CHAPTER -4
INEQUALITIES ON s-UNITARY MATRICES

In this chapter, it is shown that all standard partial orderings i.e lowener partial order, the star order and minus order or rank subractivity order are preserved under \( V [4,5,13] \). That is all partial orderings are preserved under s-unitary similarity. Some theorems relating to partial ordering on s-unitary matrices are derived. The concept of ‘0’ partial ordering on s-unitary matrices are introduced in this chapter. Some theorems relating to s-unitary matrices are also established.
4.1. S-INVARIANT PARTIAL ORDERING ON MATRICES

In this section, it is proved that all standard partial ordering are preserved under s-unitary similarity. Lowener partial order, the star order and minus order (rank subtractivity) are denoted by $\geq_L$, $\geq_*$ and $\geq_{rs}$ respectively. It is shown that all standard partial ordering are preserved under $V$ which is defined below.

**Definition 4.1.1**

For $A, B \in \mathbb{C}_{n \times n}$, we define

(i) $A \geq_L B$ if $A - B \geq 0$

(ii) $A \geq_* B$ if $B^*B = B^*A$ and $BB^* = AB^*$

(iii) $A \geq_{rs} B$ if $\text{rank}(A - B) = \text{rank}(A) - \text{rank}(B)$

The relationship between star and minus ordering are studied by Baksalary [4], Hartwig [13], Mitra [37] etc.

**Theorem 4.1.2 [15]**

For $A, B \in \mathbb{C}_{n \times n}$,

$A \geq_L B \iff \rho(A^\dagger B) \leq 1$ and $R(B) \subseteq R(A)$

where $\rho(A) = \max \{ |\lambda| / \lambda \text{ is an eigen value of } A \}$ is the spectral radius.
Theorem 4.1.3 [13]

For $A, B \in C_{n \times n}$,

$$A \succcurlyeq_{rs} B \iff A \succcurlyeq_{rs} B \quad \text{and} \quad (A - B)^\dagger = (A^\dagger - B^\dagger)$$

Other conditions to be added to rank subtractivity to equivalent to star ordering are given in [4].

Theorem 4.1.4[14]

For $A, B \in C_{n \times n}$,

$$A \succ_{rs} B \iff B = BA^\dagger B = BA^\dagger A = AA^\dagger B$$

Theorem 4.1.5

For $A, B \in C_{n \times n}$ and $V$ is the permutation matrix with units in the secondary diagonal.

$$A \succ_{L} B \iff VA \succ_{L} VB \iff AV \succ_{L} BV$$

Proof:

Let $A, B \in C_{n \times n}$ and $V$ is the permutation matrix with units in the secondary diagonal.

$$A \succ_{L} B \iff \rho(A^\dagger B) \leq 1 \quad \text{and} \quad R(B) \subseteq R(A) \quad \text{[By Theorem 4.1.2]}$$

$$\iff \rho(A^\dagger VB) \leq 1 \quad \text{and} \quad B = AA^\dagger B \quad \text{[By Theorem 1.2.31]}$$

$$\iff \rho(A^\dagger VVB) \leq 1 \quad \text{and} \quad VB = (V A)(A^\dagger V)(VB)$$

$$\iff \rho((VA)^\dagger (VB)) \leq 1 \quad \text{and} \quad R(VB) \subseteq R(VA) \quad \text{[By Remark 1.2.34 and Theorem 1.2.30]}$$

Therefore $A \succ_{L} B \iff VA \succ_{L} VB$
Now \( A_L \succ B \Leftrightarrow \rho(A^\dagger B) \leq 1 \) and \( R(B) \subseteq R(A) \) \hfill [By Theorem 4.1.2]

\( \Leftrightarrow \rho(VA^\dagger BV) \leq 1 \) and \( B = AA^\dagger B \) \hfill [By Theorem 1.2.31]

\( \Leftrightarrow \rho((AV)^\dagger (BV)) \leq 1 \) and \( BV = (AV)(AV)^\dagger (BV) \) \hfill [By Remark 1.2.34]

\( \Leftrightarrow \rho((AV)^\dagger (BV)) \leq 1 \) and \( R(BV) \subseteq R(AV) \) \hfill [By Theorem 1.2.31]

\( \Leftrightarrow AV_L \succ BV \) \hfill [By Theorem 4.1.2.]

