CHAPTER -2

S-UNITARY MATRICES

The concept of s-unitary matrices introduced for complex square matrices. Some basic characterizations are derived in this chapter. Necessary and sufficient conditions for sum of two s-unitary matrices to be s-unitary is determined. The product of two s-unitary matrices to be s-unitary is also derived. Some equivalent conditions on s-unitary matrices are also derived [30].
2.1. CHARACTERIZATIONS OF S-UNITARY MATRICES.

The s-unitary matrix is defined and its characterizations are discussed. Characterizations of s-unitary matrices analogous to that of unitary matrices are obtained. The concept of skew s-unitary matrix is introduced and established some theorems.

Definition 2.1.1

A matrix $A \in C_{n\times n}$ is said to be s-unitary if $AA^S = A^S A = I$ [26]

i.e $AA^\theta = A^\theta A = I$ where $A^\theta = \overline{A}^S$

$AVA^*V = VA^*VA = I$

$VA^*V = A^{-1}$ or $\overline{A}^S = A^{-1}$

Example 2.1.2

$$\begin{pmatrix}
  -i & 0 \\
  \sqrt{2} & \sqrt{2} \\
  1 & i \\
  \sqrt{2} & \sqrt{2}
\end{pmatrix}$$

It can be shown that $AA^\theta = A^\theta A = I$ and hence $A$ is s-unitary.

Example 2.1.3

$$\begin{pmatrix}
  -i & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & -i
\end{pmatrix}$$

is a s-unitary matrix.

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The content of the article published by Krishnamoorthy, S and Govindarasu, A in International Journal of Computational Science and Mathematics Volume 2, Number 3, 247-253 (2010) has been discussed in the context of "On secondary unitary matrices".

Theorem 2.1.4
Let $A \in C_{n \times n}$. If $A$ is s-unitary matrix, then $\overline{A}$ is also s-unitary matrix.

**Proof:** $A$ is s-unitary matrix $\Rightarrow VA^*V = A^{-1}$  
[By Definition 2.1.1]

$$V(\overline{A})^*V = V(\overline{A})^T V$$

$$= VA^*V$$

$$= VA^*V$$  
[Since $V = V$]

$$= A^{-1}$$  
[By Definition 2.1.1]

$$=(\overline{A})^{-1}$$

Hence $\overline{A}$ is s-unitary.

**Theorem 2.1.5**

Let $A \in C_{n \times n}$. If $A$ is s-unitary matrix, then $A^T$ is s-unitary matrix.

**Proof:**

$A$ is s-unitary matrix $\Rightarrow VA^*V = A^{-1}$  
[By Definition 2.1.1]

$$(A^{-1})^T = (VA^*)^T$$

$$= V^T (A^*)^T V^T$$

$$= V(A^*)^T V$$  
[Since $V^T = V$]

$$(A^T)^{-1} = V(A^*)^T V$$  
Hence $A^T$ is s-unitary matrix.

**Theorem 2.1.6**
Let \( A \in C_{n \times n} \). If \( A \) is s-unitary matrix, then \( A^* \) is s-unitary matrix.

**Proof:**

\( A \) is s-unitary matrix \( \Rightarrow VA^*V = A^{-1} \) [By Definition 2.1.1]

\[(A^{-1})^* = (VA^*V)^*\]

\[= V^* (A^*)^* V\]

\[(A^{-1})^* = V (A^*)^* V \quad \text{[Since } V^* = V \text{]}\]

\[(A^*)^{-1} = V (A^*)^* V \]

Hence \( A^* \) is s-unitary matrix.

**Theorem 2.1.7**

Let \( A \in C_{n \times n} \). If \( A \) is s-unitary matrix, then \( A^{-1} \) is s-unitary matrix.

