Chapter 3

Controllability of Higher Order Fractional Dynamical Systems

3.1 Introduction

Control system is an interconnection of components forming a system configuration that will provide a desired system response. Controllability is one of the structural properties of dynamical systems and it plays a major role in the development of modern mathematical control theory. It means the ability to move a system around the entire configuration space using only certain admissible manipulations. It deals with whether or not the state of a state-space equation can be controlled from the input. The problem of controllability of dynamical systems is widely used in analysis and design of control systems. The controllability results for linear and nonlinear systems represented by first order differential equations have been studied extensively in the literature. (see [11, 13, 52, 120] and the references therein). Sharma and George [107] investigated the controllability of the system governed by a matrix second order nonlinear differential equation in finite dimensional spaces using a trigonometric matrix approach. For a detailed study on second order matrix system, one can refer [113].

The controllability of fractional dynamical systems is an important issue for many applied problems because the use of fractional order derivatives and integrals in control theory leads to better results than those of the integer order systems. For example, the fractional order models need fractional order controllers for more effective control of dynamical systems [94]. In recent years few contributions have been devoted to the controllability results for fractional dynamical systems of order $0 < \alpha < 1$ by many
Researchers [14, 15, 17, 18, 19, 20, 22, 29, 42, 72, 82, 83, 116] using controllability Grammian matrix and rank condition. However, there is no work reported on the problem of controllability of fractional dynamical systems of order $1 < \alpha \leq 2$.

Motivated by this fact, in this chapter, we consider the fractional harmonic oscillator which is governed by the following fractional differential equation of order $1 < \alpha \leq 2$.

$$C D_0^\alpha x(t) + a^2 x(t) = f(t), \quad a > 0,$$

where the function $f(t)$ represents the forcing function. Narahari Achar et al. [85] studied the equation of motion of fractional oscillator and expressed the generalized momentum and the total energy by using fractional calculus. Yongyong and Xiu'e [124] used a new type of fractional oscillator in which the restoring force is represented by a term containing fractional derivative and retaining its characteristics. However, it should be emphasized that, to the best of our knowledge, controllability results for fractional harmonic oscillator type dynamical systems have not yet been established.

In order to fill this gap, we make an attempt to study the controllability results for fractional harmonic oscillator type dynamical systems. The necessary and sufficient conditions for the controllability results for linear systems are derived using controllability Grammian matrix which is defined by means of Mittag-Leffler matrix function. The main difficulty arising in the control problem for nonlinear dynamical systems is the lack of general methods for solving nonlinear differential equations. The well-known fixed point theorems are used to derive different types of conditions depending on the nonlinear function [12]. Sufficient conditions for the controllability of nonlinear fractional dynamical systems are established by using Schauder’s fixed point theorem with the assumption on the nonlinear function as in [17, 35]. Moreover, the sufficient conditions for the controllability of nonlinear damped fractional dynamical systems are established by using Schaefer’s fixed point theorem as in [16].

### 3.2 Preliminaries

In this section, the solution of a fractional differential equation is obtained using Laplace transform method and the properties of Mittag-Leffler function. Laplace transform
technique has been considered as an efficient way in solving ordinary differential equations with initial conditions. But for fractional initial value problems, the Laplace transform technique works effectively only for relatively simple equations because of the difficulties of calculating inversion of Laplace transforms. To overcome this difficulty, Sheng et al. [108] investigated a complicated fractional differential equation and solved numerically using numerical inverse Laplace transform algorithms. Moreover sufficient conditions to guarantee the rationality of solving fractional differential equations were employed in [50].

Consider the linear fractional differential equation of the form

\[
\begin{aligned}
\mathcal{C}D^\alpha_0 x(t) + A^2 x(t) &= f(t), \\
x(0) &= x_0, \quad x'(0) = y_0,
\end{aligned}
\tag{3.2.1}
\]

where \(1 < \alpha \leq 2, x \in \mathbb{R}^n, A\) is an \(n \times n\) matrix and \(f\) is a continuous function. Applying Laplace transform to both sides, we get

\[
s^\alpha X(s) - s^{\alpha-1}x(0) - s^{\alpha-2}x'(0) + A^2 X(s) = F(s).
\]

Then

\[
X(s) = \frac{s^{\alpha-1}}{s^\alpha I + A^2}x_0 + \frac{s^{\alpha-2}}{s^\alpha I + A^2}y_0 + \frac{F(s)}{s^\alpha I + A^2}.
\]

Taking inverse Laplace transform on both sides, we get

\[
\mathcal{L}^{-1}\{X(s)\}(t) = \mathcal{L}^{-1}\left\{s^{\alpha-1} \left(s^\alpha I + A^2\right)^{-1}\right\}(t)x_0 + \mathcal{L}^{-1}\left\{s^{\alpha-2} \left(s^\alpha I + A^2\right)^{-1}\right\}(t)y_0 \\
+ \mathcal{L}^{-1}\left\{F(s) \left(s^\alpha I + A^2\right)^{-1}\right\}(t).
\]

Using the Laplace transform of Mittag-Leffler function, we get the solution of the system (3.2.1) as

\[
x(t) = E_\alpha(-A^2t^\alpha)x_0 + tE_{\alpha,2}(-A^2t^\alpha)y_0 + f(t) * t^{\alpha-1}E_{\alpha,\alpha}(-A^2t^\alpha) \\
= \Phi_0(t)x_0 + \Phi_1(t)y_0 + \int_0^t \Phi(t-s)f(s)ds,
\]

where

\[
\begin{aligned}
\Phi_0(t) &= E_\alpha(-A^2t^\alpha), \\
\Phi_1(t) &= tE_{\alpha,2}(-A^2t^\alpha), \\
\Phi(t) &= t^{\alpha-1}E_{\alpha,\alpha}(-A^2t^\alpha).
\end{aligned}
\]
3.3 Controllability of Linear Fractional Dynamical Systems

Consider the linear fractional dynamical system represented by the fractional differential equation

\[
CD_0^\alpha x(t) + A^2 x(t) = Bu(t), \quad t \in J,
\]

\[
x(0) = x_0, \quad x'(0) = y_0,
\]

where \(1 < \alpha \leq 2\), \(x(t) \in \mathbb{R}^n\), \(u(t) \in L_2(J; \mathbb{R}^m)\) and \(A, B\) are matrices of dimensions \(n \times n, m \times n\) respectively. The solution of the system (3.3.1) is

\[
x(t) = \Phi_0(t)x_0 + \Phi_1(t)y_0 + \int_0^t \Phi(t-s)Bu(s)ds.
\]

Definition 3.3.1. The system (3.3.1) is said to be controllable on \(J\) if, for each vectors \(x_0, y_0, x_1 \in \mathbb{R}^n\), there exists a control \(u(t) \in L_2(J; \mathbb{R}^m)\) such that the corresponding solution of (3.3.1) together with \(x(0) = x_0\) satisfies \(x(T) = x_1\).

We note that our controllability definition is concerned only with steering the states but not the velocity vector \(y_0\) in (3.3.1).

Lemma 3.3.1. [27]. Let \(f_i\), for \(i = 1, 2, \ldots, n\), be \(1 \times p\) vector valued continuous functions defined on \([t_1, t_2]\). Let \(F\) be an \(n \times p\) matrix with \(f_i\) as its \(i\)th row. Then \(f_1, f_2, \ldots, f_n\) are linearly independent on \([t_1, t_2]\) if and only if the \(n \times n\) constant matrix

\[
M(t_1, t_2) = \int_{t_1}^{t_2} F(t)F^*(t)dt
\]

is positive definite.

Theorem 3.3.1. The following statements regarding the linear system (3.3.1) are equivalent:

(a) The linear system (3.3.1) is controllable on \(J\).

(b) The rows of \(\Phi(t)B\) are linearly independent.

