CHAPTER 2

MATHEMATICAL ANALYSIS

2.1 INTRODUCTION

In this Chapter, we list out the basic governing equations and its solutions which will be derived based on Bessel function concept. Basic equations which are essential to deal are the governing field equations of motion, heat conduction equation, generalized heat conduction equation, the radial displacement equation of the motion for an inviscid fluid and the electric and magnetic field equations. Solutions for the wave propagation of a transversely isotropic solid cylinder of various cross-sections are obtained by using Fourier expansion collocation method.

2.2 STRESS-STRAIN-DISPLACEMENT RELATIONS

The strain-displacement relationships of a linearly elastic material in cylindrical coordinates \((r, \theta, z)\) are

\[
\begin{align*}
e_{rr} &= u_{rr}, & e_{\theta\theta} &= r^{-1}(u_r + u_{\theta,\theta}), & e_{zz} &= u_{zz}, \\
e_{r\theta} &= \frac{1}{2}(u_{\theta,r} + r^{-1}(u_{r,\theta} - u_{\theta})), \\
e_{\theta\theta} &= \frac{1}{2}(u_{\theta,\theta} + r^{-1}u_{r,\theta}), & e_{rz} &= \frac{1}{2}(u_{z,r} + u_{z,\theta})
\end{align*}
\]

(2.1)

where \(u_r, u_{\theta}, u_z\) respectively denotes radial, circumferential and axial displacement components. The comma in the subscript denotes the partial derivatives with respect to the respective coordinates. The stress-strain relations for transversely isotropic materials are given as
\begin{align*}
\sigma_{rr} &= c_{11} e_{rr} + c_{12} e_{\theta\theta} + c_{13} e_{zz} \\
\sigma_{\theta\theta} &= c_{12} e_{rr} + c_{11} e_{\theta\theta} + c_{13} e_{zz} \\
\sigma_{zz} &= c_{13} e_{rr} + c_{11} e_{\theta\theta} + c_{33} e_{zz} \\
\sigma_{r\theta} &= 2c_{66} e_{r\theta}, \quad \sigma_{\theta z} = 2c_{44} e_{\theta z}, \quad \sigma_{r z} = 2c_{44} e_{r z} 
\end{align*} \quad (2.2)

where \( \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz} \) are the normal stresses along the radial, circumferential and axial directions respectively and \( \sigma_{r\theta}, \sigma_{\theta z}, \sigma_{r z} \) are the shearing stress components and \( c_{11}, c_{12}, c_{13}, c_{33}, c_{44} \) and \( 2c_{66} = c_{11} - c_{12} \) are the elastic constants for the transversely isotropic materials.

### 2.3 EQUATIONS OF MOTION AND ITS SOLUTION

Consider a transversely isotropic solid cylinder of infinite length of arbitrary cross-section immersed in an inviscid fluid. The system is assumed to be linear so that the three-dimensional stress equations of motion are used for both the cylinder and the fluid. In cylindrical coordinate, the three-dimensional stress equation of motion in the absence of body force for linear elastic medium by Berliner and Solecki (1996) are

\begin{align*}
\sigma_{rr,r} + r^{-1} \sigma_{r\theta,\theta} + \sigma_{rz,z} + r^{-1} \left( \sigma_{rr} - \sigma_{\theta\theta} \right) = \rho u_{r,r} \\
\sigma_{r\theta,r} + r^{-1} \sigma_{\theta\theta,\theta} + \sigma_{r\theta,z} + 2r^{-1} \sigma_{r\theta} = \rho u_{\theta,r} \\
\sigma_{rz,r} + r^{-1} \sigma_{\theta z,\theta} + \sigma_{rz,z} + r^{-1} \sigma_{rz} = \rho u_{z,r} 
\end{align*} \quad (2.3)

where \( \rho \) is the density. On substitution of equations (2.1) and (2.2) into equation (2.3) yields the following displacement equations of motion.

\begin{align*}
&c_{11} \left( u_{r,r} + r^{-1} u_{r,r} - r^{-2} u_r \right) - r^{-2} \left( c_{11} + c_{66} \right) u_{\theta,\theta} + r^{-2} c_{66} u_r = \rho u_{r,r} \\
+c_{44} u_{r,z} + \left( c_{44} + c_{13} \right) u_{z,r} + r^{-1} \left( c_{66} + c_{12} \right) u_{\theta,\theta} = \rho u_{r,r}
\end{align*}
\[ r^{-1} (c_{12} + c_{66}) u_{r,r\theta} + r^{-2} (c_{66} + c_{11}) u_{r,\theta} + c_{66} \left( u_{\theta,rr} + r^{-1} u_{\theta,\theta} - r^{-2} u_{\theta} \right) \\
+ r^{-2} c_{11} u_{\theta,\theta\theta} + c_{44} u_{\theta,zz} + r^{-1} \left( c_{44} + c_{13} \right) u_{z,\theta z} = \rho u_{\theta,z} \]

\[ c_{44} \left( u_{r,rr} + r^{-1} u_{r,\theta} + r^{-2} u_{r,zz} \right) + r^{-1} \left( c_{44} + c_{13} \right) \left( u_{r,zz} + u_{\theta,\theta z} \right) \\
+ \left( c_{44} + c_{13} \right) u_{r,zz} + c_{33} a u_{z,zz} = \rho u_{z,zz} \quad (2.4) \]