**Proof:**

\( A \) is s-unitary matrix \( \Rightarrow A^{-1} = \overline{A}^S \) [By Definition 2.1.1]

\[(A^{-1})^{-1} = (\overline{A}^S)^{-1}\]

\[= ((A^{-1})^S)\]

\[= (A^{-1})^S\]

Therefore \( (A^{-1})^{-1} = (A^{-1})^S \)

Hence \( A^{-1} \) is s-unitary matrix.

**Theorem 2.1.8**
Let $A \in C_{nxn}$. If $A$ is s-unitary matrix, then $iA$ is s-unitary matrix.

**Proof:**

$$A \text{ is s-unitary matrix } \Rightarrow A^{-1} = \overline{A}^S$$  \hspace{1cm} \text{[By Definition 2.1.1]}

$$iA^{-1} = i\overline{A}^S$$

$$-(iA)^{-1} = -i\overline{A}^S$$

$$(iA)^{-1} = i\overline{A}^S$$

Hence $iA$ is s-unitary matrix.

**Definition 2.1.9**

A matrix $A \in C_{nxn}$ is said to be skew secondary unitary matrix if $A^{-1} = -\overline{A}^S$.

**Example 2.1.10**

$$A = \begin{bmatrix}
\frac{1}{\sqrt{2}} & i \\
\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{bmatrix}$$

is a skew s-unitary matrix

**Theorem 2.1.11**

Let $A \in C_{nxn}$. If $A$ is skew s-unitary matrix, then $iA$ is skew s-unitary matrix.

**Proof:**
\[ A \text{ is skew s-unitary matrix} \Rightarrow A^{-1} = -\overline{A}^s \] 

[By Definition 2.1.9]

\[ iA^{-1} = -i\overline{A}^s \]

\[ -(iA)^{-1} = iA^s \]

\[ (iA)^{-1} = -i\overline{A}^s \]

Hence \( iA \) is skew s-unitary matrix.
2.2 SUMS AND PRODUCTS OF s-UNITARY MATRICES

In this section, necessary and sufficient conditions for sum and difference of two s-unitary matrices $A$ and $B$ to be s-unitary is proved. It is shown that product of two s-unitary matrices is a s-unitary matrix. Some theorems are proved relating to sums and products of s-unitary matrices.

**Theorem 2.2.1**

Let $A, B \in C_{n \times n}$ and $A, B$ are s-unitary matrices and $A \overline{B}^S = B^S A$, $B \overline{A}^S = A^S B$

(i) If $A \overline{B}^S + B \overline{A}^S = -I$ then $A + B$ is s-unitary

(ii) If $A \overline{B}^S + B \overline{A}^S = I$ then $A - B$ is s-unitary

**Proof:**

Given $A$ and $B$ are s-unitary matrices.

Then $A^{-1} = \overline{A}^S$, $B^{-1} = \overline{B}^S$ [By Definition 2.1.1]

(i) We have to show $(A + B)(A + B)^S = I$

\[
(A + B)(A + B)^S = (A + B)(\overline{A}^S + \overline{B}^S)
\]

\[
= A \overline{A}^S + (A \overline{B}^S + B \overline{A}^S) + B \overline{B}^S
\]

\[
= I + (-I) + I = I
\]

Similarly we can prove $(A + B)^S (A + B) = I$

Therefore $(A + B)(A + B)^S = (A + B)^S (A + B) = I$

Hence $(A + B)$ is s-unitary.

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(ii) We have to show \((A - B)(A - B)^s = I\)

\[
(A - B)(A - B)^s = (A - B)(\overline{A}^s - \overline{B}^s)
\]

\[
= A\overline{A}^s - A\overline{B}^s - B\overline{A}^s + B\overline{B}^s
\]

\[
= A\overline{A}^s - (A\overline{B}^s + B\overline{A}^s) + B\overline{B}^s
\]

\[
= I - (I) + I
\]

\[
= I
\]

Similarly we can prove \((A - B)^s (A - B) = I\)

Therefore \((A - B)(A - B)^s = (A - B)^s (A - B) = I\)

Hence \(A - B\) is s-unitary.