(c) The controllability Grammian

\[
M = \int_0^T \Phi(T-s)BB^*\Phi^*(T-s)ds
\]

is positive definite.
Proof. First we prove that \((a) \implies (b)\). Suppose that the system (3.3.1) is controllable, but the rows of \(\Phi(t)B\) are linearly dependent on \(J\). Then there exists a nonzero constant \(n \times 1\) row vector \(y^*\) such that

\[
y^* \Phi(t) B = 0, \text{ for every } t \in J.
\] (3.3.4)

We choose \(x(0) = x_0 = 0, x'(0) = y_0 = 0\). Therefore the solution of (3.3.1) becomes

\[
x(t) = \int_0^t \Phi(t-s)Bu(s)ds.
\]

Since the system (3.3.1) is controllable on \(J\), taking \(x(T) = y\), we have

\[
x(T) = y = \int_0^T \Phi(T-s)Bu(s)ds,
\]

\[
yy^* = \int_0^T y^*\Phi(T-s)Bu(s)ds.
\]

From (3.3.4), \(yy^* = 0\) and hence \(y = 0\). Hence it contradicts our assumption that \(y\) is non-zero. Now we prove that \((b) \implies (a)\). Suppose that the rows of \(\Phi(t)B\) are linearly independent on \(J\). Therefore, by Lemma (3.3.1), the \(n \times n\) constant matrix

\[
M = \int_0^T \Phi(T-s)BB^*\Phi^*(T-s)ds
\]

is positive definite. Now we define the control function as

\[
u(t) = B^*\Phi^*(T-t)M^{-1}[x_1 - \Phi_0(T)x_0 - \Phi_1(T)y_0].
\] (3.3.5)

Substituting (3.3.5) in (3.3.2), we have

\[
x(T) = \Phi_0(T)x_0 + \Phi_1(T)y_0 + \int_0^T \Phi(T-s)BB^*\Phi^*(T-s)M^{-1}
\]

\[
\times [x_1 - \Phi_0(T)x_0 - \Phi_1(T)y_0]ds
\]

\[
= \Phi_0(T)x_0 + \Phi_1(T)y_0 + MM^{-1}[x_1 - \Phi_0(T)x_0 - \Phi_1(T)y_0]
\]

\[
= x_1.
\]

Thus the system (3.3.1) is controllable. The implications \((b) \implies (c)\) and \((c) \implies (b)\) follow directly from Lemma (3.3.1). Hence the desired result.

\[ \square \]

**Remark 3.3.1.** It should be mentioned that the linear fractional dynamical system (3.3.1) reduces to the second order dynamical system for \(\alpha = 2\) and it is of the form (107)

\[
\frac{d^2x(t)}{dt^2} + A^2x(t) = Bu(t), \quad t \in J,
\]
Chapter 3

with the same initial conditions \( x(0) = x_0 \) and \( x'(0) = y_0 \). Further taking \( \alpha = 2 \) in (3.3.2) and (3.3.3), one can easily derive the solution and the controllability Grammian for the above second order dynamical system as

\[
x(t) = \cos(At) \ x_0 + A^{-1} \sin(At) \ y_0 + \int_0^t A^{-1} \sin(A(t-s)) Bu(s) ds,
\]

\[
M = \int_0^T A^{-1} \sin(A(t-s)) BB^* (A^{-1} \sin(A(t-s)))^* ds.
\]

Moreover this problem is also steering the states only but not the velocity vector in (3.3.1).

### 3.4 Controllability of Nonlinear Fractional Dynamical systems

Consider the nonlinear fractional dynamical system represented by the fractional differential equation

\[
^{C}D_{0+}^\alpha x(t) + A^2 x(t) = Bu(t) + f(t, x(t), u(t)), \quad t \in J,
\]

(3.4.1)

\[
x(0) = x_0, \quad x'(0) = y_0,
\]

where \( 1 < \alpha \leq 2 \), \( x(t) \in \mathbb{R}^n, u(t) \in L_2(J; \mathbb{R}^m) \) and \( A, B \) are defined as above and the nonlinear function \( f : J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is continuous. The solution of the system (3.4.1) is

\[
x(t) = \Phi_0(t) x_0 + \Phi_1(t) y_0 + \int_0^t \Phi(t-s) \left[ Bu(s) + f(s, x(s), u(s)) \right] ds.
\]

(3.4.2)

Let \( Z := C(J : \mathbb{R}^n) \times C(J : \mathbb{R}^m) \) be the Banach space of all continuous \( \mathbb{R}^n \times \mathbb{R}^m \)-valued functions defined on the interval \( J \) with the norm \( \|(x, u)\| = \|x\| + \|u\| \) where \( \|x\| = \sup\{|x(t)| : t \in J\} \) and \( \|u\| = \sup\{|u(t)| : t \in J\} \). For simplicity, let \( \tilde{x} = x_1 - \Phi_0(T)x_0 - \Phi_1(T)y_0 \).

For each \((z, v) \in Z\), consider the linear fractional dynamical system

\[
^{C}D_{0+}^\alpha x(t) + A^2 x(t) = Bu(t) + f(t, z(t), v(t)), \quad t \in J,
\]

\[
x(0) = x_0, \quad x'(0) = y_0,
\]

The solution is given by

\[
x(t) = \Phi_0(t) x_0 + \Phi_1(t) y_0 + \int_0^t \Phi(t-s) \left[ Bu(s) + f(s, z(s), v(s)) \right] ds.
\]
**Theorem 3.4.1.** Let the continuous function \( f \) satisfy the condition
\[
\lim_{|(x,u)| \to \infty} \frac{|f(t, x, u)|}{|(x, u)|} = 0 \tag{3.4.3}
\]
uniformly in \( t \in J \) and suppose that the linear system (3.3.1) is controllable. Then the nonlinear system (3.4.1) is controllable.

**Proof.** Define the operator \( P : Z \to Z \) by
\[
P(z, v) = (x, u),
\]
where
\[
u(t) = B^*\Phi^*(T - t)M^{-1}\left[\bar{x} - \int_0^T \Phi(T - s)f(s, z(s), v(s))ds\right]
\]
and
\[
x(t) = \Phi_0(t)x_0 + \Phi_1(t)y_0 + \int_0^t \Phi(t - s)[Bu(s) + f(s, z(s), v(s))]ds.
\]

We assume the following notations:
\[
a_1 = \sup \|\Phi(T - t)\|; \quad a_2 = \sup [\Phi_0(t)x_0 + \Phi_1(t)y_0];
\]
\[
b_1 = 4a_1^2T\|B^*\|\|M^{-1}\|; \quad b_2 = 4a_1T; \quad b = \max\{b_1, b_2\};
\]
\[
c_1 = 4a_1\|B^*\|\|M^{-1}\|\|\bar{x}\|; \quad c_2 = 4a_2; \quad c = \max\{c_1, c_2\};
\]
\[
a = \max\{a_1T\|B\|, 1\}; \quad \sup |f| = \sup\{|f(s, z(s), v(s))| : s \in J\}.
\]

From the hypothesis (3.4.3) on \( f \), we see that \( f \) satisfies the following conditions [34]: for every pair of constants \( c, b \) there exists a constant \( r > 0 \) such that if \( |(z, v)| \leq r \), then
\[
c + b \ |f(t, z, v)| \leq r, \quad \text{for all } t \in J.
\]