Therefore \( A_L \succ B \Leftrightarrow VA_L \succ VB \Leftrightarrow AV_L \succ BV \).

**Theorem 4.1.6**

For \( A, B \in C_{nn} \) and \( V \) is the permutation matrix with units in the secondary diagonal.

\( A \succ B \Leftrightarrow VA \succ VB \Leftrightarrow AV \succ BV \)

**Proof:** \( A \succ B \Leftrightarrow B^*B = B^*A \) and \( BB^* = AB^* \) \hfill [By (iii) of Definition 4.1.1]

\( \Leftrightarrow B^*VVB = B^*VVA \) and \( VBB^*V = VAB^*V \)

\( \Leftrightarrow (VB)^*(VB) = (VB)^*VA \) and \( (VB)(VB)^* = (VA)(VB)^* \)

\( \Leftrightarrow VA \succ VB \)

Therefore \( A \succ B \Leftrightarrow VA \succ VB \)

Similarly we can prove \( A \succ B \Leftrightarrow AV \succ BV \). Therefore \( A \succ B \Leftrightarrow VA \succ VB \Leftrightarrow AV \succ BV \)
Theorem 4.1.7

For $A, B \in C_{n \times n}$ and $V$ is the permutation matrix with units in the secondary diagonal.

$$A \overset{\succ}{_{rs}} B \iff VA \overset{\succ}{_{rs}} VB \iff AV \overset{\succ}{_{rs}} BV$$

Proof:

$$A \overset{\succ}{_{rs}} B \iff \text{rank}(A - B) = \text{rank}(A) - \text{rank}(B)$$

$$\iff \text{rank}V(A - B) = \text{rank}(VA) - \text{rank}(VB)$$

$$\iff \text{rank}(VA - VB) = \text{rank}(VA) - \text{rank}(VB)$$

$$\iff VA \overset{\succ}{_{rs}} VB$$

Therefore $A \overset{\succ}{_{rs}} B \iff VA \overset{\succ}{_{rs}} VB$

Similarly we can prove $A \overset{\succ}{_{rs}} B \iff AV \overset{\succ}{_{rs}} BV$.

Therefore $A \overset{\succ}{_{rs}} B \iff VA \overset{\succ}{_{rs}} VB \iff AV \overset{\succ}{_{rs}} BV$.

Hence all standard partial ordering are preserved under $V$.

Theorem 4.1.8

Lowener ordering is preserved under unitary similarity.

(i.e) $A \overset{\succ}{_{L}} B \iff P^*AP \overset{\succ}{_{L}} P^*BP$
Theorem 4.1.9

Star ordering is preserved under unitary similarity.

\[(i.e) \ A \succ_B \iff P^* A P \succ_P B P\]

Theorem 4.1.10

Minus ordering is preserved under unitary similarity.

\[(i.e) \ A \succ_{rs} B \iff P^* A P \succ_{rs} B P\]

Theorem 4.1.11

Let \( A, B \in C_{n \times n} \). Lowener ordering is preserved under s-unitary similarity.

Proof:

It is enough to prove \( A \succ_{L} B \iff VP^{-1} VAP \succ_{L} VP^{-1} VBP \) for some unitary matrix \( P \).

Now \( A \succ_{L} B \iff VA \succ_{L} VB \) \hspace{1cm} [By Theorem 4.1.5]

\( \iff P^* VAP \succ_{L} P^* VBP \) \hspace{1cm} [By Theorem 4.1.8]

\( \iff VP^* VAP \succ_{L} VP^* VBP \) \hspace{1cm} [By Theorem 4.1.5]

\( \iff (VP^{-1} V)AP \succ_{L} (VP^{-1} V)BP \)

\( \iff C \succ_{L} D \) where \( C = (VP^{-1} V)AP \) and \( D = (VP^{-1} V)BP \).