It is very clear that equations (2.4) are coupled partial differential equations of the three displacement components \( u_r, u_{\theta}, \) and \( u_z \). Based on the method adopted by Mirsky (1965) we could able to decouple the equations (2.4) and also we will arrive at a partial differential equation of fourth order, if we assume the components \( u_r, u_{\theta}, \) and \( u_z \) in the following form

\[
u_r (r, \theta, z, t) = \sum_{n=0}^{\infty} e_n \left[ (\phi_n, r + r^{-1} \psi_n, \theta) + (\phi_n, r + r^{-1} \psi_n, \theta) \right] e^{i(kz + \omega t)} \\
u_{\theta} (r, \theta, z, t) = \sum_{n=0}^{\infty} e_n \left[ (r^{-1} \phi_n, \theta - \psi_n, rz \right) + (r^{-1} \phi_n, \theta - \psi_n, rz) \right] e^{i(kz + \omega t)} \\
u_z (r, \theta, z, t) = (i/a) \sum_{n=0}^{\infty} e_n \left[ W_n + \overline{W}_n \right] e^{i(kz + \omega t)} \quad (2.5) \]

where \( \phi_n, \psi_n, W_n \) and \( \phi_n, \psi_n, \overline{W}_n \) are the potential functions for the symmetric and anti-symmetric modes of vibrations respectively, \( k \) is the wave number, \( \omega \) is the angular frequency and \( e_n = 1/2 \) when \( n = 0 \) and \( e_n = 1 \) when \( n \geq 1 \) and \( i = \sqrt{-1} \).

Substituting the equation (2.5) in equation (2.4) we obtain the following partial differential equations of second order of which two are involving the displacement potentials \( \phi_n (r, \theta) \) and \( W_n (r, \theta) \) and the third one is purely in terms of \( \psi_n (r, \theta) \).
They are

\[
\left(\tilde{c}_{11} \nabla^2 + \left(\Omega^2 - \zeta^2\right)\right)\phi_n - \left(1 + \tilde{c}_{13}\right)\zeta W_n = 0
\]
\[
\zeta \left(1 + \tilde{c}_{13}\right) \nabla^2 \phi_n + \left(\nabla^2 + \left(\Omega^2 - \zeta^2\right)\right)W_n = 0
\]

(2.6)

and

\[
\left[ \nabla^2 + (\Omega^2 - \zeta^2) / \tilde{c}_{66} \right] \psi_n = 0
\]

(2.7)

where the non-dimensional quantities occurring in equations (2.6) and (2.7) are

\[
\tilde{c}_{ij} = \frac{c_{ij}}{c_{44}}, \quad \zeta = k a, \quad \Omega^2 = \frac{\rho \omega^2 a^2}{c_{44}}, \quad x = \frac{r}{a}
\]

(2.8)

and

\[
\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + x^{-1} \frac{\partial}{\partial x} + x^{-2} \frac{\partial^2}{\partial \theta^2}
\]

(2.9)

The equation (2.6) is a linear equation, which gives a trivial solution, to obtain the non-trivial solution equate the determinant of its coefficient matrix to zero, we obtain

\[
\begin{vmatrix}
\tilde{c}_{11} \nabla^2 + \left(\Omega^2 - \zeta^2\right) & -\zeta \left(1 + \tilde{c}_{13}\right) \\
\zeta \left(1 + \tilde{c}_{13}\right) \nabla^2 & \nabla^2 + \left(\Omega^2 - \zeta^2\tilde{c}_{33}\right)
\end{vmatrix}
(\phi_n, W_n) = 0
\]

(2.10)

Expanding the determinant given in the equation (2.10) we obtain

\[
(A \nabla^4 + B \nabla^2 + C) (\phi_n, W_n) = 0
\]

(2.11)

where the coefficients are given as

\[
A = \tilde{c}_{11}
\]
\[
B = \left(1 + \tilde{c}_{11}\right) \Omega^2 + \zeta^2 \left(\tilde{c}_{13}^2 + 2\tilde{c}_{13} - \tilde{c}_{11}\tilde{c}_{13}\right)
\]
\[
C = \left(\Omega^2 - \zeta^2\right) \left(\Omega^2 - \zeta^2\tilde{c}_{33}\right)
\]

(2.12)
Solving the equation (2.11) we obtain the solutions for the symmetric and anti-symmetric modes of vibration. The solution to the symmetric modes of vibration are given as

\[ \phi_n = \sum_{i=1}^{2} A_m J_n(\alpha_i ax) \cos n\theta \]

\[ W_n = \sum_{i=1}^{2} d_i A_m J_n(\alpha_i ax) \cos n\theta \]  \hspace{1cm} (2.13)

Similarly the solution for the anti-symmetric mode is obtained from equation (2.13) by replacing \( \cos n\theta \) by \( \sin n\theta \)

\[ \overline{\phi}_n = \sum_{i=1}^{2} \overline{A}_m J_n(\alpha_i ax) \sin n\theta \]

\[ \overline{W}_n = \sum_{i=1}^{2} d_i \overline{A}_m J_n(\alpha_i ax) \sin n\theta \]  \hspace{1cm} (2.14)

where \( J_n(\alpha_i ax) \) is the Bessel function of the first kind of order \( n \), \( A_m, \overline{A}_m \) are the constants of integration.

The coefficients \( A, B, C \) occurring in equation (2.11) satisfies the algebraic equation

\[ A(\alpha a)^4 - B(\alpha a)^2 + C = 0 \]  \hspace{1cm} (2.15)

where \( (\alpha_i a)^2 \) \( (i = 1, 2) \) are the roots of equation (2.15).

The constant \( d_i \) occurring in equations (2.13) and (2.14) can be obtained by substituting equation (2.13) into equation (2.6) and on simplification we obtain

\[ d_i = \{[\Omega^2 - \zeta^2 - (\alpha_i a)^2 \overline{c}_{11}] / \zeta(1 + \overline{c}_{13}) \} \hspace{1cm} i = 1, 2 \]  \hspace{1cm} (2.16)
On solving the Bessel’s equation, equation (2.7) we will obtain the solution for the symmetric mode in the following form

\[ \psi_n = A_{3n} J_n(\alpha_n a x) \sin n \theta \]  

(2.17)

where \((\alpha_n a)^2 = (\Omega^2 - \zeta^2)c_0^6\). If \((\alpha_n a)^2 < 0\), then the Bessel function \(J_n\) is replaced by the modified Bessel function \(I_n\). Based on equation (2.17) we can write the solution of equation (2.7) for the anti-symmetric mode as

\[ \overline{\psi}_n = \overline{A}_{3n} J_n(\alpha_n ax) \cos n \theta \]  

(2.18)

where \(A_{3n}\) and \(\overline{A}_{3n}\) are the constants of integration.

