q, 2) - 6SP_q - H_q = 0 \) (mod 5).
2. \( x(1, q, 2) - RD_q - 2CT_q - 2HP_q - D_q = 0 \) (mod 11).
3. \( x(p, 1, 2) - 6TH_p - 2CT_p - 2 = 0 \) (mod 7).
4. \( y(p, 1, 2) - 6TH_p - 2 = 0 \) (mod 4).
5. \( x(1, q, 2) - 6HP_q - 6TH_p + 30C_q - 1 = 0 \) (mod 7).
6. \( y(1, q, 2) - OC_q - 4P_q - 1 = 0 \) (mod 4).
7. \( 10(z(p+1, 1, 2) - PR_p + 1) = 0 \) (mod 10).
8. \( z(p+1, 1, 2) - 2T_q + 1 - p = 0. \)
9. \( 2z(2p, p, 2) \) is a perfect square.
10. \( 3z(2p, p, 2) \) is a Nasty number.
11. \( Z(p+2, 1, 2) - 2T_p - G_p - 3 - p = 0 \).
12. \( Z(p+2, 1, 2) - PR_p - G_p + 3 - p = 0 \).

**CHOICE 2**

Let \( a = 3 \). The corresponding equation to be solved is

\[ x^2 - 3y^2 = 2z^3 \]

(6)

If \((x_o, y_o, z_o)\) is any given solution of (6), then \(((n+1)x_o, 3ny_o, nx_o - (3n-1)y_o, z_o)\) will also satisfy (6).

Substituting \( a = 3 \) in (3), (4) and (2), we get

\[ x = x(p, q, 3) = p^3 + 9p^2q + 9pq^2 \]
\[ y = y(p, q, 3) = p^3 + 9p^2q^3 + 3p^2q + 3q^3 \]
\[ z = z(p, q, 2) = p^2 - 3q^2 \]

**PROPERTIES**

1. \( x(p, 1, 3) - y(p, 1, 3) - 6 \) is a Nasty number.
2. \( 6(x(p, 1, 3) - y(p, 1, 3) - 6) \) is a perfect square.
3. \( y(p, 1, 3) - 6TH_p - 3 = 0 \) (mod 7).
4. \( z(p, p, 3) \) is a Nasty number.
5. \( 6(z(p, p, 3)) \) is a perfect square.
6. \( 3(270C_q - x(1, q, 3) - 3z(1, q, 3) + 4) \) is a cubic number.
7. \( y(p, 1, 3) + z(p, 1, 3) - 6TH_p + PR_p = 0 \) (mod 6).
8. \( x(p+1, 1, 3) - TH_p - 18T_p - 28 = 0 \) (mod 19).
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INTEGRAL SOLUTIONS OF TERNARY CUBIC EQUATION \( x^3 + y^3 + 4z^2 = 3xy(x + y) \)

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The Ternary Cubic Diophantine equation given by \( x^3 + y^3 + 4z^2 = 3xy(x + y) \) is analyzed for its non-zero distinct integral solutions. A few interesting relations among the solutions are presented.

1. INTRODUCTION

It is known that finding integral solutions for the ternary cubic homogeneous or non-homogeneous Diophantine equations have been an interest to many mathematicians since antiquity as can be seen from [1, 2, 3]. In [4, 5, 6], integral solutions of binary cubic non-homogeneous Diophantine equations have been studied. In [3], two parameter solutions for the equation \( x^2 + y^2 = z^3 \) are given. In [7-13], a few special cases of Ternary cubic Diophantine equations are studied.

In this communication, we look into the integral solutions of \( x^3 + y^3 + 4z^2 = 3xy(x + y) \) using the linear transformations. A few interesting relations among the solutions are presented.

KEYWORDS : Ternary Cubic Diophantine equation, Integral solutions.
MSC : 11 D 25
METHOD OF ANALYSIS

The Diophantine equations to be solved is
\[ x^3 + y^3 + 4z^2 = 3xy(x + y) \]  \hspace{1cm} (1)

which, on using the linear transformations,
\[ x = u + v, \quad y = u - v, \quad z = 2ku \quad (u \neq v \neq 0) \]  \hspace{1cm} (2)
simplifies to
\[ Y^2 = 3X^2 + \sigma^2 \]  \hspace{1cm} (3)
where \[ Y = u - 2k^2, \quad X = v, \quad \text{and} \quad \sigma = 2k^2 \]

Now, the general solution \((X_s, Y_s)\) of the Pellian \((3)\), is represented by
\[ Y_s + \sqrt{3}X_s = \sigma(2 + \sqrt{3})^{s+1}, s = 0, 1, 2, ... \]  \hspace{1cm} (4)

Since irrational roots occurs in pairs, we have
\[ Y_s - \sqrt{3}X_s = \sigma(2 - \sqrt{3})^{s+1}, s = 0, 1, 2, ... \]  \hspace{1cm} (5)

