Chapter 2

Preliminaries

Abstract

This chapter introduces the basic concept of fuzziness and distinct fuzzy numbers. The fundamental ideas of fuzzy logic as well as its fuzzy membership functions are discussed. In addition some classical representations on fuzzy numbers, which are ultimately required for this work, are also included.


2.1 Introduction

Fuzzy systems are based on fuzzy logic, a generalization of conventional (Boolean) logic that has been extended to handle the concept of partial truth - truth values between “completely true” and “completely false”. It was introduced by L. A. Zadeh of University of California, Berkeley, U.S.A., in the 1960’s, as a means to model the uncertainty of natural language. Zadeh himself says that rather than regarding fuzzy theory as a single theory, we should regard the process of “fuzzification” as a methodology to generalize any specific theory from a crisp (discrete) to a continuous (fuzzy) form.

The organization of the chapter is detailed below. In section 2.2 some preliminaries such as fuzzy sets and membership functions which are useful in the subsequent sections are stated in addition to the classification of fuzzy membership functions. In section 2.3 the general discussion of fuzzy logic are studied in detail. In section 2.4 some interpretation on fuzzy membership functions are discussed, finally in sections 2.5 and 2.6, details of the fuzzy numbers and its arithmetic operations are acknowledged.

2.2 Fuzzy Sets and Membership Functions

The theory of fuzzy sets, now encompasses a well organized corpus of
basic notions including (and not restricted to) aggregation operations, a generalized theory of relations, specific measures of information content, a calculus of fuzzy numbers. Fuzzy sets are also the cornerstone of a non-additive uncertainty theory, namely possibility theory, and of a versatile tool for both linguistic and numerical modeling: fuzzy rule-based systems. Numerous works now combine fuzzy concepts with other scientific disciplines as well as modern technologies. In mathematics fuzzy sets have triggered new research topics in connection with category theory, topology, algebra, analysis. Fuzzy sets are also part of a recent trend in the study of generalized measures and integrals, and are combined with statistical methods. Furthermore, fuzzy sets have strong logical underpinnings in the tradition of many-valued logics.

Fuzzy set-based techniques are also an important ingredient in the development of information technologies. In the field of information processing fuzzy sets are important in clustering, data analysis and data fusion, pattern recognition and computer vision. Fuzzy rule-based modeling has been combined with other techniques such as neural nets and evolutionary computing and applied to systems and control engineering, with applications to robotics, complex process control and supervision. In the field of information systems, fuzzy sets play a role in the development of intelligent and flexible man-machine interfaces and the storage of imprecise linguistic information.

In Artificial Intelligence various forms of knowledge representation and automated reasoning frameworks benefit from fuzzy set-based
techniques, for instance in interpolative reasoning, nonmonotonic reasoning, diagnosis, logic programming, constraint-directed reasoning, etc. Fuzzy expert systems have been devised for fault diagnosis, and also in medical science. In decision and organization sciences, fuzzy sets has had a great impact in preference modeling and multicriteria evaluation, and has helped bringing optimization techniques closer to the users needs. Applications can be found in many areas such as management, production research, and finance. Moreover concepts and methods of fuzzy set theory have attracted scientists in many other disciplines pertaining to human-oriented studies such as cognitive psychology and some aspects of social sciences.

In classical set theory, a subset \( U \) of a set \( S \) can be considered as a mapping from the elements of \( S \) to the elements of the set \( 0, 1 \), consisting of the two elements \( 0 \) and \( 1 \),

\[
i.e.: \quad U : S \rightarrow 0, 1
\]

This mapping may be represented as a set of ordered pairs, with exactly one ordered pair present for each element of \( S \). The first element of the ordered pair is an element of the set \( S \), and the second element is an element of the set \( 0, 1 \). The value zero is used to represent non-membership, and the value one is used to represent membership. The truth or falsity of the statement:

\[
x \text{ is in } U
\]