So let \( S(r) = \{(z, v) \in Z : |(z, v)| \leq r\} \). Then if \( (z, v) \in S(r) \), we have
\[
|u(t)| \leq \|B^*\|a_1\|M^{-1}\|\|\bar{x}\| + a_1T \sup |f| \leq \frac{c_1}{4a} + \frac{b_1}{4a} \sup |f| \leq \frac{1}{4a} (c + b \sup |f|) \leq \frac{r}{4a},
\]
and
\[
|x(t)| \leq a_2 + a_1\|B\|T\|u\| + a_1T \sup |f| \\
\leq \frac{c}{4} + \frac{b}{4} (c + b \sup |f|) + \frac{b}{4} \sup |f| \\
\leq \frac{c}{2} + \frac{b}{2} \sup |f| = \frac{1}{2} (c + b \sup |f|) \leq \frac{r}{2}.
\]
Therefore $P$ maps $S(r)$ into itself. Since $f$ is continuous, the operator is continuous and hence $P$ is completely continuous by the application of the Arzela-Ascoli theorem. Since $S(r)$ is closed, bounded and convex, the Schauder fixed point theorem guarantees that $P$ has a fixed point $(z, v) \in S(r)$ such that $P(z, v) = (z, v) = (x, u)$. Hence we have

$$x(t) = \Phi_0(t)x_0 + \Phi_1(t)y_0 + \int_0^t \Phi(t-s) [Bu(s) + f(s, x(s), u(s))] \, ds.$$  

Thus $x(t)$ is the solution of the system (3.4.1) and it is easy to verify that $x(T) = x_1$. Hence the system is controllable on $J$. □

If $\alpha_i \in L^1(J), i = 1, \ldots, q$, then $\|\alpha_i\|$ is the $L^1$ norm of $\alpha_i(\cdot)$, that is,

$$\|\alpha_i\| = \int_J |\alpha_i(s)| \, ds,$$

and we assume

$$k_1 = \max\{\|\Phi(t-s)\| : 0 \leq s \leq t \leq T\}; \quad b_i = 2k_1\|\alpha_i\|; \quad c_i = \max\{a_i, b_i\};$$

$$k_2 = \max\{\|\Phi(t-s)\| \|B\| T, 1\}; \quad d_1 = 2k_2\|B^*\Phi^*(T-t)\| \|M^{-1}\| |\bar{x}|;$$

$$d_2 = 2[\|\Phi_0(t)x_0\| + \|\Phi_1(t)y_0\| ]; \quad d = \max\{d_1, d_2\};$$

$$a_i = 2k_2\|B^*\Phi^*(T-t)\| \|M^{-1}\| \|\Phi(T-s)\| \|\alpha_i\|.$$  

**Theorem 3.4.2.** Let measurable functions $\varphi_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^+$ and $L^1$-functions $\alpha_i : J \rightarrow \mathbb{R}^+$, $i = 1, \ldots, q$, be such that

$$\|f(t, x, u)\| \leq \sum_{i=1}^q \alpha_i(t)\varphi_i(x, u). \quad (3.4.4)$$

Then the controllability of (3.3.1) implies the controllability of (3.4.1) if

$$\limsup_{r \to \infty} \left( r - \sum_{i=1}^q c_i \sup\{\varphi_i(x, u) : \|(x, u)\| \leq r\} \right) = +\infty. \quad (3.4.5)$$

**Proof.** Define the operator $Q$ on $Z$ as follows

$$Q(x, u) = (z, v),$$

Please purchase PDF Split-Merge on www.verypdf.com to remove this watermark.
where
\[ v(t) = B^*\Phi^*(T-t)M^{-1}\left[ x - \int_0^T \Phi^*(T-s)f(s, x(s), u(s))ds \right] \]  \hspace{1cm} (3.4.6)
and
\[ z(t) = \Phi_0(t)x_0 + \Phi_1(t)y_0 + \int_0^t \Phi(t-s)[Bv(s) + f(s, x(s), u(s))]ds. \]  \hspace{1cm} (3.4.7)

Under our regularity assumptions on \( f \), \( Q \) is continuous. Now the controllability result is reduced to the existence of fixed point \( Q(x, u) = (x, u) \) with \( x(T) = x_1 \).

Let
\[ \psi_i(r) = \sup\{\varphi_i(x, u) : \|(x, u)\| \leq r\}. \]

Since (3.4.5) holds, there exists \( r_0 > 0 \) such that
\[ r_0 - \sum_{i=1}^q c_i \psi_i(r_0) \geq d \quad \text{or} \quad \sum_{i=1}^q c_i \psi_i(r_0) + d \leq r_0. \]

Also
\[ S_{r_0} = \{(x, u) \in Z : \|(x, u)\| \leq r_0\}. \]

If \((x, u) \in S_{r_0}\), from (3.4.6) and (3.4.7), we have
\[
\|v\| \leq \|B^*\Phi^*(T-t)\| \|M^{-1}\| \times \left[ \|\bar{x}\| + \int_0^T \|\Phi(T-s)\| \left( \sum_{i=1}^q \alpha_i(s)\varphi_i(x(s), u(s)) \right) \right] ds \\
\leq \|B^*\Phi^*(T-t)\| \|M^{-1}\| \left[ \|\bar{x}\| + \int_0^T \|\Phi(T-s)\| \left( \sum_{i=1}^q \alpha_i(s)\psi_i(r_0) \right) \right] ds \\
\leq \frac{d_1}{2k_2} + \frac{1}{2k_2} \sum_{i=1}^q c_i \psi_i(r_0) \leq \frac{1}{2k_2} \left( d + \sum_{i=1}^q c_i \psi_i(r_0) \right) \\
\leq \frac{r_0}{2k_2},
\]
\[
\|z\| \leq \|\Phi_0(t)x_0\| + \|\Phi_1(t)y_0\| \\
+ \int_0^t \left\|\Phi(t-s)\right\| \left[ \|B\| \|v\| + \sum_{i=1}^q \alpha_i(s)\varphi_i(x(s), u(s)) \right] ds \\
\leq \frac{d}{2} + k_2\|v\| + \int_0^t k_1 \left( \sum_{i=1}^q \alpha_i(s)\psi_i(r_0) \right) ds \leq \frac{d}{2} + \frac{r_0}{2} + \frac{1}{2} \sum_{i=1}^q c_i \psi_i(r_0) \\
\leq \frac{1}{2} \left( d + \sum_{i=1}^q c_i \psi_i(r_0) \right) + \frac{r_0}{2} \leq \frac{r_0}{2} + \frac{r_0}{2} = r_0.
\]
Hence $Q$ maps $S_r$ into itself. Next we need to show that $Q(S_r)$ is equicontinuous for all $r > 0$. Now, for every $(x, u) \in S_r$ and $s_1, s_2 \in J$ with $s_1 < s_2$, we have

$$
\|v(s_1) - v(s_2)\| \\
\leq \|B\| \|M^{-1}\| \|\Phi^*(T - s_1) - \Phi^*(T - s_2)\| \\
\times \left[\|\bar{x}\| + \int_0^T \|\Phi(T - s)\| \left(\sum_{i=1}^q \alpha_i(s)\varphi_i(x, u(s))\right) \, ds\right] \\
\leq \|B\| \|M^{-1}\| \|\Phi^*(T - s_1) - \Phi^*(T - s_2)\| \\
\times \left[\|\bar{x}\| + \|\Phi(T - s)\| \sum_{i=1}^q \|\alpha_i\|\psi_i(r)\right], \quad (3.4.8)
$$

Thus the right hand sides of $(3.4.8)$ and $(3.4.9)$ do not depend on particular choices of $(x, u)$. Hence it is clear that $Q(S_r)$ is equicontinuous for all $r > 0$. By the Arzela-Ascoli theorem, $Z$ is a compact operator. Since $S_{r_0}$ is nonempty, closed, bounded and convex, the Schauder fixed point theorem guarantees the existence of a fixed point.