### 2.3.1 Analysis of fluid

Acoustic pressure or sound pressure is the local pressure deviation from the ambient atmospheric pressure caused by sound wave. The acoustic pressure in air can be measured using a microphone and in water using a hydrophone. In cylindrical polar coordinates \(r, \theta\) and \(z\) the acoustic pressure and radial displacement equations of motion for an inviscid fluid as per Achenbach (1973) are

\[ p^f = -B^f \left( u^f_{r,r} + r^{-1} \left( u^f_{r} + u^f_{\theta,\theta} \right) + u^f_{z,z} \right) \]  

(2.19)

and \(c_f^{-2} \Delta = \Delta_r\)  

(2.20)

where \(\Delta_r = u^f_{r,r} + r^{-1} \left( u^f_{r} + u^f_{\theta,\theta} \right) + u^f_{z,z}\)

(2.21)

\(B^f\) is the adiabatic bulk modulus, \(\rho^f\) is the density and \(c_f = \left( \frac{B^f}{\rho^f} \right)^{1/2}\) is the acoustic phase velocity in the fluid and \(u^f_r, u^f_\theta\) and \(u^f_z\) are the radial, circumferential
and axial displacement components in fluid. The relation between the displacements and the potential functions in the fluid and in the respective directions are

\[ u_r^f = \phi_r^f, \quad u_\theta^f = r^{-1} \phi_\theta^f, \quad u_z^f = \phi_z^f \]  

(2.22)

Now seek the solution of the radial displacement equation of motion, equation (2.20) in the form

\[ \phi_r^f (r, \theta, z, t) = \sum_{n=0}^{\infty} \varepsilon_n [\phi_n^f (r) \cos n\theta + \tilde{\phi}_n^f (r) \sin n\theta] e^{i(\xi z + \omega t)} \]  

(2.23)

where \( \phi_n^f (r) \) and \( \tilde{\phi}_n^f (r) \) are respectively, the radial potential functions in the fluid for the symmetric and anti-symmetric modes.

The fluid that represents the oscillating waves propagating away from the arbitrary cross-sectional cylinder is given by

\[ \phi_n^f = A_{4n} H_n^{(1)} (\delta ax) \]  

(2.24)

where \( H_n^{(1)} \) is the Henkel function of the first kind of order \( n \) and \( A_{4n} \) is the constant of integration. The corresponding non-dimensional quantities are

\[ (\delta a)^2 = \frac{\Omega^2}{\rho B'}, \quad \bar{\rho} = \frac{\rho}{\rho f}, \quad \bar{B'} = \frac{B'}{c_{44}} \]  

(2.25)

If equation (2.23) along with equation (2.24) is substituted in equation (2.19) then the acoustic pressure for the fluid can be expressed as

\[ p_f = \sum_{n=0}^{\infty} \varepsilon_n A_{4n} \Omega^2 \bar{\rho} H_n^{(1)} (\delta ax) \cos n\theta e^{i(\xi z + \omega t)} \]  

(2.26)

where \[ \Omega^2 = \rho \omega^2 a^2 / c_{44} \], \( \bar{z} = z / a \), \( T_a = [t \sqrt{c_{44} / \rho}] / a \).
2.3.2 Analysis of thermo-elastic cylinder

Consider a homogeneous transversely isotropic, thermally conducting elastic cylinder of infinite length with uniform temperature in the undisturbed state. The heat conduction equation in the absence of body force for transversely isotropic materials, given by Sharma and Sharma (2002), is

\[ K_1 \left( T_{rr} + r^{-1}T_r + r^{-2}T_{\theta\theta} \right) + K_3 T_{zz} - \rho c_e T_j = T_0 \left( \beta_1 \left( e_{rr} + e_{\theta\theta} \right) + \beta_3 e_{zz} \right) \]  

(2.27)

Where \( \beta_1 = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_3 \), \( \beta_3 = 2c_{13}\alpha_1 + c_{33}\alpha_3 \)

Here, \( T(r, \theta, z, t) \) is the temperature change, \( \alpha_1, \alpha_3 \) and \( K_1, K_3 \) are respectively, the coefficients of linear thermal expansion and thermal conductivities along and perpendicular to the axis of symmetry, \( c_v \) is the specific heat at constant strain, and \( T_0 \) is the uniform temperature in the undisturbed state.

The relation between stress and strain components and the temperature change in terms of elastic constants and thermal expansion coefficients for the homogeneous transversely isotropic materials, as per Sharma and Sharma (2002), are

\[
\begin{align*}
\sigma_{rr} &= c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz} - \beta_1 T \\
\sigma_{\theta\theta} &= c_{12}e_{rr} + c_{11}e_{\theta\theta} + c_{13}e_{zz} - \beta_1 T \\
\sigma_{zz} &= c_{13}e_{rr} + c_{33}e_{\theta\theta} + c_{33}e_{zz} - \beta_3 T \\
\sigma_{r\theta} &= 2c_{66}e_{r\theta}, \quad \sigma_{r\theta} = 2c_{44}e_{r\theta}, \quad \sigma_{rz} = 2c_{44}e_{rz}
\end{align*}
\]

(2.28)

Substituting equations (2.1) and (2.28) into equations (2.3) and (2.27) we obtain the three-dimensional equations of motion and heat conduction in the following form
Propagation of harmonic waves in polygonal and arbitrary cross-sectional cylinders is obtained by assuming the solution of the displacement components in terms of derivatives of potentials introduced by Berliner and Solecki (1996). The temperature distribution for a transversely isotropic material in the form of Ponnumsy (2007) can be written as,

\[ T(r, \theta, z, t) = (c_{44} / \beta_i a^2) \sum_{n=0}^{\infty} \varepsilon_n (T_n + \bar{T}_n) e^{i(kz + \omega t)} \]  

(2.30)

where \( T_n \) and \( \bar{T}_n \) are respectively, the temperature potential functions for the symmetric and anti-symmetric modes and \( \varepsilon_n = 1/2 \) for \( n = 0 \), \( \varepsilon_n = 1 \) for \( n \geq 1 \) and \( i = \sqrt{-1} \).