is determined by finding the ordered pair whose first element is \( x \). The
statement is true if the second element of the ordered pair is 1, and
the statement is false if it is 0. Similarly, a fuzzy subset (or fuzzy set)
F of a set S can be defined as a set of ordered pairs, each with the first
element from S, and the second element from the interval [0, 1], with
exactly one ordered pair present for each element of S. This defines
a mapping between elements of the set S and values in the interval
[0, 1]. The value zero is used to represent complete non-membership,
the value one is used to represent complete membership, and values
in-between are used to represent intermediate degrees of membership.
The set S is referred to as the universe of discourse for the fuzzy subset
F. Frequently, the mapping is described as a function, the membership
function of F. The ordinary sets are considered as special cases of
fuzzy sets with the membership functions equal to the characteristic
functions. They are called crisp sets. The above definition of fuzzy
set brings the equivalence between a fuzzy set as such, intuitively a
set-based concept, and its membership function, a mapping from the
universe of discourse to the unit interval [0, 1], or, more generally, to
some lattice L. Here, the operations with fuzzy sets are defined by the
operations with functions. We define a fuzzy set as a family of (crisp)
sets, where each member of the family corresponds to a specific grade
of membership from the unit interval [0, 1]. Doing this we define easily
the corresponding membership function. This approach is intuitively
well understandable and practically easily tractable, as it is natural
to work with (crisp) sets with the membership grade greater or equal
to some level. Moreover, this approach seems to be more elegant to
some mathematicians who are rather reluctant to speak about “sets
"having in mind "functions ". The fuzzy set based on the concept of family of nested sets, enjoys, among others, the following advantages:

- It makes possible to create consistent (mathematical) theory,
- No "artificial "identification of a fuzzy set with its membership function is necessary,
- Nonfuzzy sets can be naturally embedded into the fuzzy sets,
- Nonfuzzy concepts may be extended to represent fuzzy ones,
- Any fuzzy problem can be viewed as a family of non fuzzy ones,
- Practical tractability is achieved.

A classical set (or crisp set) is defined as a collection of elements x in X. Each element can either belong or not belong to a set A, A subset of X. The membership of elements x to the subset A of X can be expressed by a characteristic function in which 1 indicates membership and 0, non membership. For a normalized fuzzy set, the characteristic function allows various degrees of membership for the elements of a given set.

2.2.1 Definition
A fuzzy set A in X is a set of ordered pairs: \( A = \{(x, \mu_A(x)) \mid x \in X\} \) where \( \mu_A(x) \) is called the membership function of x in A, which maps X to the membership Space \([0, 1]\). A is non fuzzy and \( \mu_A(x) \) is identical to the characteristic function of a non fuzzy set. If X is the set of all positive real numbers.
2.2.2 Definition
Let $X$ be a universal set. Then the fuzzy subset $A$ of $X$ is defined by its membership function $\mu_A(x) \rightarrow [0, 1]$ which assign a real number in the interval $[0, 1]$ to each element $x$ in $X$, where the value of $\mu_A(x)$ at $x$ shows the grade of membership of $x$ in $A$.

2.3 Classification of fuzzy Membership Functions

Fuzzy sets are sets without clear or crisp boundaries. The elements they contain may only have a partial degree of membership. They are therefore not the same as classical sets in the sense that the sets are not closed. Some examples of vague fuzzy sets and their respective units include the following.

- Loud noises (sound intensity)
- Ambient (brightness)
- High speeds (velocity)
- Desirable actions (decision of control space)

Fuzzy sets can be combined through fuzzy rules to represent specific actions/behavior and it is this property of fuzzy logic that will be utilized when implementing a fuzzy logic controller in subsequent
A membership function (MF) is a curve that defines how each point in the input space is mapped to the set of all real numbers from 0 to 1. This is really the only stringent condition brought to bear on a MF. A classical set may be for example written as:

\[ A = \{ x \mid x > 3 \} \]

Now if X is the universe of discourse with elements x then a fuzzy set A in X is defined as a set of ordered pairs:

\[ A = \{ (x, \mu_A(x)) \mid x \text{ in } X \} \]

Note that in the above expression \( \mu_A(x) \) may be called the membership function of x in A and that each element of X is mapped to a membership value between 0 and 1.

![Figure 2.1: Example of fuzzy set](image)

Example of Fuzzy Set. S, small; MS, medium small; M, medium, ML, medium large; L, large. Typical membership function shapes include triangular, trapezoidal and Gaussian functions. The shape is chosen
on the basis of how well it describes the set it represents. Below in figure some example of fuzzy sets can be observed.