From (4) and (5), we get
\[ Y_s = k^2f \]
\[ X_s = \frac{k^2g}{\sqrt{3}} \]

where
\[ f = ((2 + \sqrt{3})^{s+1} + (2 - \sqrt{3})^{s+1}) \quad \text{and} \quad g = ((2 + \sqrt{3})^{s+1} - (2 - \sqrt{3})^{s+1}) \]

Thus, the integral solutions of \((1)\) are
\[ x_s = 2k^2 + k^2f + \frac{k^2g}{\sqrt{3}} \]
\[ y_s = 2k^2 + k^2f - \frac{k^2g}{\sqrt{3}} \]
\[ z_s = 2k(k^2f + 2k^2) \]

The recurrence relations satisfied by the solutions of \(x_s, \ y_s, \ \text{and} \ z_s\) are respectively given by
\[ x_{s+2} - 4x_{s+1} + x_s = -4k^2 \]
\[ y_{s+2} - 4y_{s+1} + y_s = -4k^2 \]
\[ z_{s+2} - 4z_{s+1} + z_s = -8k^2, \ s = 0,1,2,\ldots \]

A few observations are presented below:

1. \( k(x_s + y_s) = z_s \)
2. \( (x_s + y_s - 4k^2)^2 - 3(x_s - y_s)^2 \) is a perfect square.
3. Each of the following expressions is a Nasty number.
   (i) \[ 6[3(2kx_s - z_s)^2 + 16k^6] \]
   (ii) \[ 6(x_s + y_s - 4k^2)^2 \]
   (iii) \[ 3(x_{2s+1} + y_{2s+1}) \]
4. \( 4k[x_{3s+2} + y_{3s+2} - 16k^2 + 3(x_s + y_s)] \) is a cubical integer.
5. \( 8k^2[x_{4s+3} + y_{4s+3} + 8k^2(x_{2s+1} + y_{2s+1} + 1) - 32k^4] \) is a quartic integer.
6. In particular, the following are the few observations when \( k=1 \):
   (i) \( x_s = y_{s+1} \)
   (ii) \( y_s x_{s+1} - x_s y_{s+1} = 0 \text{ (mod 8)} \)
   (iii) \( y_{2s} \equiv y_{2s-1} \text{ (mod 16)}, s = 1,2,3,\ldots \)
   (iv) \( y_{3s} \equiv y_{3s-1} \text{ (mod 60)} \)
   (v) \( x_{3s-1} \equiv x_{3s-2} \text{ (mod 60)} \)
   (vi) \( x_{2s} + y_{2s} + z_{2s} = 0 \text{ (mod 24)} \)
   (vii) \( z_{2s+1} \text{ divides } x_{2s+1} + y_{2s+1} \)
   (viii) \( z_s - x_s = x_{s-1}, s = 1,2,3,\ldots \)
   (ix) \( 16 \text{ divides } y_s^2 + z_s^2 \)
   (x) Each of the following expressions is a Nasty number
      (i) \[ 6(x_{2s-1} + y_{2s-1} + z_{2s-1}) \]
      (ii) \[ \frac{3}{8}(y_s z_s)^2 \]
      (iii) \[ \frac{3}{8}(x_s z_s)^2 \]
   (xi) Each of the following expressions is a Perfect square
      (i) \[ 3z_{2s} \]
      (ii) \[ 2z_{2s+1} \]
   (xii) (i) \( \Delta^m z_s \equiv 0 \text{ (mod 4)}, \text{ for all } m \)
(ii) \[ \Delta x_s = x_{s+1} - y_{s+1} \]
(iii) \[ \Delta y_{s+1} = \Delta x_s \text{ where } \Delta \text{ is forward difference operator.} \]

**REMARK**

Suppose in (2), \( z = ku \), then the integral solutions of (1) are obtained as below

\[
\begin{align*}
x_s &= \frac{k^2 f}{4} + \frac{k^2 g}{2\sqrt{12}} \\
y_s &= \frac{k^2 f}{4} - \frac{k^2 g}{2\sqrt{12}} \\
z_s &= \frac{k^2 f}{4}
\end{align*}
\]

where
\[
\begin{align*}
f &= [(7 + 2\sqrt{12})^{s+1} + (7 - 2\sqrt{12})^{s+1} + 2] \\
g &= [(7 + 2\sqrt{12})^{s+1} - (7 - 2\sqrt{12})^{s+1}]
\end{align*}
\]

The few observations noticed are given below:

(i) \( 2k(x_s + y_s) = 4z_s \)
(ii) \( 24(x_s + y_s)^2 \) is a Nasty number.
(iii) \( k^2(x_{2s+1} + y_{2s+1}) = 2(x_s + y_s)^2 \)
(iv) \( k[2(x_{3s+2} + y_{3s+2} - k^2) + 6(x_s + y_s)] \) is a cubical integer.

One may search for other patterns of solutions for the cubic equation under consideration and search for the properties.

**REFERENCES**

\(x^3 + y^3 = a(x^2 - y^2) + b(x + y)\). International Journal of Mathematics, Computer Sciences and Information Technology, 1(1)(2000), 135-136.