Therefore Lowener ordering is preserved under s-unitary similarity.
Theorem 4.1.12

Let $A, B \in C_{n \times n}$. Star ordering is preserved under s-unitary similarity.

Proof:

It is enough to prove $A \succ_B \Leftrightarrow VP^{-1}VAP \succ VP^{-1}VB$ for some unitary matrix $P$.

\[
A \succ_B \Leftrightarrow VA \succ VB \quad \text{[By Theorem 4.1.6]}
\]

\[
\Leftrightarrow P^*VAP \succ P^*VB \quad \text{[By Theorem 4.1.9]}
\]

\[
\Leftrightarrow VP^*VAP \succ VP^*VB \quad \text{[By Theorem 4.1.6]}
\]

\[
\Leftrightarrow (VP^{-1}V)AP \succ (VP^{-1}V)BP
\]

\[
\Leftrightarrow C \succ D \quad \text{where } C = (VP^{-1}V)AP \text{ and } D = (VP^{-1}V)BP.
\]

Therefore Star ordering is preserved under s-unitary similarity.

Theorem 4.1.13

Let $A, B \in C_{n \times n}$. Minus ordering is preserved under s-unitary similarity.

Proof:

It is enough to prove $A \succ_B \Leftrightarrow VP^{-1}VAP \succ VP^{-1}VB$ for some unitary matrix $A$.

\[
A \succ_B \Leftrightarrow VA \succ VB \quad \text{[By Theorem 4.1.7]}
\]

\[
\Leftrightarrow P^*VAP \succ P^*VB \quad \text{[By Theorem 4.1.10]}
\]
\[
\Leftrightarrow VP^r VAP \geq VP^r VBP \\
\text{[By Theorem 4.1.7]}
\]

\[
\Leftrightarrow (VP^{-1}V)AP \geq (VP^{-1}V)BP \\
\]

\[
\Leftrightarrow C \geq D \text{ where } C = (VP^{-1}V)AP \text{ and } D = (VP^{-1}V)BP.
\]

Therefore minus ordering is preserved under s-unitary similarity.
4.2 PARTIAL ORDERING OF s-UNITARY MATRICES

Some characterizations of the star partial ordering and ranksubtractivity for matrices were discussed by Hartwig, R.E and Styan, G.P.H in [13]. A relationship between star and minus ordering was settled by Baksalary, J.K. in [4]. Jurgen Grob observed some remarks on partial ordering of hermitian matrices in [24].

In [22] Jorma K. Merkoski and Xiaogi Liu have developed star partial ordering on normal matrices. They found several characterizations of $A \succ^* B$ in the case of normal matrices. In this section we introduce the concept of partial ordering on s-unitary matrices. We have proved theorems relating to partial ordering of s-unitary matrices. Also theorems relating to lowener partial ordering and star partial ordering are given [28]. In section 4.1 we defined Lowener, Star and Minus partial ordering. For proving some results in the partial ordering on s-unitary matrices, we need the following theorem.

**Theorem 4.2.1**

Let $A, B \in C_{n \times n}$ . If $A \succ^* L B$ then $A - B$ is s-hermitian.

**Proof:**

$A \succ^* L B \Leftrightarrow V A \succ^* L B$ \hspace{1cm} [By Theorem 4.1.5]

$\Rightarrow V A - V B \succeq 0$

$\Rightarrow V (A - B) \succeq 0$

Hence $A - B$ is s-hermitian positive definite.

$(A - B)^* = V (A - B)V \Rightarrow A - B$ is s-hermitian.

The content of the article published by Krishnamoorthy, S and Govindarasu, A in International Journal of Algebra, Vol 6, No.5, 227-231(2012) has been discussed in the context of “On the partial ordering of s-unitary matrices”.