Moreover, for all $(x, u) \in S_r$,

$$
\|v\| \\
\leq \|B\| \|\Phi^*(T - t)\| \|M^{-1}\| \left[\|\bar{x}\| + \int_0^T \|\Phi(T - s)\| \left(\sum_{i=1}^q \alpha_i(s)\psi_i(r)\right) \, ds\right] \\
\leq \|B\| \|\Phi^*(T - t)\| \|M^{-1}\| \left[\|\bar{x}\| + \|\Phi(T - s)\| \sum_{i=1}^q \|\alpha_i\|\psi_i(r)\right].
$$

Thus the right hand sides of $(3.4.8)$ and $(3.4.9)$ do not depend on particular choices of $(x, u)$. Hence it is clear that $Q(S_r)$ is equicontinuous for all $r > 0$. By the Arzela-Ascoli theorem, $Z$ is a compact operator. Since $S_{r_0}$ is nonempty, closed, bounded and convex, the Schauder fixed point theorem guarantees the existence of a fixed point.

Also $x(T) = x_1$ shows that $x(t)$ steers from $x_0$ to $x_1$ at time $T$. Hence the system $(3.4.1)$ is controllable on $J$. \qed
Chapter 3

43

To apply the above theorem, one usually has to construct $\alpha'$'s and $\varphi'$'s such that (3.4.4) must be satisfied. These constructions are different for different situations. However an obvious construction of $\alpha'$'s and $\varphi'$'s is easily achieved by letting $q = 1$, $\alpha_1 = \alpha = 1$ and

$$
\varphi_1(x, u) = \varphi(x, u) = \sup \{ \| f(t, x(t), u(t)) \| : t \in J \}.
$$

In this case, (3.4.5) holds if

$$
\liminf_{r \to \infty} \left( \frac{1}{r} \sup \{ \varphi(x, u) : \| (x, u) \| \leq r \} \right) < \frac{1}{c_1}.
$$

(3.4.10)

**Corollary 3.4.1.** Suppose that there exist $\alpha(t), \beta(t) \in \mathbb{L}^1(J)$ and monotonically non-decreasing functions $\varphi, \psi$ such that, for every $(t, x, u) \in J \times \mathbb{R}^n \times \mathbb{R}^m$,

$$
\| f(t, x, u) \| \leq \alpha(t) (\varphi(\| x \|) + \psi(\| u \|)) + \beta(t).
$$

(3.4.11)

Let

$$
c = c_2 = \max \{ 2k_2 B^* \Phi^* M^{-1} \| \Phi(T - s) \| \| \alpha \|, 2k_2 \| \alpha \| \}.
$$

Then (3.4.1) is controllable if (3.3.1) is controllable and

$$
\limsup_{r \to \infty} (r - c(\varphi(r) + \psi(r))) = +\infty.
$$

In particular, this is true if

$$
\liminf_{r \to \infty} (1/r) [\varphi(r) + \psi(r)] < 1/c.
$$

Proof. In order to prove the corollary, it is enough to show that condition (3.4.5) holds under the following settings,

$$
q = 2, \quad \alpha = \beta, \quad \alpha_2 = \alpha; \quad \varphi_1 = 1; \quad \varphi_2 = \varphi(\| x \|) + \psi(\| u \|).
$$

However this is trivial since

$$
\limsup_{r \to \infty} (r - \sup \{ c_1 + c(\varphi(\| x \|) + \psi(\| u \|)) : \| (x, u) \| \leq r \})
\geq \limsup_{r \to \infty} (r - c_1 - c(\varphi(r) + \psi(r))) = +\infty.
$$

The remaining part of the proof is obvious. □
3.5 Controllability of Nonlinear Damped Fractional Dynamical systems

Consider the nonlinear fractional dynamical system represented by the fractional differential equation

\[
C \frac{D_0^\alpha}{0^+} x(t) + A^2 x(t) = Bu(t) + f(t, x(t), C \frac{D_0^\beta}{0^+} x(t)), \quad t \in J,
\]

\[
x(0) = x_0, \quad x'(0) = y_0,
\]

with \(1 < \alpha \leq 2, \quad 0 < \beta \leq 1\), \(A\) and \(B\) are defined as above and the nonlinear function \(f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) is continuous. The solution of (3.5.1) is given by

\[
x(t) = \Phi_0(t)x_0 + \Phi_1(t)y_0 + \int_0^t \Phi(t-s) \left[ Bu(s) + f(s, x(s), C \frac{D_0^\beta}{0^+} x(s)) \right] ds.
\]

3.5.1 Basic Assumptions

Now we make the following assumptions to obtain the controllability results for the systems:

(H1) For each \(t \in J\), the function \(f(t, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) is continuous and the function \(f(\cdot, x, y) : J \to \mathbb{R}^n\) is strongly measurable for each \(x, y \in \mathbb{R}^n\).

(H2) For every positive constant \(k\), there exists \(h_k \in L^1(J)\) such that

\[
\sup_{\|x\|, \|y\| \leq k} \|f(t, x, y)\| \leq h_k(t), \quad \text{for every } t \in J.
\]

(H3) There exists a continuous function \(m_1 : J \to [0, \infty)\) such that

\[
\|f(t, x, y)\| \leq m_1(t)\Omega (\|x\| + \|y\|), \quad t \in J, \quad x, y \in \mathbb{R}^n,
\]

where \(\Omega : (0, \infty) \to (0, \infty)\) is a continuous nondecreasing function and

\[
\int_0^T m(s)ds < \int_r^\infty \frac{ds}{\Omega(s)}.
\]

and

\[
r = n_1\|x_0\| + n_2\|y_0\| + n_3^2T\|B\|\|B^*\||M^{-1}| \left[ \|\bar{x}\| + n_3 \int_0^T m_1(s)\Omega(w(s))ds \right].
\]
(H4) There exists a constant \( M > 0 \) and a continuous function \( m_2 : J \to [0, \infty) \) such that
\[
\frac{k_2 t^{-\beta}}{\Gamma(1 - \beta)} + \frac{n_5}{\Gamma(1 - \beta)} \int_0^t (t - \xi)^{-\beta} m_1(\xi) \Omega(w(\xi)) d\xi \leq M m_2(t) \Omega(w(t)),
\]
where
\[
\begin{align*}
n_1 &= \sup\{\|\Phi_0(t)\|, t \in J\}; & n_2 &= \sup\{\|\Phi_1(t)\|, t \in J\}; \\
n_3 &= \sup\{\|\Phi(t - s)\|, t, s \in J\}; & n_4 &= \sup\{\|A^2 \Phi(t)\|, t \in J\}; \\
n_5 &= \sup\{\|\Phi_2(t - s)\|, t, s \in J\}; & \bar{x} &= x_1 - \Phi_0(T)x_0 - \Phi_1(T)y_0; \\
m(t) &= \max\{n_3 m_1(t), M m_2(t)\}; & \Phi_2(t) &= t^{\alpha - 1} E_{\alpha, \alpha - 1}(-A^2 t^\alpha); \\
n_6 &= n_4 \|x_0\| + n_1 \|y_0\| + n_3 n_5 T \|B\| \|B^*\| \|M^{-1}\| \left[ \|\bar{x}\| + n_3 \int_0^T h_q(s) ds \right]; \\
r &= n_1 \|x_0\| + n_2 \|y_0\| + n_2^2 T \|B\| \|B^*\| \|M^{-1}\| \left[ \|\bar{x}\| + n_3 \int_0^T m_1(s) \Omega(w(s)) ds \right].
\end{align*}
\]

Lemma 3.5.1. (Schaefer Theorem \cite{109}) Let \( X \) be a normed space, \( T \) a continuous mapping of \( X \) into \( X \), which is compact on each bounded subset \( S \) of \( X \). Then either
(i) the equation \( x = \lambda T(x) \) has a solution for \( \lambda = 1 \) or (ii) the set of all such solutions \( x \), for \( 0 < \lambda < 1 \) is unbounded.

Theorem 3.5.1. Assume that the hypotheses (H1) – (H4) hold and suppose that the linear system (3.3.1) is controllable. Then the nonlinear system (3.5.1) is controllable on \( J \).