2.4 Fuzzy Logic

The degree to which the statement \( x \) is in \( F \) is true is determined by finding the ordered pair whose first element is \( x \). The degree of truth of the statement is the second element of the ordered pair. In practice, the terms “membership function” and fuzzy subset get used interchangeably. That is a lot of mathematical baggage, so here is an example. Let us talk about people and “tallness” expressed as their \( \text{HEIGHT} \). In this case the set \( S \) (the universe of discourse) is the set of people. Let us define a fuzzy subset tall, which will answer the question “to what degree is person \( x \) tall?” Zadeh describes \( \text{HEIGHT} \) as a linguistic variable, which represents our cognitive category of “tallness”. The values of this linguistic variable are fuzzy subsets as tall, very tall, or short. To each person in the universe of discourse, we assign a degree of membership in the fuzzy subset tall. The easiest way to do this is with a membership function based on the real function \( h \)
(“height of a person in cm”) which is defined for each person $x \in S$.

$$tall(x) = \begin{cases} 
0 & \text{if } h(x) < 150 \\
\frac{h(x) - 150}{50} & \text{if } 150 < h(x) \leq 200 \\
1 & \text{if } h(x) > 200
\end{cases}$$

The fuzzy subset verytall may be defined by a nonlinear function of $h(x)$:

$$verytall(x) = \begin{cases} 
0 & \text{if } h(x) < 170 \\
\frac{(h(x) - 170)^2}{50^2} & \text{if } 170 < h(x) \leq 220 \\
1 & \text{if } h(x) > 220
\end{cases}$$

On the other hand, the fuzzy subset short is defined as follows:

$$Short(x) = \begin{cases} 
0 & \text{if } h(x) < 150 \\
\frac{200 - h(x)}{50} & \text{if } 150 < h(x) \leq 200 \\
1 & \text{if } h(x) > 200
\end{cases}$$

A Graphs of these membership functions look like in figure: 2.2. Given this definition, here are some example values:
Expressions like “x is A” can be interpreted as degrees of truth, e.g., “Hideki is tall = 0.46”.

### 2.4.1 Remark

Membership functions used in most applications almost never have as simple a shape as tall(x) in the above stated example. At minimum, they tend to be triangles pointing up, and they can be much more complex than that. Also, the discussion characterizes membership functions as if they always are based on a single criterion, but this is not always the case, although it is quite common. One could, for example, want to have the membership function for tall depend on both a person’s height and their age, e.g. “somebody is tall for his age”. This is perfectly legitimate, and occasionally used in practice. It is referred to as a two-dimensional membership function, or a “fuzzy
relation ”. It is also possible to have even more criteria, or to have
the membership function depend on elements from two completely
different universes of discourse.

2.5 Fuzzy Numbers and its Representations

Among the various types of fuzzy sets, of special significance are fuzzy
sets that are defined on the set $\mathbb{R}$ of real numbers. Membership
functions of these sets, which have the form

$$ A : \mathbb{R} \rightarrow [0, 1] $$

clearly have a quantitative meaning and may, under certain conditions,
be viewed as fuzzy numbers or fuzzy intervals. To view them in this
way, they should capture our intuitive conceptions of approximate
numbers or intervals, such as “numbers that are close to a given
real number ” or “numbers that are around a given interval of real
numbers.” Such concepts are essential for characterizing states of
fuzzy variables and, consequently, play an important role in many
applications, including fuzzy control, decision making, approximate
reasoning, optimization, and statistics with imprecise probabilities.
Fuzzy numbers are fuzzy subsets of the real line. They have a peak
or plateau with membership grade 1, over which the members of
the universe are completely in the set. The membership function is increasing towards the peak and decreasing away from it. Fuzzy numbers are used very widely in fuzzy control applications. A typical case is the triangular fuzzy number which is one form of the fuzzy number. Slope and trapezoidal functions are also used, as well as exponential curves similar to Gaussian probability densities.