73
Theorem 4.2.2

Let $A$ and $B$ are $s$-unitary matrices and hermitian. Then $A \geq_L B$ iff $A^{-1} \geq_L B^{-1}$

Proof:

$$A \geq_L B \Rightarrow VA \geq_L VB$$  \hspace{1cm} [By Theorem 4.1.5]

$$\Rightarrow VA^* \geq_L VB^*$$ \hspace{1cm} [Since $A$ and $B$ are hermitian]

Postmultiplying the above equation by $V$ we get,

$$\Rightarrow VA^*V \geq_L VB^*V$$

$$\Rightarrow A^{-1} \geq_L B^{-1}$$ \hspace{1cm} [Since $A$ and $B$ are $s$-unitary matrices]

Conversely, if $A^{-1} \geq_L B^{-1}$

$$\Rightarrow VA^*V \geq_L VB^*V$$ \hspace{1cm} [By Definition 2.1.1]

Post multiplying the above equation by $V$ we get,

$$\Rightarrow VA^*VV \geq_L VB^*VV$$

$$\Rightarrow VA^* \geq_L VB^*$$ \hspace{1cm} [Since $V^2 = I$]

$$\Rightarrow VA \geq_L VB$$ \hspace{1cm} [Since $A$ and $B$ are hermitian matrices]

$$\Rightarrow A \geq_L B$$

Therefore $A \geq_L B$ iff $A^{-1} \geq_L B^{-1}$
Theorem 4.2.3

Let $A$ and $B$ are s-unitary matrices and let $A \gtrsim_{L} B$. If $A$ is involutary then $B$ is also involutary and vice versa.

Proof:

\[ A \gtrsim_{L} B \Leftrightarrow VA \gtrsim_{L} VB \]  
[By Theorem 4.1.5]

\[ \Rightarrow VA - VB \geq 0 \]

\[ \Rightarrow V(A - B) \geq 0 \]

Hence $A - B$ is s-hermitian positive definite.

By theorem 4.2.1

\[ \Rightarrow A - B \text{ is s-hermitian.} \]

\[ \Rightarrow (A - B)^* = V(A - B)V \]

\[ \Rightarrow A^* - B^* = V(A - B) \]

\[ \Rightarrow VA^{-1}V - VB^{-1}V = V(A - B)V \]

\[ \Rightarrow VA^{-1}V - VB^{-1}V = VAV - VBV \]

Pre multiplying and Postmultiplying the above equation by $V$ we get,

\[ \Rightarrow A^{-1} - B^{-1} = A - B \quad -----(4.2.4) \]

[Since $V^2 = I$]

Assume that $A$ is involutary, Premultiplying by (4.2.4) $A$ we get,

\[ I - AB^{-1} = I - AB \]

\[ \Rightarrow AB^{-1} = AB \quad -----------------(4.2.5) \]
Pre multiplying (4.2.5) by $A^{-1}$ we get $B^{-1} = B$

Premultiplying the above equation by $B$ we get, $I = B^2$

Therefore $B$ is involutary. If we assume $B$ is involutary then we can show $A$ is involutary.

**Theorem 4.2.6**

Let $A \geq B$. If $A$ is s-unitary matrix and s-hermitian then $B$ is s-hermitian.

**Proof:**

Since $A \geq B$ we have $A - B$ is s-hermitian. [By Theorem 4.2.1]

$$ (A - B)^* = V(A - B)V $$

Taking conjugate transpose on both sides we get

$$ \Rightarrow [(A - B)^*]^* = [V(A - B)V]^* $$

$$ = V^* (A - B)^* V^* $$

$$ A - B = V (A^* - B^*) V $$ [Since $V^* = V$]

$$ = V A^* V - V B^* V $$

$$ A - B = A^{-1} - V B^* V $$ [Since $A$ is s-unitary matrix]

Since $A$ is s-hermitian, therefore $A = A^{-1}$ \Rightarrow $B = V B^* V$.

Hence $B$ is s-hermitian.
Theorem 4.2.7

Let $A$ and $B$ are s-unitary matrices. Then $A \succ_B A$ iff $A^{-1} \succ_B A^{-1}$.