Proof. Consider the Banach space \( X = \left\{ x : x \in C(J, \mathbb{R}^n) \text{ and } C^D_{0+} x \in C(J, \mathbb{R}^n) \right\} \) with norm \( \|x\|^* = \max\{\|x\|, \|C^D_{0+} x\|\} \). For an arbitrary function \( x(\cdot) \), define the control by
\[
u(t) = B^* \Phi^*(T - t) M^{-1} \left[ \bar{x} - \int_0^T \Phi(T - s) f(s, x(s), C^D_{0+} x(s)) ds \right].
\]

We now show that, when using this control, the nonlinear operator \( F : X \to X \) defined by
\[
(Fx)(t) = \Phi_0(t)x_0 + \Phi_1(t)y_0 + \int_0^t \Phi(t - s) Bu(s) ds \\
+ \int_0^t \Phi(t - s) f(s, x(s), C^D_{0+} x(s)) ds
\]
has a fixed point. This fixed point is then a solution to (3.5.1). Substituting the control \( u(t) \) in the above equation, we get

\[
(Fx)(t) = \Phi_0(t) x_0 + \Phi_1(t) y_0 + \int_0^t \Phi(t-s) BB^* \Phi^*(T-s) M^{-1} \]

\[
\times \left[ \vec{x} - \int_0^T \Phi(T-\xi)f(\xi,x(\xi),C^D_{0+} x(\xi)) \, d\xi \right] \, ds
\]

\[
+ \int_0^t \Phi(t-s) f(s,x(s),C^D_{0+} x(s)) \, ds.
\]

Clearly \((Fx)(T) = x_1\) which means that the control \( u \) steers the system from the initial state \( x_0 \) to \( x_1 \) in time \( T \) provided we obtain a fixed point of the nonlinear operator \( F \).

The first step is to obtain an a priori bound of the set

\[ \zeta(F) = \{ x \in X : x = \lambdaFx \text{ for some } \lambda \in (0,1) \} \].

Let \( x \in \zeta(F) \). Then \( x = \lambdaFx \) for some \( 0 < \lambda < 1 \). Thus, for each \( t \in J \), we have

\[
x(t) = \lambda \Phi_0(t) x_0 + \lambda \Phi_1(t) y_0 + \lambda \int_0^t \Phi(t-s) f(s,x(s),C^D_{0+} x(s)) \, ds
\]

\[
+ \lambda \int_0^t \Phi(t-s) BB^* \Phi^*(T-s) M^{-1} \left[ \vec{x} - \int_0^T \Phi(T-\xi)f(\xi,x(\xi),C^D_{0+} x(\xi)) \, d\xi \right] \, ds.
\]

Then

\[
\| x(t) \| \leq n_1 \| x_0 \| + n_2 \| y_0 \| + n_3 \int_0^t m_1(s) \Omega(\| x(s) \| + \| C^D_{0+} x(s) \|) \, ds
\]

\[
+ n_3 T \| \| B \| \| B^* \| \| M^{-1} \| \left[ \| \vec{x} \| + n_3 \int_0^T m_1(s) \Omega(\| x(s) \| + \| C^D_{0+} x(s) \|) \, ds \right]
\]

\[
\equiv r + n_3 \int_0^t m_1(s) \Omega(\| x(s) \| + \| C^D_{0+} x(s) \|) \, ds.
\]

Denoting the right-hand side of the above inequality by \( r_1(t) \), we have \( r_1(0) = r \),

\[
\| x(t) \| \leq r_1(t)
\]

and

\[
r_1'(t) = n_3 m_1(t) \Omega(\| x(t) \| + \| C^D_{0+} x(t) \|).
\]
Also

\[ x'(t) = -\lambda A^2 \Phi(t) x_0 + \lambda \Phi_0(t) y_0 + \lambda \int_0^t \Phi_2(t-s) f(s, x(s), C_{D_0^x} x(s)) ds + \lambda \int_0^t \Phi_2(t-s) B B^* \Phi(T-s) M^{-1} \times \left[ \bar{x} - \int_0^T \Phi(T-\xi) f(\xi, x(\xi), C_{D_0^x} x(\xi)) d\xi \right] ds \]

and

\[ \|x'(t)\| \leq n_4 \|x_0\| + n_1 \|y_0\| + n_5 \int_0^t m_1(s) \Omega(\|x(s)\| + \|C_{D_0^x} x(s)\|) ds + n_3 n_5 T \|B\| \|B^*\| \|M^{-1}\| \left[ \|\bar{x}\| + n_3 \int_0^T m_1(s) \Omega(\|x(s)\| + \|C_{D_0^x} x(s)\|) ds \right] \]

\[ \equiv n_6 + n_5 \int_0^t m_1(s) \Omega(\|x(s)\| + \|C_{D_0^x} x(s)\|) ds. \]

Hence it follows that

\[ \|C_{D_0^+} x(t)\| \leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{\beta-1} \|x'(s)\| ds \]

\[ \leq \frac{n_6}{\Gamma(1-\beta)} \int_0^t (t-s)^{\beta-1} ds + \frac{n_5}{\Gamma(1-\beta)} \int_0^t (t-s)^{\beta-1} \left( \int_0^s m_1(\xi) \Omega(\|x(\xi)\| + \|C_{D_0^+} x(\xi)\|) d\xi \right) ds \]

\[ \leq \frac{n_6}{\Gamma(2-\beta)} t^{1-\beta} + \frac{n_5}{\Gamma(2-\beta)} \int_0^t (t-\xi)^{1-\beta} m_1(\xi) \Omega(\|x(\xi)\| + \|C_{D_0^+} x(\xi)\|) d\xi \int_0^t (t-s)^{\beta-1} ds \]

Denoting the right-hand side of the above inequality by \( r_2(t) \), we have \( r_2(0) = 0 \) and

\[ \|C_{D_0^+} x(t)\| \leq r_2(t) \]

and

\[ r_2'(t) = \frac{n_6}{\Gamma(1-\beta)} t^{\beta-1} + \frac{n_5}{\Gamma(1-\beta)} \int_0^t (t-\xi)^{\beta-1} m_1(\xi) \Omega(\|x(\xi)\| + \|C_{D_0^+} x(\xi)\|) d\xi. \]

Let \( w(t) = r_1(t) + r_2(t), \ t \in J. \) Then \( w(0) = r_1(0) + r_2(0) = r \) and

\[ w'(t) = r_1'(t) + r_2'(t) \leq m(t) \Omega(w(t)) \]
which implies, that for each \( t \in J \),
\[
\int_{w(0)}^{w(t)} \frac{ds}{\Omega(s)} \leq \int_{0}^{T} m(s)ds < \int_{r}^{\infty} \frac{ds}{\Omega(s)}.
\]
From the above inequality, we see that there exists a constant \( K \) such that
\[
w(t) = r_1(t) + r_2(t) \leq K, \quad t \in J.
\]
Then \( \|x(t)\| \leq r_1(t) \) and \( \|C_{D_{0+}^\beta} x(t)\| \leq r_2(t), \quad t \in J \), and hence
\[
\|x\|^* = \max\{\|x\|, \|C_{D_{0+}^\beta} x\|\} \leq K
\]
and the set \( \zeta(F) \) is bounded. Next we prove that the operator \( F : X \to X \) is completely continuous.