In order to convert the variable quantities with variable units occurring in this problem, to the non-dimensional one, the following expressions may be considered:

\[ x = \frac{r}{a}, \quad \zeta = k a, \quad \Omega^2 = \frac{\rho \omega^2 a^2}{c_{44}}, \quad \frac{1}{c_{ij}} = \frac{c_{ij}}{c_{44}} \]

\[ \bar{\beta} = \frac{\beta_i}{\beta_3}, \quad \bar{K}_i = \frac{(\rho c_{44})}{\beta_3 T_0} K_i, \quad \bar{d} = \frac{\rho c_{44}}{\beta_3 T_0} i = 1, 3 \]  

(2.31)
Substituting equations (2.5), (2.30) and (2.31) into equation (2.29) and on simplification we obtain the following four second order partial differential equations in terms of the potential functions \( \phi_n \), \( W_n \), \( T_n \) and \( \psi_n \)

\[
\left( \tilde{c}_{11} \nabla^2 + \left( \Omega^2 - \zeta^2 \right) \right) \phi_n - \zeta \left( 1 + \tilde{c}_{13} \right) W_n - \bar{\beta} T_n = 0
\]

\[
\zeta \left( 1 + \tilde{c}_{13} \right) \nabla^2 \phi_n + \left( \nabla^2 + \left( \Omega^2 - \zeta^2 \tilde{c}_{33} \right) \right) W_n - \zeta T_n = 0
\]

\[
\bar{\beta} \nabla^2 \phi_n - \zeta W_n + \left( \bar{d} + jK_i \nabla^2 - jK_3 \zeta^2 \right) T_n = 0
\]

(2.32)

and

\[
\left( \nabla^2 + \left( \Omega^2 - \zeta^2 \right) \right) \psi_n = 0
\]

(2.33)

where the expression for \( \nabla^2 \) is same as given in equation (2.9). Simultaneous partial differential equations given by equation (2.32) can be written in the form of a determinant equation as

\[
\begin{vmatrix}
\tilde{c}_{11} \nabla^2 + \left( \Omega^2 - \zeta^2 \right) & -\zeta \left( 1 + \tilde{c}_{13} \right) & -\bar{\beta} \\
\zeta \left( 1 + \tilde{c}_{13} \right) \nabla^2 & \nabla^2 + \left( \Omega^2 - \zeta^2 \tilde{c}_{33} \right) & -\zeta \\
\bar{\beta} \nabla^2 & -\zeta & \bar{d} + iK_i \nabla^2 - iK_3 \zeta^2 \\
\end{vmatrix} = 0
\]

(2.34)

Expansion of equation (2.34) yields a simple partial differential equation of the sixth order namely

\[
\left( A \nabla^6 + B \nabla^4 + C \nabla^2 + D \right) \left( \phi_n, W_n, T_n \right) = 0
\]

(2.35)
where the coefficients A, B, C, D occurring in equation (2.35) are obtained as

\[ A = i c_{11} K_1 \]

\[ B = i K_1 \left( \left( \Omega^2 - \bar{c}_{33} \zeta^2 \right) \bar{c}_{11} + \left( \Omega^2 - \zeta^2 \right) + \zeta^2 \left( 1 + \bar{c}_{13} \right)^2 \right) + \bar{c}_{11} \left( \bar{d} - i \bar{K}_3 \zeta^2 \right) + \bar{\beta}^2 \]

\[ C = \bar{c}_{11} \left( \Omega^2 - \bar{c}_{33} \zeta^2 \right) \left[ \bar{a} - \zeta^2 \left( 1 + i \bar{K}_3 \right) \right] + \left( \Omega^2 - \zeta^2 \right) \left[ \left( \bar{d} - i \bar{K}_3 \zeta^2 \right) + i \bar{K}_1 \left( \Omega^2 - \bar{c}_{33} \zeta^2 \right) \right] + \left( 1 + \bar{c}_{13} \right)^2 \left[ \zeta^2 \left( \bar{d} - i \bar{K}_3 \zeta^2 \right) + \zeta^2 \bar{\beta} \right] + \bar{\beta}^2 \left( 1 + \bar{c}_{13} \right) + \bar{\beta}^2 \left( \Omega^2 - \bar{c}_{33} \zeta^2 \right) \]

\[ D = \left( \Omega^2 - \zeta^2 \right) \left[ \left( \Omega^2 - \bar{c}_{33} \zeta^2 \right) \left( \bar{a} - i \bar{K}_3 \zeta^2 \right) - \zeta^2 \right] \tag{2.36} \]

The sixth order partial differential equation, equation (2.35) is factorized into a cubic equation for \((\alpha_i^2) \) \((i = 1, 2, 3)\). Based on the Bessel functions concept, the solution for the symmetric mode are obtained as

\[ \phi_n = \sum_{i=1}^{3} A_{mn} J_n (\alpha_i ax) \cos n \theta \]

\[ W_n = \sum_{i=1}^{3} d_i A_{mn} J_n (\alpha_i ax) \cos n \theta \]

\[ T_n = \sum_{i=1}^{3} e_i A_{mn} J_n (\alpha_i ax) \cos n \theta \tag{2.37} \]