Proof: Assuming that $A \succ_B A$ then we have

(i) $B^* B = B^* A$ and (ii) $BB^* = AB^*$ [By Definition 4.1.1]

From (i) $VB^* BV = VB^* AV$

$\Rightarrow VB^* VBV = VB^* VVAV$ [Since $V^2 = I$]

$\Rightarrow (B^{-1})(VBV) = (B^{-1})(VAV)$ [Since $B$ is s-unitary]

Taking conjugate transpose of the above equation we get

$(VBV)^* (B^{-1})^* = (VAV)^* (B^{-1})^*$

$\Rightarrow (V^* B^* V^*)(B^{-1})^* = (V^* A^* V^*)(B^{-1})^*$

$\Rightarrow (VB^* V)(B^{-1})^* = (V^* A^* V)(B^{-1})^*$ [Since $V^* = V$]

$\Rightarrow (B^{-1})(B^{-1})^* = (A^{-1})(B^{-1})^* \quad (4.2.8) \quad [A \text{ and } B \text{ are s-unitary matrices}]$

From (ii) $VB^* BV = VAB^* V$

$\Rightarrow VBVVB^* V = VAVVVB^* V$ [Since $V^2 = I$]

$\Rightarrow (VBV)(VB^* V) = (V^* A^* V)(VB^* V)$

$\Rightarrow (VBV)(B^{-1}) = (V^* A^* V)(B^{-1}) \quad [Since \; B \; is \; s-unitary \; matrix]$

Taking conjugate transpose of the above equation we get
\[(B^{-1})^* (V B V)^* = (B^{-1})^* (V A V)^*\]

\[\Rightarrow (B^{-1})^* (V^* B^* V^*) = (B^{-1})^* (V^* A^* V^*)\]

\[\Rightarrow (B^{-1})^* (V B^* V) = (B^{-1})^* (V A^* V)\]

\[\Rightarrow (B^{-1})^* (B^{-1}) = (B^{-1})^* (A^{-1}) \quad \text{----(4.2.9)} \quad [A \text{ and } B \text{ are s-unitary matrices}]\]

From (4.2.8) and (4.2.9) we have \(A \succ B \Rightarrow A^{-1} \succ B^{-1}\).

Similarly we can prove \(A^{-1} \succ B^{-1} \Rightarrow A \succ B\)

**Theorem 4.2.10**

Let \(A \succ B\) and \(A \succeq B\). If \(A\) and \(B\) are s-unitary matrices then \(A = B\).

**Proof:** Assuming that \(A \succ B\) then we have

(i) \(B^* B = B^* A\) and (ii) \(BB^* = AB^*\) \quad [By Definition 4.1.1]

Therefore \(B^* (A - B) = 0 \quad \text{----(4.2.11)}\)

Taking conjugate transpose of equation(4.2.11) we get

\[(A - B)^* B = 0 \quad \text{----(4.2.12)}\]

Since \(A \succeq B\) we have \((A - B)^* = V (A - B)V\) \quad [By Theorem 4.2.1]

Hence from (4.2.1) \(V (A - B)V B = 0\)

\[\Rightarrow V AVB - V BVB = 0\]

\[\Rightarrow V AVB = V BVB\]

Taking conjugate transpose of the above equation we get
\( B^*V^*A^*V^* = B^*V^*B^*V^* \)

\[ \Rightarrow B^*VA^*V = B^*VB^*V \]

\[ \Rightarrow B^*A^{-1} = B^*B^{-1} \quad \text{[since } A \text{ and } B \text{ are s-unitary]} \]

\[ \Rightarrow A^{-1} = B^{-1} \]

\[ \Rightarrow A = B. \]

**Theorem 4.2.13**

Let \( A \) and \( B \) be s-unitary matrices. Then \( A \gtrsim B \) iff \( A^{-1} \gtrsim B^{-1} \).