**Theorem 2.2.2**

Let \(A \in C_{n \times n}\). If \(A\) is s-unitary and \(V\) is a permutations matrix with units in the secondary diagonal and all other elements are zero and \(VA = AV\) then \(VA\) is unitary.

**Proof:**

\(A\) is s-unitary matrix \(\Rightarrow VA^*V = A^{-1}\) \hspace{1cm} [By Definition 2.1.1]

\[V(A^*V^*) = A^{-1}\] \hspace{1cm} [Since \(V^* = V\)]

\[V(VA)^* = A^{-1}\]

Premultiplying both sides by \(A\) we get

\[AV(VA)^* = AA^{-1}\]
\[(VA)(VA)^* = I\]  \[\text{[Since} VA = AV]  \]

Also, \[VAV^* = A^{-1}\]  \[\text{[Since} V = V]  \]

\[(V^*A^*)V = A^{-1}\]  \[\text{[Since} V^* = V]  \]

Postmultiplying both sides by \(A\) we get

\[(AV)^*(VA) = A^{-1}A\]  \[\text{[Since} (AV)^* = V^*A^*]  \]

\[(VA)^*(VA) = I\]  \[\text{[Since} VA = AV]  \]

Therefore \((VA)(VA)^* = (VA)^*(VA) = I\)

Hence \(VA\) is unitary.

**Theorem 2.2.3**

Let \(A \in \mathbb{C}_{nxn}\). If \(A\) is s-unitary and \(V\) is a permutation matrix with units in the secondary diagonal and all other elements are zero and \(VA = AV\) then \(AV\) is unitary.

**Proof:**

\(A\) is s-unitary matrix \(\Rightarrow VA^*V = A^{-1}\) \[\text{[By Definition 2.1.1]}\]

\[(V^*A^*)V = A^{-1}\]  \[\text{[Since} V^* = V]  \]

\[(AV)^*V = A^{-1}\]  \[\text{[Since} (AV)^* = V^*A^*]  \]

Post multiplying both sides by \(A\), we get

\[(AV)^*VA = A^{-1}A\]

\[(AV)^*(AV) = I\]  \[\text{[Since} VA = AV]  \]
Also, \( VA^*V = A^{-1} \)

\[ V(A^*V^*) = A^{-1} \]  \[ \text{[Since } V^* = V \text{] } \]

\[ V(VA)^* = A^{-1} \]  \[ \text{[Since } (VA)^* = A'V^* \text{] } \]

Pre multiplying both sides by \( A \), we get

\[ (AV)(AV)^* = AA^{-1} = I \]  \[ \text{[Since } AV = VA \text{] } \]

Therefore \( (AV)(AV)^* = (AV)^*(AV) = I \)

Hence \( AV \) is unitary.

**Theorem 2.2.4**

Let \( A, B \in C_{n \times n} \). If \( A \) and \( B \) are s-unitary matrices then \( AB \) is a s-unitary matrix.

**Proof:**

\( A \) is s-unitary matrix \( \Rightarrow VA^*V = A^{-1} \)  \[ \text{[By Definition 2.1.1] } \]

\( B \) is s-unitary matrix \( \Rightarrow VB^*V = B^{-1} \)  \[ \text{[By Definition 2.1.1] } \]

\[ V(AB)^*V = V(B^*A^*)V \]

\[ = (VB^*V)(VA^*V) \]  \[ \text{[Since } V^2 = I \text{] } \]

\[ = B^{-1}A^{-1} \]

\[ = (AB)^{-1} \]

Therefore \( V(AB)^*V = (AB)^{-1} \)

Hence \( AB \) is s-unitary matrix.
Theorem 2.2.5

Let $A \in C_{n \times n}$ and $V$ is a permutations matrix with units in the secondary diagonal and all other elements are zero. If $A$ is s-unitary then so are $VA$ and $AV$.