Similarly, in the same way we shall write the solution for the anti-symmetric mode in the following form. It is obtained by replacing \( \cos n \theta \) by \( \sin n \theta \) in equation (2.37)

\[ \phi_n = \sum_{i=1}^{3} \overline{A}_{mn} J_n (\alpha_i ax) \sin n \theta \]

\[ \overline{W}_n = \sum_{i=1}^{3} d_i \overline{A}_{mn} J_n (\alpha_i ax) \sin n \theta \]

\[ \overline{T}_n = \sum_{i=1}^{3} e_i \overline{A}_{mn} J_n (\alpha_i ax) \sin n \theta \tag{2.38} \]

where \( d_i, e_i, A_m \) and \( \overline{A}_m \) are all constants of integration.
Here \((\alpha_i \alpha)^2 > 0 \ (i = 1, 2, 3)\) are the roots of algebraic equation

\[
A(\alpha a)^6 - B(\alpha a)^4 + C(\alpha a)^2 - D = 0
\]  
(2.39)

It is well known that \(J_n(0)\) is zero except for \(n = 0\), so the solution of the equation (2.39) corresponding to the root \((\alpha_i \alpha)^2 = 0\) is not considered here. For other roots the use of Bessel function of first kind of order \(n\), \(J_n\) depends on whether the roots \((\alpha_i \alpha)^2\) \(i = 1, 2, 3\) are real or complex. If the roots \((\alpha_i \alpha)^2\) are imaginary then the solution of the equation (2.39) depends on the modified Bessel function \(I_n\), of first kind of order \(n\).

If equation (2.37) is substituted in equation (2.32) we will arrive at the simultaneous equations involving the constants \(d_i\) and \(e_i\) are given as

\[
\zeta (1 + \tilde{c}_{13}) d_i + \overline{\rho} e_i = - \left( \tilde{c}_{11} (\alpha_i \alpha)^2 - \Omega^2 + \zeta^2 \right) \\
\left( \Omega^2 - \tilde{c}_{33} \zeta^2 \right) - (\alpha_i \alpha)^2 \right) d_i - \zeta e_i = (\alpha_i \alpha)^2 \left( 1 + \tilde{c}_{13} \right) \zeta
\]  
(2.40)

Equation (2.33) is an equation corresponding to the case of an axial displacement of the solid cylinder in terms of the potential function \(\psi_n\). So the solution of equation (2.33) for the symmetric mode is obtained as

\[
\psi_n = A_{4n} J_n (\alpha_4 a x) \sin n\theta
\]  
(2.41)

where \((\alpha_4 a)^2 = (\Omega^2 - \zeta^2) / \tilde{c}_{66}\). The solution for the anti-symmetric mode is obtained from the equation (2.41) by replacing \(\sin n\theta\) by \(\cos n\theta\), so

\[
\overline{\psi}_n = A_{4n} J_n (\alpha_4 a x) \cos n\theta
\]  
(2.42)
2.3.3 Analysis of magneto- electro-elastic cylinder

The relation connecting the stress, electric displacement and magnetic induction in terms of strain, electric field and magnetic field for the transversely isotropic materials by Buchanan (2003), the normal stress components are

\[
\sigma_{rr} = c_{11} e_{rr} + c_{12} e_{\theta\theta} + c_{13} e_{zz} - e_{31} E_z - q_{31} H_z \\
\sigma_{\theta\theta} = c_{12} e_{rr} + c_{11} e_{\theta\theta} + c_{13} e_{zz} - e_{31} E_z - q_{31} H_z \\
\sigma_{zz} = c_{13} e_{rr} + c_{13} e_{\theta\theta} + c_{33} e_{zz} - e_{33} E_z - q_{33} H_z
\]

(2.43)

The shearing stress components are

\[
\sigma_{r\theta} = 2c_{66} e_{r\theta} \\
\sigma_{r\theta} = 2c_{44} e_{r\theta} - e_{15} E_\theta - q_{15} H_\theta \\
\sigma_{rz} = 2c_{44} e_{r\theta} - e_{15} E_r - q_{15} H_r
\]

(2.44)

The electric displacement components in terms of strain, electric and magnetic field components are

\[
D_r = e_{15} e_{rz} + e_{11} E_r + m_{11} H_r \\
D_\theta = e_{15} e_{\theta\theta} + e_{11} E_\theta + m_{11} H_\theta \\
D_z = e_{31} (e_{rr} + e_{\theta\theta}) + e_{33} e_{zz} + e_{33} E_z + m_{33} H_z
\]

(2.45)

The relationship between magnetic induction components in terms of strain, electric and magnetic field components are

\[
B_r = q_{15} e_{rz} + m_{11} E_r + \mu_{11} H_r \\
B_\theta = q_{15} e_{\theta\theta} + m_{11} E_\theta + \mu_{11} H_\theta \\
B_z = q_{31} (e_{rr} + e_{\theta\theta}) + q_{33} e_{zz} + m_{33} E_z + \mu_{33} H_z
\]

(2.46)

Here \( D_r, D_\theta, D_z \) denotes the electric displacement components, \( B_r, B_\theta, B_z \) are the magnetic induction components, \( e_{31}, e_{33}, e_{15} \) are the piezo-electric material
coefficients, \( q_{31}, q_{33}, q_{45} \) denotes the piezo-magnetic material coefficients, \( \varepsilon_{11}, \varepsilon_{33} \) are the dielectric coefficients, \( \mu_{11}, \mu_{33} \) denotes the magnetic permeability coefficients \( m_{11}, m_{33} \) are the magneto-electric material coefficients.

The electric field vector \( E_i = (E_r, E_\theta, E_z) \) and magnetic field vector \( H_i = (H_r, H_\theta, H_z) \) are respectively, related to the electric potential \( \chi \) and magnetic potential \( \vartheta \) as in Buchanan (2003) in the form

\[
E_r = -\chi_r, \quad E_\theta = -r^{-1}\chi_\theta, \quad E_z = -\chi_z
\]

and

\[
H_r = -\vartheta_r, \quad H_\theta = -r^{-1}\vartheta_\theta, \quad H_z = -\vartheta_z
\]

The complete equations governing the behavior of a piezo-electric cylinder have been given by Paul and Raju (1982) in terms of displacement and electric potential and the extension to coupled magneto-electro-elasticity in cylindrical coordinates is straightforward. The governing equations that relate the magnetic field to the magnetic potential are identical, in form, to those that relate the electric field to the electric potential. It follows that all equations that govern electric displacement are similar to those that govern magnetic induction.