**Proof:**

\[ A \gtrsim B \Rightarrow B = BA^{(i)}A = AA^{(i)}B = BA^{(i)}B \quad \text{[By Definition 4.1.1]} \]

\[ \Rightarrow VBV = VBA^{(i)}AV = VAA^{(i)}BV = VBA^{(i)}BV \quad \text{-----------------------(4.2.14)} \]

Taking conjugate transpose of the above equation-(4.2.14) we get

\[ \Rightarrow (VBV)^* = (VBA^{(i)}AV)^* = (VAA^{(i)}BV)^* = (VBA^{(i)}BV)^* \]

\[ \Rightarrow V^*B^*V^* = V^*A^*(A^{(i)})^*B^*V^* = V^*B^*(A^{(i)})^*A^*V^* = V^*B^*(A^{(i)})^*B^*V^* \]

\[ \Rightarrow VB^*V = VA^*(A^{(i)})^*B^*V = VB^*(A^{(i)})^*A^*V = VB^*(A^{(i)})^*B^*V \]

\[ \Rightarrow VB^*V = VA^*(A^{(i)})^*B^*V = VB^*(A^{(i)})^*A^*V = VB^*(A^{(i)})^*B^*V \]

\[ \Rightarrow VB^*V = VA^*(VAV)^{(i)}VB^*V = VB^*(VAV)^{(i)}A^*V = VB^*(VAV)^{(i)}VB^*V \]

\[ \Rightarrow B^{-1} = A^{-1}(A^{-1})^*B^{-1} = B^{-1}(A^{-1})A^{-1} = B^{-1}(A^{-1})^*B^{-1} \]
\[ A^{-1} \geq_{rs} B^{-1} \]

Hence \[ A_{rs} \geq B_{rs} \Rightarrow A^{-1}_{rs} \geq B^{-1}_{rs} \].

Similarly we can prove \[ A^{-1}_{rs} \geq B^{-1}_{rs} \Rightarrow A_{rs} \geq B_{rs} \].

Therefore \[ A_{rs} \geq B_{rs} \text{ iff } A^{-1}_{rs} \geq B^{-1}_{rs} \].
4.3 ‘θ’ PARTIAL ORDERING OF s-UNITARY MATRICES

In this section the concept of ‘θ’ partial ordering on s-unitary matrices is introduced. Relation between ‘θ’ partial ordering, star partial ordering and lowener partial ordering are obtained. Some theorems relating to the above are derived.

Definition 4.3.1 ['θ’ PARTIAL ORDERING] [29]

Let \( A, B \in C_{nn} \), \( A \overset{\theta}{\leq} B \) iff \( A^\theta A = A^\theta B \) and \( AA^\theta = BA^\theta \).

Theorem 4.3.2

Let \( A \) and \( B \) be s-hermitian matrices and nonnegative definite.

Then \( A \overset{\theta}{\leq} B \) iff \( A \overset{\theta}{\leq} B \)

Proof:

Assuming \( A \overset{\theta}{\leq} C \) and \( B \overset{\theta}{\leq} D \)

\[ C \overset{\theta}{\leq} D \Rightarrow C^\theta C = C^\theta D \]

\[ \Rightarrow (A^\theta)^2 A^2 = (A^\theta)^2 B^2 \]

\[ \Rightarrow (A^\theta)^2 A^2 = (A^\theta)^2 B^2 \Rightarrow A^\theta AA = A^\theta BB \]

\[ A^\theta AA = A^\theta AB \quad \text{[Since } A^\theta A = A^\theta B \text{]} \]

\[ \Rightarrow A^\theta AA = A^\theta BA \quad \text{[Since } BA = AB \text{]} \]

\[ \Rightarrow A^\theta A = A^\theta B . \quad \text{Hence } A \overset{\theta}{\leq} B . \]

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81
Conversely, assume $A \leq_{\theta} B$

$$\Rightarrow A^\theta A = A^\theta B \quad \text{(4.3.3)} \quad \text{[By Definition 4.3.1]}$$

Post multiplying (4.3.3) by $A$ we get

$$\Rightarrow A^\theta A^2 = A^\theta BA$$

$$= A^\theta AB \quad \text{[Since $BA = AB$]}$$

$$\Rightarrow A^\theta A^2 = A^\theta BB \quad \text{[Since $A^\theta A = A^\theta B$]}$$

Pre multiplying the above equation by $A^\theta$ we get

$$(A^\theta)^2 A^2 = (A^\theta)^2 B^2$$

$$\Rightarrow (A^2)^\theta A^2 = (A^2)^\theta B^2$$

Hence $A^2 \leq_{\theta} B^2$.