Let \( B_q = \{ x \in X; \|x\|^* \leq q \} \). We first show that \( F \) maps bounded sets into equicontinuous family in \( B_q \). Let \( x \in B_q \) and \( t_1, t_2 \in J \). Then if \( 0 < t_1 < t_2 \leq T \),
\[
\|(F x)(t_2) - (F x)(t_1)\|
\]
\[
\leq \|\Phi_0(t_2) - \Phi_0(t_1)\| \|x_0\| + \|\Phi_1(t_2) - \Phi_1(t_1)\| \|y_0\|
\]
\[
+ \|\int_{0}^{t_1} [\Phi(t_2 - s) - \Phi(t_1 - s)] f(s, x(s), C_{D_{0+}^\beta} x(s))ds\|
\]
\[
+ \|\int_{t_1}^{t_2} \Phi(t_2 - s)f(s, x(s), C_{D_{0+}^\beta} x(s))ds\| + \|\int_{0}^{t_1} [\Phi(t_2 - s) - \Phi(t_1 - s)]BB^*\]
\[
\times \Phi^*(T - s)M^{-1} \left[ \bar{x} - \int_{0}^{T} \Phi(T - \xi)f(\xi, x(\xi), C_{D_{0+}^\beta} x(\xi))d\xi \right] ds\|
\]
\[
+ \|\int_{t_1}^{t_2} \Phi(t_2 - s)BB^*\Phi^*(T - s)M^{-1} \left[ \bar{x} - \int_{0}^{T} \Phi(T - \xi)f(\xi, x(\xi), C_{D_{0+}^\beta} x(\xi))d\xi \right] ds\|
\]
\[
\leq \|\Phi_0(t_2) - \Phi_0(t_1)\| \|x_0\| + \|\Phi_1(t_2) - \Phi_1(t_1)\| \|y_0\|
\]
\[
+ \int_{0}^{t_1} \|\Phi(t_2 - s) - \Phi(t_1 - s)\| h_q(s)ds + \int_{t_1}^{t_2} \|\Phi(t_2 - s)\| h_q(s)ds
\]
\[
+ \int_{0}^{t_1} \|\Phi(t_2 - s) - \Phi(t_1 - s)\| \|B\| \|B^*\| \|\Phi^*(T - s)\| \|M^{-1}\|
\]
\[
\times \left[ \|\bar{x}\| + n_3 \int_{0}^{T} h_q(\xi)d\xi \right] ds + \int_{t_1}^{t_2} \|\Phi(t_2 - s)\| \|B\| \|B^*\| \|\Phi^*(T - s)\| \|M^{-1}\|
\]
\[
\times \left[ \|\bar{x}\| + n_3 \int_{0}^{T} h_q(\xi)d\xi \right] ds.
\]
\[(3.5.2)\]
And
\[
\|(Fx)'(t)\| \leq \|A^2\Phi(t)\|\|x_0\| + \|\Phi_0(t)\|\|y_0\| + \int_0^t \|\Phi_2(t-s)\| h_q(s) ds
\]
\[
+ \int_0^t \|\Phi_2(t-s)\| B \|B^*\| \|\Phi^*(T-s)\|\|M^{-1}\|
\]
\[
\times \left[\|\bar{x}\| + \int_0^T \|\Phi(T-\xi)\| h_q(\xi) d\xi\right] ds
\]
\[
\leq n_4\|x_0\| + n_1\|y_0\| + n_5\int_0^t h_q(s) ds
\]
\[
+ n_3n_5T B \|B^*\|\|M^{-1}\| \left[\|\bar{x}\| + n_3\int_0^T h_q(\xi) d\xi\right] ds
\]
\leq n_6 + n_5\int_0^t h_q(s) ds.
\]
Hence it follows that
\[
\|CD_{0+}^\beta (Fx)(t_2) - CD_{0+}^\beta (Fx)(t_1)\|
\]
\[
= \left\| \frac{1}{\Gamma(1-\beta)} \int_0^{t_2} (t_2-s)^{-\beta} (Fx)'(s) ds - \frac{1}{\Gamma(1-\beta)} \int_0^{t_1} (t_1-s)^{-\beta} (Fx)'(s) ds \right\|
\]
\[
\leq \frac{1}{\Gamma(1-\beta)} \left\| \int_0^{t_2} (t_2-s)^{-\beta} (Fx)'(s) ds \right\|
\]
\[
+ \frac{1}{\Gamma(1-\beta)} \left\| \int_0^{t_1} ((t_2-s)^{-\beta} - (t_1-s)^{-\beta}) (Fx)'(s) ds \right\|
\]
\[
\leq \frac{1}{\Gamma(1-\beta)} \left( \int_0^{t_2} (t_2-s)^{-\beta} ds + \int_0^{t_1} (t_1-s)^{-\beta} (\int_0^s h_q(\xi) d\xi) ds \right.
\]
\[
+ \frac{n_6}{\Gamma(1-\beta)} \int_0^{t_2} (t_2-s)^{-\beta} ds
\]
\[
+ \frac{n_5}{\Gamma(1-\beta)} \int_0^{t_1} ((t_2-s)^{-\beta} - (t_1-s)^{-\beta}) (\int_0^s h_q(\xi) d\xi) ds
\]
\[
+ \frac{n_5}{\Gamma(1-\beta)} \int_0^{t_1} ((t_2-s)^{-\beta} - (t_1-s)^{-\beta}) (\int_0^s h_q(\xi) d\xi) ds
\]
\[
\leq \frac{n_6}{\Gamma(2-\beta)} (t_2^{1-\beta} - t_1^{1-\beta}) + \frac{1}{\Gamma(1-\beta)} \int_0^{t_2} (t_2-s)^{-\beta} \left( \int_0^s h_q(\xi) d\xi\right) ds
\]
\[
+ \frac{1}{\Gamma(2-\beta)} \int_0^{t_1} ((t_2-\xi)^{1-\beta} - (t_2-t_1)^{1-\beta} - (t_1-\xi)^{1-\beta}) h_q(\xi) d\xi. \quad (3.5.3)
\]
The right-hand sides of (3.5.2) and (3.5.3) tend to zero as \( t_2 \to t_1 \). Thus \( F \) maps \( B_q \) into an equicontinuous family of functions. It is easy to see that the family \( FB_q \) is
uniformly bounded. Next we show that $F$ is a compact operator. It suffices to show that the closure of $FB_q$ is compact.

Let $0 \leq t \leq T$ be fixed and $\epsilon$ be a real number satisfying $0 < \epsilon < t$. For $x \in B_q$, we define

$$(F_\epsilon x)(t) = \Phi_0(t)x_0 + \Phi_1(t)y_0 + \int_0^{t-\epsilon} \Phi(t-s)f(s, x(s), C_{D_{0+}}x(s))ds$$

$$+ \int_0^{t-\epsilon} \Phi(t-s)BB^*\Phi^*(T-s)M^{-1}$$

$$\times \left[ \bar{x} - \int_0^T \Phi(T-\xi)f(\xi, x(\xi), C_{D_{0+}}x(\xi))d\xi \right] ds.$$ 

Note that using the same methods as in the procedure above, we obtain the boundedness and equicontinuous property of $F_\epsilon$ which implies that the set $S_\epsilon(t) = \{(F_\epsilon x)(t), x \in B_q\}$ is relatively compact in $X$ for every $0 < \epsilon < t$.

Moreover, for every $x \in B_q$,

$$
\| (Fx)(t) - (F_\epsilon x)(t) \|
\leq \left\| \int_{t-\epsilon}^{t} \Phi(t-s)f(s, x(s), C_{D_{0+}}x(s))ds \right\| + \left\| \int_{t-\epsilon}^{t} \Phi(t-s)BB^*\Phi^*(T-s)M^{-1} \right\|
\times \left[ \bar{x} - \int_0^T \Phi(T-\xi)f(\xi, x(\xi), C_{D_{0+}}x(\xi))d\xi \right] ds

\leq \int_{t-\epsilon}^{t} \| \Phi(t-s) \| h_q(s)ds

+ \int_{t-\epsilon}^{t} \| \Phi(t-s) \| \| B \| \| B^* \| \| \Phi^*(T-s) \| \| M^{-1} \| \left\| \bar{x} \right\| + n_3 \int_0^T h_q(\xi)d\xi ds.