The dynamical equations for the magneto-electro-elastic solid cylinder in cylindrical coordinates as per Yu and Wu (2009), in the absence of body forces, electric charge and current density are

\[
D_{r,r} + r^{-1}D_r + D_{\theta,\theta} + D_{z,z} = 0
\]

\[
B_{r,r} + r^{-1}B_r + B_{\theta,\theta} + B_{z,z} = 0
\]

(2.49)
Upon substituting equations (2.1), (2.2), (2.43) to (2.48) into equation (2.3) and (2.49) will lead to the following set of governing equations in terms of displacements, electric potential and magnetic potential.

\[
\begin{align*}
c_{11} (u_{rr} + r^{-1}u_{r} - r^{-2}u_{rr}) + c_{66} r^{-2}u_{r,\theta \theta} + c_{44} u_{r,zz} + (c_{12} + c_{66}) r^{-1}u_{\theta ,r} \\
- (c_{11} + c_{66}) r^{-2}u_{\theta ,\theta} + (c_{13} + c_{44}) u_{z,rr} + (e_{31} + e_{15}) \chi_{,r} + (q_{31} + q_{15}) \phi_{,r} &= \rho u_{rr} \\
(c_{66} + c_{12}) r^{-1}u_{r,rr} + (c_{11} + c_{66}) r^{-2}u_{r,\theta} + c_{66} (u_{\theta ,r} + r^{-1}u_{\theta ,r} - r^{-2}u_{\theta}) \\
+ c_{44} u_{\theta ,zz} + c_{11} r^{-2}u_{\theta ,\theta} + (c_{13} + c_{44}) r^{-1}u_{z,zz} + (e_{31} + e_{15}) r^{-1} \chi_{,\theta} + (q_{31} + q_{15}) \phi_{,\theta} &= \rho u_{\theta ,r} \\
(c_{44} + c_{13}) (u_{r,rr} + r^{-1}u_{r} + r^{-1}u_{\theta ,\theta}) + c_{44} (u_{z,rr} + r^{-1}u_{z} + r^{-2}u_{z,\theta}) + c_{33} u_{zz} \\
+ e_{33} \chi_{,zz} + q_{33} \phi_{,zz} + e_{15} (\chi_{,rr} + r^{-1} \chi_{,r} + r^{-2} \chi_{,\theta}) + q_{15} (\phi_{,rr} + r^{-1} \phi_{,r} + r^{-2} \phi_{,\theta}) &= \rho u_{z,rr} \\
e_{15} (u_{z,rr} + r^{-1}u_{z} + r^{-2}u_{z,\theta}) + (e_{31} + e_{15}) (u_{r,rr} + r^{-1}u_{rr} + r^{-2}u_{r,\theta}) + e_{33} u_{zz} \\
-e_{33} \chi_{,zz} - m_{33} \phi_{,zz} - e_{11} (\chi_{,rr} + r^{-1} \chi_{,r} + r^{-2} \chi_{,\theta}) - m_{11} (\phi_{,rr} + r^{-1} \phi_{,r} + r^{-2} \phi_{,\theta}) &= 0 \\
q_{15} (u_{z,rr} + r^{-1}u_{z} + r^{-2}u_{z,\theta}) + (q_{31} + q_{15}) (u_{r,rr} + r^{-1}u_{z} + r^{-2}u_{r,\theta}) + q_{33} u_{zz} \\
-m_{33} \chi_{,zz} - \mu_{33} \phi_{,zz} - \mu_{11} (\chi_{,rr} + r^{-1} \chi_{,r} + r^{-2} \chi_{,\theta}) - m_{11} (\phi_{,rr} + r^{-1} \phi_{,r} + r^{-2} \phi_{,\theta}) &= 0 \quad (2.50)
\end{align*}
\]

To obtain propagation of harmonic waves along the symmetric axis, we assume solutions of the displacement components to be expressed in terms of derivatives of potentials introduced by Paul and Raman (1991); and from the constitutive equation we can easily guess the form of the electric and magnetic potential as
where \( \omega \) is the angular velocity, \( k \) the axial wave number, \( \phi(r, \theta), \psi(r, \theta), W(r, \theta) \) are the displacement potentials and \( U(r, \theta) \) and \( V(r, \theta) \) are the electric and magnetic potentials respectively.

The non-dimensional quantities used in this chapter are

\[
\zeta = ka, \quad x = r/a, \quad \Omega^2 = \rho \omega^2 a^2 / c_{44}, \quad \bar{e}_{ij} = e_{ij} / c_{44}
\]

\[
\bar{\epsilon}_{ij} = \epsilon_{ij} / \epsilon_{33}, \quad \bar{q}_{ij} = q_{ij} / q_{33}, \quad \bar{m}_{ij} = m_{ij} c_{44} / q_{33},
\]

\[
\bar{\epsilon}_{ij} = \epsilon_{ij} c_{44} / e_{33}^2, \quad \bar{\mu}_{ij} = \mu_{ij} c_{44} / q_{33}^2
\] (2.52)