2.5.1 Some Interpretation on Fuzzy numbers

A fuzzy number is a quantity whose value is imprecise, rather than exact as is the case with “ordinary” (single-valued) numbers. Any fuzzy number can be thought of as a function whose domain is a specified set (usually the set of real numbers, and whose range is the span of non-negative real numbers between, and including, 0 and 1000. Each numerical value in the domain is assigned a specific “grade of membership” where 0 represents the smallest possible grade, and 1000 is the largest possible grade. In many respects, fuzzy numbers depict the physical world more realistically than single-valued numbers. Suppose, for example, that you are driving along a highway where the speed limit is 55 miles an hour (mph). You try to hold your speed at exactly 55 mph, but your car lacks “cruise control”, so your speed varies from moment to moment. If you graph your instantaneous speed over a period of several minutes and then plot the result in rectangular coordinates, you will get a function that looks like one of the curves shown below.

The curve (Fig. 2.3) represents a triangular fuzzy number; the curve(Fig. 2.4) shows a trapezoidal fuzzy number; the curve (Fig. 2.5)
illustrates a bell-shaped fuzzy number. These three functions, known as membership functions, are all convex (the grade starts at zero, rises to a maximum, and then declines to zero again as the domain increases). However, some fuzzy numbers have concave, irregular, or even chaotic membership functions. There is no restriction on the shape of the membership curve, as long as each value in the
domain corresponds to one and only one grade in the range, and the grade is never less than 0 nor more than 1000. Fuzzy numbers are used in statistics, computer programming, engineering (especially communications), and experimental science. The concept takes into account the fact that all phenomena in the physical universe have a degree of inherent uncertainty.

2.6 Arithmetic Operations on Fuzzy Numbers

In this section, we present the method for developing fuzzy arithmetic and its employs the extension principle, by which operations on real numbers are extended to operations on fuzzy numbers. We assume in this section that fuzzy numbers are represented by continuous
membership functions. All the three fuzzy numbers and its arithmetic operations and alpha cut also are given as follows.

2.6.1 Triangular fuzzy number

1. Definition

A triangular fuzzy number \( A \) can be defined as a triplet \( A = [a_1, a_2, a_3] \) and its membership function is defined as follows:

\[
\mu_A(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\
\frac{a_3-x}{a_3-a_2} & \text{if } a_2 \leq x \leq a_3
\end{cases}
\]

The Pictorial representation of triangular membership function \( A \) is given below in the figure.

![Figure 2.6: Triangular membership function \( \mu_A(x) \)](image-url)
2. Definition ($\alpha$ Cut)

Given a fuzzy set $A$ in $X$ and any real number $\alpha$ in $[0, 1]$, then the $\alpha$-cut or $\alpha$-level or cut worthy set of $A$, denoted by $^\alpha A$ is the crisp set $^\alpha A = \{ x \in X : \mu_A(x) \geq \alpha \}$. The strong $\alpha$ cut, denoted by $^\alpha+ A$ is the crisp set $^\alpha+ A = \{ x \in X : \mu_A(x) \geq \alpha \}$. For example, let $A$ be a fuzzy set whose membership function is given as follows:

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2} & \text{if } a_2 \leq x \leq a_3 \end{cases}$$

To find the $\alpha$-cut of $A$, we first set $\alpha \in [0,1]$ to both left and right reference functions of $A$. That is,

$$\alpha = \frac{x-a_1}{a_2-a_1} \quad \text{and} \quad \alpha = \frac{a_3-x}{a_3-a_2}$$

Expressing $x$ in terms of $\alpha$, we have $x = (a_2 - a_1)\alpha + a_1$ and $x = a_3 - (a_3 - a_2)\alpha$ and which gives the $\alpha$-cut of $A$ is $^\alpha A = [(a_2 - a_1)\alpha + a_1, a_3 - (a_3 - a_2)\alpha]$.

3. Example

In the case of triangular fuzzy number $A = (-5, -1, 1)$ (Figure: 2.7), the membership function will be
The Pictorial representation of triangular membership function $\mu_A(x)$ is given below in the figure.

\[
\mu_A(x) = \begin{cases} 
0 & \text{if } x < -5 \\
\frac{x+5}{4} & \text{if } -5 \leq x \leq -1 \\
\frac{1-x}{2} & \text{if } -1 \leq x \leq 1 \\
1 & \text{if } x > 1
\end{cases}
\]

Then $\alpha$-cut interval from the fuzzy number is $\frac{x+5}{4} = \alpha \Rightarrow x = 4\alpha - 5$;
$\frac{1-x}{2} = \alpha \Rightarrow x = -2\alpha + 1$.