Proof:

$A$ is s-unitary matrix $\Rightarrow VA^*V = A^{-1}$  

[By Definition 2.1.1]

$(VA)^{-1} = A^{-1}V$

$= VA^*VV$  

[Since $A^{-1} = VA^*V$]

$(VA)^{-1} = V(VA)^*V$  

[Since $(VA)^* = A^*V^*$]

Hence $VA$ is s-unitary.

Also  

$(AV)^{-1} = VA^{-1}$

$= V(VA^*)V$

$= V(V^*A^*)V$  

[Since $VA^* = V^*A^*$]

Therefore $(AV)^{-1} = V(AV)^*V$

Hence $AV$ is s-unitary.

Theorem 2.2.6

Let $A \in C_{n \times n}$ If $A$ is s-unitary then $A^n$ is s-unitary for all integer $n$.

Proof:

Case (i) $n = 0$

$A^0 = I$ is s-unitary.
**Case (ii)** \( n \) is a positive integer.

\[
(A^n)^{-1} = (A\ldots A)^{-1}
\]

\[
= A^{-1}A^{-1}A^{-1} \ldots A^{-1} \quad (n \text{ times})
\]

\[
= (VA^*V)(VA^*V)\ldots (VA^*V) \quad (n \text{ times})
\]

\[
\]

\[
= V(A^*A^*A^*\ldots A^*)V \quad \text{[Since } V^2 = I \text{ ]}
\]

\[
= V(A^*)^nV
\]

\[
(A^n)^{-1} = V(A^n)^*V.
\]

Hence \( A^n \) is s-unitary.

**Case (iii):** \( n \) is a negative integer.

First let us prove the theorem for \( n = -1 \).

\[
A^{-1} = VA^*V \quad \text{[By Definition 2.1.1]}
\]

Taking inverse on both sides

\[
(A^{-1})^{-1} = (VA^*V)^{-1}
\]

\[
= V(A^*)^{-1}V
\]

\[
= V(A^{-1})^*V
\]

Hence \( A^{-1} \) is s-unitary.

Let \( n \) is any negative integer.
Let \( m = -n \). Since \( \mathbf{A}^{-1} \) is s-unitary, \((\mathbf{A}^{-1})^m\) is s-unitary (m>0) by what we have proved in case (i)

\[ \mathbf{A}^n = \mathbf{A}^{-m} \]

\[ = (\mathbf{A}^{-1})^m \]

Hence \( \mathbf{A}^n \) is s-unitary.

**Theorem 2.2.7**

If \( M^n \) is the collection of all \( n \times n \) s-unitary matrices then it is a group under multiplication.

**Proof:**

Since then \( I_n \in M^n \), it is not empty.

If \( A, B \in M^n \) then \( AB \in M^n \).

Matrix multiplication is associative. The identity matrix \( I_n \) is s-unitary.

If \( A \in M^n \) then \( A^{-1} \in M^n \).

Hence \( M^n \) is a group under multiplication.
2.3 SOME EQUIVALENT CONDITIONS ON s-UNITARY MATRICES

In this section, equivalent conditions on s-unitary matrices are derived [30].

**Theorem 2.3.1**

Let $A \in C_{n \times n}$. If $A$ is s-unitary matrix, then $AA^*$ and $A^*A$ are s-unitary matrices.

Proof:

$$(AA^*)^{-1} = (A^*)^{-1}A^{-1}$$

$$= (A^{-1})^*A^{-1}$$

$$= (VA^*V)(VA^*V)$$

[By Definition 2.1.1]

$$= V(A^*)^*(VV)A^*V$$

[Since $VV^* = V$]

$$= VA(VV)A^*V$$

$$= (VA^*V)$$

$$= (AA^*)^*V$$

[Since $V^* = V$]

Hence $AA^*$ is s-unitary.

$$(A^*A)^{-1} = A^{-1}(A^*)^{-1}$$

$$= A^{-1}(A^{-1})^*$$

$$= (VA^*V)(VA^*V)^*$$

[Since $A^{-1} = VA^*V$]

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\[(VA^*V)(AV) = (VA^*V)(VAV) = VA^*(VV)AV = VA^*AV = V(A^*A)^*V\]

Hence \(A^*A\) is s-unitary.