Making use of the three displacement components, an electric potential and a magnetic potential components given in equation (2.51) into equation (2.50) we find that \( \phi, \psi, W, U \) and \( V \) satisfies the four simultaneous partial differential equation of second order. They are
\[
\left( \tilde{c}_{11} \nabla^2 + \left( \Omega^2 - \zeta^2 \right) \right) \phi - \zeta \left( 1 + \tilde{c}_{13} \right) W - \zeta \left( \tilde{e}_{31} + \tilde{e}_{15} \right) U - \zeta \left( \tilde{q}_{31} + \tilde{q}_{15} \right) V = 0
\]

\[
\zeta \left( 1 + \tilde{c}_{13} \right) \nabla^2 \phi + \left( \nabla^2 + \left( \Omega^2 - \zeta^2 \tilde{c}_{33} \right) \right) W + \left( \tilde{e}_{15} \nabla^2 - \zeta^2 \tilde{e}_{33} \right) \phi + \left( \tilde{q}_{15} \nabla^2 - \zeta^2 \tilde{q}_{33} \right) V = 0
\]

\[
\zeta \left( \tilde{e}_{31} + \tilde{e}_{15} \right) \nabla^2 \phi + \left( \tilde{e}_{15} \nabla^2 - \zeta^2 \right) W - \left( \tilde{e}_{11} \nabla^2 - \zeta^2 \tilde{e}_{33} \right) U - \left( \tilde{m}_{11} \nabla^2 - \zeta^2 \tilde{m}_{33} \right) V = 0
\]

and

\[
\left( \nabla^2 + \left( \Omega^2 - \zeta^2 \right) / \tilde{c}_{66} \right) \psi = 0
\] (2.54)

where the expression for \( \nabla^2 \) is same as given by equation (2.9)

Simultaneous partial differential equation, equation (2.53) in terms of the potentials \( \phi(r, \theta), W(r, \theta), U(r, \theta) \) and \( V(r, \theta) \) can be written as a determinant equation of the form

\[
\begin{vmatrix}
\tilde{c}_{11} \nabla^2 + \left( \Omega^2 - \zeta^2 \right) & -\zeta \left( 1 + \tilde{c}_{13} \right) & -\zeta \left( \tilde{e}_{31} + \tilde{e}_{15} \right) & -\zeta \left( \tilde{q}_{31} + \tilde{q}_{15} \right) \\
\zeta \left( 1 + \tilde{c}_{13} \right) \nabla^2 & \nabla^2 + \left( \Omega^2 - \zeta^2 \tilde{c}_{33} \right) & \tilde{e}_{15} \nabla^2 - \zeta^2 & \tilde{q}_{15} \nabla^2 - \zeta^2 \\
\zeta \left( \tilde{e}_{31} + \tilde{e}_{15} \right) \nabla^2 & \tilde{e}_{15} \nabla^2 - \zeta^2 & \zeta^2 \tilde{e}_{33} - \tilde{e}_{11} \nabla^2 & \zeta^2 \tilde{m}_{33} - \tilde{m}_{11} \nabla^2 \\
\zeta \left( \tilde{q}_{31} + \tilde{q}_{15} \right) \nabla^2 & \tilde{q}_{15} \nabla^2 - \zeta^2 & \zeta^2 \tilde{m}_{33} - \tilde{m}_{11} \nabla^2 & \zeta^2 \tilde{m}_{33} - \tilde{m}_{11} \nabla^2 \\
\end{vmatrix} (\phi, W, U, V) = 0
\] (2.55)

The determinant equation, equation (2.55) on simplification reduces to the following eighth order partial differential equation

\[
\left( AW^8 + BW^6 + CW^4 + DW^2 + E \right) (\phi, W, U, V) = 0
\] (2.56)

where the constants \( A, B, C, D, E \) are as follows
\[ A = c_{11} g_1 \]

\[ B = \tilde{c}_{11} \left[ g_2 + g_3 q_4 - g_5 - 2\tilde{e}_{15} g_6 + 2\tilde{q}_{15} g_7 + 2\zeta^2 \tilde{e}_{15} q_{13} m_{33} \right] + g_2 g_8 + g_4 g_9^2 \]

\[ + 2\zeta g_9 \left( g_{10} g_{11} + g_{12} g_{13} \right) + \zeta^2 g_{10} \left( g_{10} q_{14} + 2 g_{12} g_{15} \right) - \zeta^2 g_{12}^2 g_{16} \]

\[ C = \tilde{c}_{11} \left[ \zeta^2 \left( g_6 - g_7 \right) g_{17} - 2\tilde{e}_{15} g_{18} - 2\tilde{q}_{15} g_{19} + g_2 g_3 \right] \]

\[ + g_8 \left[ g_2 + g_3 q_4 - g_5 - 2\tilde{e}_{15} g_6 + 2\tilde{q}_{15} g_7 + 2\zeta^2 \tilde{e}_{15} q_{13} m_{33} \right] \]

\[ + g_2 g_9^2 + 2\zeta g_9 \left[ g_{10} (g_6 - g_{20}) + g_{12} (g_7 + g_{21}) \right] \]

\[ - \zeta^2 \left[ \tilde{\mu}_{15} g_{10}^2 g_3 - 2\tilde{m}_{11} g_{10} g_{12} g_3 + \tilde{e}_{11} g_{12}^2 g_3 \right] \]

\[ + \zeta^4 \left[ g_{10}^2 g_{22} - 2 g_{10} g_{12} g_{23} + g_{12}^2 g_{25} \right] \]

\[ D = \tilde{c}_{11} \left[ \zeta^2 g_{17} g_3 + \zeta^4 \left( g_{18} - g_{19} \right) \right] + g_8 \left[ \zeta^2 \left( g_6 - g_7 + g_{17} \right) - 2\zeta^2 \left( \tilde{e}_{15} g_{18} + \tilde{q}_{15} g_{19} \right) + g_2 g_3 \right] \]

\[ + \zeta^4 \left[ g_9^2 g_{24} + \tilde{\mu}_{33} g_{10}^2 g_3 - 2\tilde{m}_{33} g_{10} g_{12} g_3 + \tilde{e}_{33} g_{12}^2 g_3 \right] - 2\zeta^3 g_9 \left[ g_{10} g_{18} + g_{12} g_{19} \right] \]

\[ - \zeta^6 \left[ g_{10}^2 - 2 g_{10} g_{12} + g_{12}^2 \right] \]

\[ E = \zeta^2 g_8 \left[ g_3 g_{17} + \zeta^2 \left( g_{18} - g_{19} \right) \right] \]