$\alpha A = [x_1^\alpha, x_2^\alpha] = [4\alpha - 5, -2\alpha + 1]$.

If $\alpha = 0.5$, substituting 0.5 for $\alpha$, we get $^\alpha A$.

i.e., $^\alpha A = [x_1^{(0.5)}, x_2^{(0.5)}] = [-3, 0]$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig27.png}
\caption{$\alpha = 0.5$ cut of triangular fuzzy number $A$.}
\end{figure}
2.6.2 Trapezoidal fuzzy number

1. Definition

A trapezoidal fuzzy number $A$ can be defined as a $A = [a_1, a_2, a_3, a_4]$ and its membership function is defined as follows:

$$\mu_A(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\
1 & \text{if } a_2 \leq x \leq a_3 \\
\frac{x-a_4}{a_3-a_4} & \text{if } a_2 \leq x \leq a_3
\end{cases}$$

The Pictorial representation of trapezoidal membership function $\mu_A(x)$ is given below in the figure.

![Figure 2.8: Trapezoidal membership function $\mu_A(x)$.](image)

2. Definition

We define a ranking function $R: F(R) \to \mathbb{R}$, which maps each fuzzy number into the real line, $F(R)$ represents the set of all trapezoidal fuzzy numbers. If $R$ be any ranking function, then $R(A) = \frac{a_1 + a_2 + a_3 + a_4}{4}$. 
3. Definition (Arithmetic operations)

Let $A = [a_1, a_2, a_3, a_4]$ and $B = [b_1, b_2, b_3, b_4]$ be two trapezoidal fuzzy numbers then the arithmetic operations on $A$ and $B$ as follows:

**Addition:** $A + B = [a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4]$

**Subtraction:** $A - B = [a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1]$

**Multiplication:**

(i) If $R(B) > 0$;

$$A \cdot B = \begin{cases} \frac{a_1}{4}(b_1 + b_2 + b_3 + b_4), \frac{a_2}{4}(b_1 + b_2 + b_3 + b_4), \\ \frac{a_1}{4}(b_1 + b_2 + b_3 + b_4), \frac{a_2}{4}(b_1 + b_2 + b_3 + b_4). \end{cases}$$

(ii) If $R(B) < 0$;

$$A \cdot B = \begin{cases} \frac{a_3}{4}(b_1 + b_2 + b_3 + b_4), \frac{a_4}{4}(b_1 + b_2 + b_3 + b_4), \\ \frac{a_3}{4}(b_1 + b_2 + b_3 + b_4), \frac{a_4}{4}(b_1 + b_2 + b_3 + b_4). \end{cases}$$

4. Definition ($\alpha$ Cut)

Given a fuzzy set $A$ in $X$ and any real number $\alpha$ in $[0, 1]$, then the $\alpha$-cut or $\alpha$-level or cut worthy set of $A$, denoted by $^\alpha A$ is the crisp set $^\alpha A = \{x \in X : \mu_A(x) \geq \alpha\}$. The strong $\alpha$ cut, denoted by $^{\alpha^+} A$ is the crisp set $^{\alpha^+} A = \{x \in X : \mu_A(x) \geq \alpha\}$. For example, let $A$ be a fuzzy set whose membership function is given as follows:
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\[ \mu_A(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\
1 & \text{if } a_2 \leq x \leq a_3 \\
\frac{x-a_4}{a_3-a_4} & \text{if } a_2 \leq x \leq a_3 
\end{cases} \]

To find the \( \alpha \)-cut of \( A \), we first set \( \alpha \in [0,1] \) to both left and right reference functions of \( A \). That is,

\[ \alpha = \frac{x-a_1}{a_2-a_1} \quad \text{and} \quad \alpha = \frac{x-a_4}{a_3-a_4} \]

Expressing \( x \) in terms of \( \alpha \), we have \( x = (a_2 - a_1)\alpha + a_1 \) and \( x = a_4 + (a_3 - a_4)\alpha \) and which gives the \( \alpha \)-cut of \( A \) is \( \alpha A = [(a_2 - a_1)\alpha + a_1, a_4 + (a_3 - a_4)\alpha] \).