**Theorem 4.3.4**

Let $A V \leq_{\theta} V A$. If $A$ is s-unitary then $A$ is unitary.

**Proof:**

Let $A V \leq_{\theta} V A$

$$\Rightarrow (A V)^\theta A V = (A V)^\theta V A \quad \text{[By Definition 4.3.1]}$$

$$\Rightarrow V A^\theta A V = V A^\theta V A$$

$$\Rightarrow V I V = V A^\theta V A \quad \text{[Since $A$ is s-unitary]}$$

$$\Rightarrow I = V A^\theta V A$$
\[ \Rightarrow A^{-1} = VA^\theta V \]

\[ = VVA^\theta V \]

Therefore \( A^{-1} = A^* \)

Hence \( AA^* = I \) \( \text{--------------------------}(4.3.5) \)

\[ AV \lesssim_{\theta} VA \Rightarrow AV(AV)^\theta = VA(AV)^\theta \quad [\text{By definition 4.3.1}] \]

\[ \Rightarrow AVVA^\theta = VAVA^\theta \]

\[ \Rightarrow AA^\theta = VAVA^\theta \]

\[ I = VAVVA^\theta V \quad [\text{Since } A \text{ is s-unitary}] \]

\[ \Rightarrow I = VAA^\theta V \]

\[ \Rightarrow VIV = A^* A \]

Therefore \( A^* A = I \) \( \text{--------------------------}(4.3.6) \)

From (4.3.5) and (4.3.6) we have \( A^* A = AA^* = I \)

Therefore \( A \) is unitary.

**Theorem 4.3.7**

If \( A \lesssim_{L} B \quad \text{and} \quad A \lesssim_{\theta} B \) then \( A = B \).

**Proof:**

Given \( A \lesssim_{\theta} B \)

\[ A \lesssim_{\theta} B \Rightarrow A^\theta A = A^\theta B \quad \text{and} \quad AA^\theta = BA^\theta \quad [\text{By definition 4.3.1}] \]
\[ \Rightarrow A^\theta (B - A) = 0 \]

Taking conjugate secondary transpose on both sides we get

\[ (B - A)^\theta A = 0 \] \hspace{1cm} (4.3.8)

Since \( A \lesssim B \) we have \( (B - A)^* B = V (B - A)V \)

\[ (4.3.8) \Rightarrow V (B - A)^* VA = 0 \]

\[ \Rightarrow VV (B - A)VVA = 0 \]

\[ \Rightarrow I (B - A)IA = 0 \] \hspace{1cm} [Since \( V^2 = I \)]

\[ \Rightarrow BA - A^2 = 0 \]

\[ \Rightarrow (B - A) A = 0 \]

\[ \Rightarrow BA = A^2 \]

\[ \Rightarrow B = A \]

Hence the theorem.

**Theorem 4.3.9**

If \( A \) and \( B \) are s-unitary matrices then \( A \lesssim \theta B \) \Rightarrow \( A^{-1} \lesssim \theta B^{-1} \)

**Proof:** Given \( A \lesssim \theta B \).

\[ A \lesssim \theta B \Rightarrow A^\theta A = A^\theta B \quad \text{and} \quad AA^\theta = BA^\theta \] \hspace{1cm} [By definition 4.3.1]

By definition \( A^\theta A = A^\theta B \)

\[ \Rightarrow A^{-1} A = A^{-1} B \] \hspace{1cm} [Since \( A \) is s-unitary]
Taking conjugate secondary transpose on both sides we get

\[ \Rightarrow (A^{-1}A)^\theta = (A^{-1}B)^\theta \]

\[ \Rightarrow A^\theta (A^{-1})^\theta = B^\theta (A^{-1})^\theta \]