where

\[ g_1 = \tilde{e}_{11} \tilde{\mu}_{11} - \tilde{m}_{11}^2 + \tilde{e}_{15} \tilde{\mu}_{11} + q_{15} e_{11} - 2\tilde{e}_{15} \tilde{q}_{15} m_{11} \]

\[ g_2 = \zeta^2 \left( 2\tilde{m}_{11} m_{33} - \tilde{e}_{33} \tilde{\mu}_{11} - \tilde{e}_{11} \tilde{\mu}_{33} \right) \]

\[ g_3 = \Omega^2 - \zeta^2 \tilde{e}_{33} \]

\[ g_{11} = \tilde{e}_{15} \tilde{\mu}_{11} - \tilde{q}_{15} m_{11} \]

\[ g_{19} = \zeta^2 \left( m_{33} - \tilde{e}_{33} \right) \]

\[ g_4 = \tilde{e}_{11} \mu_{11} - m_{11}^2 \]

\[ g_{12} = \tilde{q}_{31} + q_{15} \]

\[ g_{20} = \zeta^2 \left( \tilde{e}_{15} m_{33} - \tilde{m}_{33} q_{15} \right) \]
\[ g_5 = \zeta^2 \left( e_{15} \mu_{33} + q_{15} e_3 \right) \]
\[ g_{13} = q_{15} e_{11} - e_{15} m_{11} \]
\[ g_{21} = \zeta^2 \left( e_{15} m_{33} + q_{15} e_3 \right) \]
\[ g_6 = \zeta^2 \left( \mu_{11} - m_{11} \right) \]
\[ g_{14} = \mu_{11} + q_{15} \]
\[ g_{22} = \mu_{33} + 2q_{15} \]
\[ g_7 = \zeta^2 \left( m_{11} - e_{11} \right) \]
\[ g_{15} = m_{11} + e_{15} q_{15} \]
\[ g_{23} = m_{33} + q_{15} + e_{15} \]
\[ g_8 = \Omega^2 - \zeta^2 \]
\[ g_{16} = e_{11} + e_{15} \]
\[ g_{24} = e_{33} m_{33} - m_{33} \]
\[ g_9 = \zeta \left( 1 + c_{11} \right) \]
\[ g_{17} = \zeta^2 \left( e_{33} \mu_{33} - m_{33} \right) \]
\[ g_{18} = \zeta^2 \left( m_{33} - \mu_{33} \right) \]
\[ g_{25} = e_{33} + 2e_{15} \quad (2.57) \]

Factorizing the partial differential equation given in equation (2.56) into a bi-quadratic equation for \((\alpha, \alpha)^2\) \(i = 1, 2, 3, 4\), the solution for the system are obtained as

\[ \phi = \sum_{i=1}^{4} A_{i} J_{n} (\alpha, r) \cos n\theta \]
\[ W = \sum_{i=1}^{4} a_{i} A_{i} J_{n} (\alpha, r) \cos n\theta \]
\[ U = \sum_{i=1}^{4} b_{i} A_{i} J_{n} (\alpha, r) \cos n\theta \]
\[ V = \sum_{i=1}^{4} c_{i} A_{i} J_{n} (\alpha, r) \cos n\theta \quad (2.58) \]

Here \((\alpha, r)^2\), \(i = 1, 2, 3, 4\) are the non-zero roots of the equation.

\[ A\alpha^8 - B\alpha^6 + C\alpha^4 - D\alpha^2 + E = 0 \quad (2.59) \]

The constants \(a_i, b_i, c_i\) defined in equation (2.58) can be calculated from equation (2.53), and they are given by
\[ a_i = \frac{\zeta\left( g_{10}(\zeta^2 + (\alpha_i a)^2 \tilde{q}_{12}) - g_{15}(\zeta^2 + (\alpha_i a)^2 \tilde{e}_{15})\right)}{g_y(\zeta^2 + (\alpha_i a)^2 \tilde{e}_{15}) - \zeta g_{10}(\alpha_i a)^2} - g_{13} \],

\[ b_i = \frac{\zeta g_{52}^2 \left( (\zeta^2 + (\alpha_i a)^2 \tilde{q}_{12})_1 \right) - \zeta g_{10}(\alpha_i a)^2 - g_{13}(\alpha_i a)^2}{g_y(\zeta^2 + (\alpha_i a)^2 \tilde{e}_{15}) - \zeta g_{10}(\alpha_i a)^2} - g_{13} \],

\[ c_i = \frac{(\tilde{c}_{11}(\alpha_i a)^2 - g_{13})(\alpha_i a)^2 - g_{13} - g_{10}^2(\alpha_i a)^2}{g_y(\zeta^2 + (\alpha_i a)^2 \tilde{e}_{15}) - \zeta g_{10}(\alpha_i a)^2} - g_{13} \] (2.60)

The non-appearance of the function \( \psi \) in equation (2.53) shows that the axial vibration is independent of the radial and circumferential vibrations. Hence, the solution of equation (2.54) is obtained as

\[ \psi = A_n J_n(\alpha_n a x) \sin n\theta \] (2.61)

where \( (\alpha_n a)^2 = (\Omega^2 - \zeta^2)/\tilde{c}_{66} \). If \( (\alpha_n a)^2 < 0 \), the Bessel function \( J_n \) is replaced by the modified Bessel function \( I_n \).

### 2.4 CONCLUSION

This Chapter deals with the mathematical modeling of wave propagation of elastic, thermo-elastic, and magneto-electro-elastic cylinders. The solutions to the solid elastic cylinder, thermo-elastic cylinder and magneto-electro-elastic cylinders are analyzed in detail for the symmetric and anti-symmetric modes respectively. Also, the equation of the motion of the fluid is formulated using the constitutive equations of an inviscid fluid and their solutions are presented for both symmetric and anti-symmetric cases. In the subsequent Chapters, these solutions are used to study vibrational characteristics of different kinds of solids.

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