5. Example

In the case of the trapezoidal fuzzy number \( A = (2, 4, 6, 8) \)(Figure: 2.9), the membership function value will be

\[ \mu_A(x) = \begin{cases} 
\frac{x-2}{2} & \text{if } 2 \leq x \leq 4 \\
1 & \text{if } 4 \leq x \leq 6 \\
\frac{x-8}{2} & \text{if } 6 \leq x \leq 8 \\
0 & \text{if } x > 8 
\end{cases} \]

The Pictorial representation of trapezoidal membership function
\( \mu_A(x) \) is given below in the figure.

![Figure 2.9 α = 0.5 cut of trapezoidal fuzzy number A.](image)

Then \( \alpha \) - cut interval from the fuzzy number is \( \frac{x-2}{2} = \alpha \Rightarrow x = 2\alpha + 2; \)
\( \frac{x-8}{2} = \alpha \Rightarrow x = -2\alpha + 8. \)
\( \alpha A = [x_1^\alpha, x_2^\alpha] = [2\alpha + 2, -2\alpha + 8]. \)
If \( \alpha = 0.5 \), substituting 0.5 for \( \alpha \), we get \( \alpha A \).
i.e., \( \alpha A = [x_1^{(0.5)}, x_2^{(0.5)}] = [3, 7]. \)

**2.6.3 Piece Wise Quadratic Fuzzy Number**

A fuzzy number is a convex normalized fuzzy set of the real line \( \mathbb{R} \)
whose membership function is piecewise continuous.

1. **Definition**

A piece wise quadratic fuzzy number \( A \) can be defined as \( A = [a_1, a_2, a_3, a_4, a_5] \) is a defined by the membership function as follows:
2. Definition

We define a ranking function \( R: F(R) \rightarrow \mathbb{R} \), which maps each fuzzy number into the real line, \( F(R) \) represents the set of all piecewise quadratic fuzzy numbers. If \( R \) be any ranking function, then

\[
\mu_A(x) = \begin{cases} 
\frac{(x-a_1)^2}{2(a_2-a_1)^2} & \text{if } a_1 \leq x \leq a_2 \\
\frac{-(x-a_3)^2}{2(a_3-a_2)^2} + 1 & \text{if } a_2 \leq x \leq a_3 \\
\frac{-(x-a_3)^2}{2(a_4-a_3)^2} + 1 & \text{if } a_3 \leq x \leq a_4 \\
\frac{(x-a_5)^2}{(a_5-a_4)^2} & \text{if } a_4 \leq x \leq a_5
\end{cases}
\]

The Pictorial representation of piecewise quadratic membership function \( \mu_A(x) \) is given below in the figure.

\[\text{Figure 2.10: Piecewise quadratic membership function } \mu_A(x).\]

3. Definition (Arithmetic operations)

Let \( A = [a_1, a_2, a_3, a_4, a_5] \) and \( B = [b_1, b_2, b_3, b_4, b_5] \) be two trapezoidal fuzzy numbers then the arithmetic operations on \( A \) and \( B \) as follows:
Addition: \[ A + B = [a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5] \]

Subtraction: \[ A - B = [a_1 - b_5, a_2 - b_4, a_3 - b_3, a_4 - b_2, a_5 - b_1] \]

Multiplication:
(i) If \( R(B) > 0; \)
\[ A \bullet B = \left\{ \frac{1}{2}(a_5b_1 + a_1b_5), \frac{1}{2}(a_4b_2 + a_2b_4), a_3b_3, \frac{1}{2}(a_2b_4 + a_4b_2), \frac{1}{2}(a_1b_5 + a_5b_1) \right\} \]
(ii) If \( R(B) < 0; \)
\[ A \bullet B = \left\{ \frac{1}{2}(a_1b_5 + a_5b_1), \frac{1}{2}(a_2b_4 + a_4b_2), a_3b_3, \frac{1}{2}(a_4b_2 + a_2b_4), \frac{1}{2}(a_5b_1 + a_1b_5) \right\} \]

4. Remark
1. If \( R(A) > R(B) \) then A is called fuzzy maximum then B.
2. If \( R(A) < R(B) \) then A is called fuzzy minimum then B.