Therefore \( A^{-1}(A^{-1})^\theta = B^{-1}(A^{-1})^\theta \) \( \text{(4.3.10)} \)

\[ AA^\theta = BA^\theta \]

\[ \Rightarrow AA^{-1} = BA^{-1} \quad \text{[Since } A \text{ is s-unitary}] \]

Taking conjugate secondary transpose on both sides we get

\[ \Rightarrow (AA^{-1})^\theta = (BA^{-1})^\theta \]

\[ \Rightarrow (A^{-1})^\theta A^\theta = (A^{-1})^\theta B^\theta \]

Therefore \( (A^{-1})^\theta A^{-1} = (A^{-1})^\theta B^{-1} \) \( \text{(4.3.11)} \)

From \( \textbf{(4.3.10)} \) and \( \textbf{(4.3.11)} \) we get \( A^{-1} \leq B^{-1} \)

Therefore \( A \leq B \Rightarrow A^{-1} \leq B^{-1} \)

\textbf{Theorem 4.3.12}

If \( A \) and \( B \) are s-unitary matrices then \( A \leq B \Rightarrow VA \leq VB \)

\textbf{Proof:}

Given \( A \leq B \)

\( A \leq B \Rightarrow A^\theta A = A^\theta B \quad \text{and } AA^\theta = BA^\theta \) \[ \text{[By Definition 4.3.1]} \]
By definition $A^\theta A = A^\theta B$

\[ \Rightarrow VA^*VA = VA^*VB \quad [\text{Since } A \text{ is s-unitary}] \]

\[ \Rightarrow (AV)^*VA = (AV)^*VB \]

\[ \Rightarrow (VA)^*VA = (VA)^*VB \quad \text{----------(4.3.13)} \quad [\text{Since } (VA)^* = (AV)^*] \]

By definition $AA^\theta = BA^\theta$

\[ \Rightarrow AVA^*V = BVA^*V \quad [\text{Since } A \text{ is s-unitary}] \]

Pre multiplying and Post multiplying the above equation by $V$ we get,

\[ \Rightarrow VAVA^*VV = VBVA^*VV \]

\[ \Rightarrow VAVA^* = VBVA^* \quad [\text{Since } V^2 = I] \]

\[ \Rightarrow VA(AV)^* = VB(AV)^* \]

\[ \Rightarrow VA(VA)^* = VB(VA)^* \quad \text{----------(4.3.14)} \quad [\text{Since } (VA)^* = (AV)^*] \]

From (4.3.13) and (4.3.14) we have $VA^* \leq VB$.

Therefore $A^\theta B \Rightarrow VA^* \leq VB$.

**Theorem 4.3.15**

If $A$ and $B$ are s-unitary matrices then $A^\theta B \Rightarrow VA^* \leq VB$.

**Proof:**

$A^\theta B \Rightarrow A^*A = A^*B$ and $AA^* = BA^*$

Take $A^*A = A^*B$
\( \Rightarrow (VA^*V)VA = (VA^*V)VB \)

\( \Rightarrow A^0VA = A^0VB \) \hspace{1cm} [Since \( A \) is s-unitary]

\( \Rightarrow VA^0VA = VA^0VB \)

\( \Rightarrow (AV)^0VA = (AV)^0VB \)

\( \Rightarrow (VA)^0VA = (VA)^0VB \) \hspace{1cm} \text{(4.3.16)}

Take \( AA^* = BA^* \)

\( \Rightarrow AVVA^* = BVVA^* \)

\( \Rightarrow AV(VA^*V) = BV(VA^*V) \)

\( \Rightarrow AV(A^0) = BV(A^0) \) \hspace{1cm} [Since \( A \) is s-unitary]

\( \Rightarrow AV(A^0V) = BV(A^0V) \)

\( \Rightarrow AV(VA)^0 = BV(VA)^0 \)

\( \Rightarrow VA(VA)^0 = VB(VA)^0 \) \hspace{1cm} \text{(4.3.17)}

From \((4.3.16)\) and \((4.3.17)\) we get \( VA \leq_{\theta} VB \).

Therefore \( A \leq_{\ast} B \Rightarrow VA \leq_{\theta} VB \).