Also

$$
\| (Fx)'(t) - (F_\epsilon x)'(t) \|
\leq \left\| \int_{t-\epsilon}^{t} \Phi_2(t-s)f(s, x(s), C_{D_{0+}}x(s))ds \right\| + \left\| \int_{t-\epsilon}^{t} \Phi_2(t-s)BB^*\Phi^*(T-s)M^{-1} \right\|
\times \left[ \bar{x} - \int_0^T \Phi(T-\xi)f(\xi, x(\xi), C_{D_{0+}}x(\xi))d\xi \right] ds

\leq \int_{t-\epsilon}^{t} \| \Phi_2(t-s) \| h_q(s)ds

+ \int_{t-\epsilon}^{t} \| \Phi_2(t-s) \| \| B \| \| B^* \| \| \Phi^*(T-s) \| \| M^{-1} \| \left\| \bar{x} \right\| + n_3 \int_0^T h_q(\xi)d\xi ds.

Since $\|(Fx)(t) - (F_\epsilon x)(t)\| \to 0$ and $\|(Fx)'(t) - (F_\epsilon x)'(t)\| \to 0$ as $\epsilon \to 0$, this implies
that
\[
\|C^{D_{0+}^\beta}(Fx)(t) - C^{D_{0+}^\beta}(F_x)(t)\| \\
\leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \|((Fx)'(t) - (F_x)'(t))\| \, ds \to 0 \text{ as } \varepsilon \to 0.
\]

So relatively compact sets \(S_n(t) = \{(Fx)(t), \ x \in B_q\} \) are arbitrarily close to the set \(\{(Fx)(t), \ x \in B_q\} \). Hence \(\{(Fx)(t), \ x \in B_q\} \) is compact in \(X\) by the Ascoli-Arzela theorem.

Next it remains to show that \(F\) is continuous. Let \(\{x_n\}\) be a sequence in \(X\) such that \(\|x_n - x\| \to 0\) as \(n \to \infty\). Then there is an integer \(k\) such that \(\|x_n\| \leq k, \|C^{D_{0+}^\beta}x_n\| \leq k\) for all \(n\) and \(t \in J\). So \(\|x(t)\| \leq k, \|C^{D_{0+}^\beta}x(t)\| \leq k\) and \(x, C^{D_{0+}^\beta}x \in \dot{X}\). By (H1),
\[
f(t, x_n(t), C^{D_{0+}^\beta}x_n(t)) \to f(t, x(t), C^{D_{0+}^\beta}x(t)),
\]
for each \(t \in J\). Since
\[
\|f(t, x_n(t), C^{D_{0+}^\beta}x_n(t)) - f(t, x(t), C^{D_{0+}^\beta}x(t))\| \leq 2h_k(t),
\]
we have, by the dominated convergence theorem,
\[
\|(Fx_n)(t) - (Fx)(t)\| \\
= \sup_{t \in J} \left\| \int_0^t \Phi(t-s) \left[f(s, x_n(s), C^{D_{0+}^\beta}x_n(s)) - f(s, x(s), C^{D_{0+}^\beta}x(s))\right] \, ds \\
+ \int_0^t \Phi(t-s)BB^*\Phi^*(T-s)M^{-1} \int_0^T \Phi(T-\xi) \\
\times \left[f(\xi, x_n(\xi), C^{D_{0+}^\beta}x_n(\xi)) - f(\xi, x(\xi), C^{D_{0+}^\beta}x(\xi))\right] \, d\xi \, ds \right\|
\leq \int_0^T \|\Phi(t-s) \left[f(s, x_n(s), C^{D_{0+}^\beta}x_n(s)) - f(s, x(s), C^{D_{0+}^\beta}x(s))\right] \| \, ds \\
+ \int_0^T \|\Phi(t-s)BB^*\Phi^*(T-s)M^{-1} \int_0^T \Phi(T-\xi) \\
\times \left[f(\xi, x_n(\xi), C^{D_{0+}^\beta}x_n(\xi)) - f(\xi, x(\xi), C^{D_{0+}^\beta}x(\xi))\right] \, d\xi \, ds \to 0 \text{ as } n \to \infty.
\]
Also
\[
\| (F x_n)'(t) - (F x)'(t) \|
\]
\[
= \sup_{t \in J} \left\| \int_0^t \Phi_2(t-s) \left[ f(s, x_n(s), cD_{0+}^\beta x_n(s)) - f(s, x(s), cD_{0+}^\beta x(s)) \right] ds \right. \\
+ \int_0^t \Phi_2(t-s) BB^* \Phi^*(T-s) M^{-1} \int_0^T \Phi(T-\xi) \times \left[ f(\xi, x_n(\xi), cD_{0+}^\beta x_n(\xi)) - f(\xi, x(\xi), cD_{0+}^\beta x(\xi)) \right] d\xi dT \right.
\]
\[
\leq \int_0^T \left\| \Phi_2(t-s) \left[ f(s, x_n(s), cD_{0+}^\beta x_n(s)) - f(s, x(s), cD_{0+}^\beta x(s)) \right] \right\| ds \\
+ \int_0^T \left\| \Phi_2(t-s) BB^* \Phi^*(T-s) M^{-1} \int_0^T \Phi(T-\xi) \times \left[ f(\xi, x_n(\xi), cD_{0+}^\beta x_n(\xi)) - f(\xi, x(\xi), cD_{0+}^\beta x(\xi)) \right] d\xi dT \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
This implies that
\[
\| cD_{0+}^\beta (F x_n)(t) - cD_{0+}^\beta (F x)(t) \|
\]
\[
\leq \frac{1}{\Gamma(1-\beta)} \int_0^T (t-s)^{-\beta} \| (F x_n)'(t) - (F x)'(t) \| ds \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
Thus \( F \) is continuous. Finally the set \( \zeta(F) = \{ x \in X; x = \lambda F x, \lambda \in (0,1) \} \) is bounded as shown in the first step. By Schaefer’s theorem, the operator \( F \) has a fixed point in \( X \). This fixed point is then the solution of (3.5.1). Hence the system (3.5.1) is controllable on \([0, T]\).

\section{3.6 Examples}

In this section, we apply the results obtained in the previous sections to the following nonlinear and fractional damped dynamical systems.

\textbf{Example 3.6.1.} Consider the fractional dynamical system
\[
\begin{align*}
&cD_{0+}^{3/2} x_1(t) - 2x_1(t) + 3x_2(t) = u_1(t) + \frac{x_1}{x_1^2 + u_1^2 + \sin t}, \\
&cD_{0+}^{3/2} x_2(t) - 4x_1(t) + 5x_2(t) = u_2(t) + \frac{x_1}{x_1^2 + u_2^2 + t},
\end{align*}
\]
(3.6.1)

with initial conditions \[
\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } t \in [0, 2].
\]
It has the following form
\[ \begin{align*}
CD_{0, t}^{3/2} x(t) + A^2 x(t) &= Bu(t) + f(t, x, u), \quad t \in [0, 2], \\
x(0) &= x_0, \quad x'(0) = y_0,
\end{align*} \tag{3.6.2} \]
where \( A^2 = \begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad f(t, x, u) = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 + u_1^2 + \sin t \\ x_2 \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \]
and \( u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \). Choose \( x(2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \). Using Mittag-Leffler matrix function for a given matrix \( A^2 \), we get
\[ \Phi(2 - s) = \begin{bmatrix} L_1(s) & L_2(s) \\ L_3(s) & L_4(s) \end{bmatrix}, \]
where
\[ \begin{align*}
L_1(s) &= (2 - s)^{1/2} \left[ 4E_{3, 3/2}(-2s^{3/2}) - 3E_{3, 3/2}(-2s^{3/2}) \right], \\
L_2(s) &= (2 - s)^{1/2} \left[ 3E_{3, 3/2}(-2s^{3/2}) - 3E_{3, 3/2}(-2s^{3/2}) \right], \\
L_3(s) &= (2 - s)^{1/2} \left[ 4E_{3, 3/2}(-2s^{3/2}) - 4E_{3, 3/2}(-2s^{3/2}) \right], \\
L_4(s) &= (2 - s)^{1/2} \left[ 4E_{3, 3/2}(-2s^{3/2}) - 3E_{3, 3/2}(-2s^{3/2}) \right].
\end{align*} \]
By simple matrix calculation, one can see that the controllability Grammian matrix
\[ M = \int_0^2 \Phi(2 - s)BB^*\Phi^*(2 - s) \, ds \]
\[ = \int_0^2 \begin{bmatrix} L_1^2(s) + L_2^2(s) & L_1(s)L_3(s) + L_2(s)L_4(s) \\ L_1(s)L_3(s) + L_2(s)L_4(s) & L_3^2(s) + L_4^2(s) \end{bmatrix} \, ds \]
\[ = \begin{bmatrix} 5.5201 & 4.4008 \\ 4.4008 & 3.8628 \end{bmatrix} \]
is positive definite. Further the nonlinear function \( f \) is bounded, continuous and satisfies conditions of Theorem (3.4.1). Observe that the control defined by
\[ u(t) = B^*\Phi^*(2 - t)M^{-1} \left[ \bar{x} - \int_0^2 \Phi(2 - s)f(s, x(s), u(s)) \, ds \right] \]
steers the system from \[
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\] to \[
\begin{bmatrix}
2 \\
2
\end{bmatrix}
\]. Hence the system (3.6.1) is controllable on [0, 2].

**Example 3.6.2.** Consider the fractional dynamical system

\[
\begin{aligned}
C^{D_{0+}^{5/4}} x_1(t) + x_2(t) &= u(t) \exp(x_2(t)), \\
C^{D_{0+}^{5/4}} x_2(t) + x_1(t) &= u(t) + 2 \cos x_1(t),
\end{aligned}
\tag{3.6.3}
\]

with initial conditions

\[
\begin{bmatrix}
x_1(0) \\
x_2(0)
\end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
x_1'(0) \\
x_2'(0)
\end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

for \( t \in [0, 1] \).

It has the following form

\[
\begin{aligned}
C^{D_{0+}^{5/4}} x(t) + A^2 x(t) &= Bu(t) + f(t, x, u), \quad t \in [0, 1], \\
x(0) &= x_0, \quad x'(0) = y_0,
\end{aligned}
\tag{3.6.4}
\]

where \( A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( f(t, x, u) = \begin{bmatrix} u(t) \exp(x_2(t)) \\ 2 \cos x_1(t) \end{bmatrix} \) and \( x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \).

Choose \( x(1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \). Using Mittag-Leffler matrix function for a given matrix \( A^2 \), we get

\[
\Phi(1 - t) = \begin{bmatrix} P_1(t) & P_2(t) \\ P_2(t) & P_1(t) \end{bmatrix},
\]

where

\[
\begin{align*}
P_1(t) &= \frac{(1-t)^{1/4}}{2} \left[ E_{5/4,5/4}((1-t)^{5/4}) + E_{5/4,5/4}(-(1-t)^{5/4}) \right], \\
P_2(t) &= \frac{(1-t)^{1/4}}{2} \left[ E_{5/4,5/4}((1-t)^{5/4}) - E_{5/4,5/4}(-(1-t)^{5/4}) \right].
\end{align*}
\]

By simple matrix calculation, one can see that the controllability Grammian matrix

\[
M = \int_0^1 \Phi(1-s)BB^*\Phi^*(1-s) \, ds
\]

\[
= \int_0^1 \begin{bmatrix}
P_2^2(s) & P_1(s)P_2(s) \\
P_1(s)P_2(s) & P_1^2(s)
\end{bmatrix} \, ds
\]

\[
= \begin{bmatrix}
0.1514 & -0.3448 \\
-0.3448 & 0.9462
\end{bmatrix}
\]
is positive definite. Further the nonlinear function \( f \) satisfies the conditions of Theorem (3.4.2). Hence the system (3.6.3) is controllable on \([0, 1]\) if \((3.4.10)\) holds. Observe that the control defined by

\[
u(t) = B^*\Phi^*(2 - t)M^{-1}\left[\bar{x} - \int_0^2 \Phi(2 - s)f(s, x(s), u(s))ds\right]\

steers the system from \([1, 3]\) to \([-1, 0]\).

**Example 3.6.3.** Consider the fractional damped dynamical system

\[
\begin{align*}
^{C}D_{0+}^{7/4}x_1(t) - x_1(t) &= u_2(t) + \frac{\exp(-2t) \left(|x_1| + |^{C}D_{0+}^{1/2}x_1(t)|\right)}{1 + |x_2(t)|}, \\
^{C}D_{0+}^{7/4}x_2(t) - x_2(t) &= u_1(t) + \frac{\exp(-2t) \left(|x_2| + |^{C}D_{0+}^{1/2}x_2(t)|\right)}{1 + |x_1(t)|},
\end{align*}
\]

with initial conditions \(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\) and \(\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\) for \(t \in [0, 3]\).

It has the following form

\[
^{C}D_{0+}^{7/4}x(t) + A^2x(t) = Bu(t) + f(t, x(t), {^{C}D}_{0+}^{1/2}x(t)), \quad t \in [0, 3]
\]

\[
x(0) = x_0, \quad x'(0) = y_0,
\]

where \(A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\), \(B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\), \(x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\), \(u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}\) and

\[
f(t, x, {^{C}D}_{0+}^{1/2}x) = \begin{bmatrix} \frac{\exp(-2t) \left(|x_1| + |^{C}D_{0+}^{1/2}x_1(t)|\right)}{1 + |x_2(t)|} \\ \frac{\exp(-2t) \left(|x_2| + |^{C}D_{0+}^{1/2}x_2(t)|\right)}{1 + |x_1(t)|} \end{bmatrix}.
\]

Choose \(x(3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\). Using Mittag-Leffler matrix function for a given matrix \(A^2\), we get

\[
\Phi(3 - t) = \begin{bmatrix} N(t) & 0 \\ 0 & N(t) \end{bmatrix},
\]
where $N(t) = (3 - t)^{3/4} E_{7/4,7/4}((3 - t)^{7/4})$. By simple matrix calculation, one can see that the controllability Grammian matrix

\[
M = \int_0^3 \Phi(3-s)BB^*\Phi(3-s)\, ds
\]

is positive definite. Further the nonlinear function $f$ is continuous and satisfies the hypotheses of Theorem (3.5.1). Observe that the control defined by

\[
u(t) = B^*\Phi^*(1-t)M^{-1}\left[\bar{x} - \int_0^1 \Phi(1-s)f(s,x(s),C^{1/2}D_{0+}^{1/2}x(s))\, ds\right]
\]

steers the system (3.6.5) from \[\begin{bmatrix} 1 \\ 0 \end{bmatrix}\] to \[\begin{bmatrix} 1 \\ 1 \end{bmatrix}\]. Hence the system (3.6.5) is controllable on $[0,3]$.

### 3.7 Conclusion

In this chapter, we have considered the fractional harmonic oscillator type dynamical systems represented by the fractional differential equations of order $1 < \alpha \leq 2$. Controllability results for this type of fractional dynamical systems are obtained. To establish these results, first we have derived the necessary and sufficient conditions for the controllability of linear fractional dynamical systems using controllability Grammian matrix and Mittag-leffler matrix function. Under different types of conditions depending on the nonlinear functions, we have established the sufficient conditions for the controllability of nonlinear and nonlinear damped fractional dynamical systems by using Schauder’s and Schaefer’s fixed point theorems respectively. Finally some numerical examples are presented to verify the results.

**********