**Theorem 2.3.2**

Let \(A \in C_{n \times n}\). Any two of the following imply the other:

(i) \(A\) is s-unitary  (ii) \(A\) is hermitian  (iii) \(VA\) or \(AV\) is involutary.

**Proof:**

(i) and (ii) \(\Rightarrow\) (iii)

\[VA^*V = A^{-1}\]  
[By Definition 2.1.1]

\[VAV = A^{-1} \quad (2.3.3)\]  
[Since \(A\) is hermitian]

Post multiplying (2.3.3) by \(A\) we get

\[VAVA = A^{-1}A\]

\[(VA)^2 = I\]

Therefore \(VA\) is involutary.

Premultiplying (2.3.3) by \(A\) we get

\[AVA^*V = AA^{-1}\]
\[(AV)^2 = I\]

Therefore \(AV\) is involutary.

(ii) and (iii) \(\Rightarrow\) (i)

Given \(VA\) is involutary.

\[(VA)^2 = I\]

\[VAVA = I\]

\[VAV = A^{-1}\]

\[VA^*V = A^{-1}\]  
\[\text{[Since } A \text{ is hermitian]}\]

Hence \(A\) is s-unitary.

Given \(AV\) is involutary

\[(AV)^2 = I\]

\[AVAV = I\]

\[VAV = A^{-1}\]

\[VA^*V = A^{-1}\]  
\[\text{[Since } A \text{ is hermitian]}\]

Hence \(A\) is s-unitary.

(i) and (iii) \(\Rightarrow\) (ii)

Given \(VA\) is involutary.

\[(VA)^2 = I\]

\[VAVA = I\]

\[VAV = A^{-1}\]
\[ VAV = VA^*V \]

\[ A = A^* \]

Hence \( A \) is hermitian.

**Theorem 2.3.4**

Let \( A \in C_{nn} \). Any two of the following imply the other:

(i) \( A \) is s-unitary
(ii) \( A \) is s-hermitian
(iii) \( A \) is involutary.

**Proof**:

(i) and (ii) \( \Rightarrow \) (iii)

\[ VA^*V = A^{-1} \]

[By Definition 2.1.1]

\[ A = A^{-1} \]

[Since \( A \) is s-hermitian]

Therefore \( A^2 = I \)

Hence \( A \) is involutary.

(ii) and (iii) \( \Rightarrow \) (i)

\[ A^2 = I \Rightarrow AA = I \]

\[ AVA^*V = I \]

[Since \( A \) is s-hermitian]

\[ VA^*V = A^{-1} \]

Hence \( A \) is s-unitary.

(iii) and (i) \( \Rightarrow \) (ii)

Given \( A \) is involutary.

\[ A^2 = I \Rightarrow AA = I \]
\[ A = A^{-1} \]

\[ = VA^*V \]  \hspace{1cm} \text{[By Definition 2.1.1]} \]

Therefore \( A = VA^*V \)

Hence \( A \) is s-hermitian.

**Remark 2.3.5**

For any matrix \( A \) commutes with \( V \) iff \( A^* \) commutes with \( V \)

\[ VA = AV \iff (VA)^* = (AV)^* \]

\[ \iff A^*V = VA^* \]

**Theorem 2.3.6**

Let \( A \in C_{n \times n} \). Any two of the following imply the other:

(i) \( A \) is unitary  \hspace{0.5cm} (ii) \( A \) is s-unitary  \hspace{0.5cm} (iii) \( AV = VA \)

**Proof :**

(i) and (ii) \( \Rightarrow \) (iii)

\( A \) is s-unitary  \( \Rightarrow \)

\[ VA^*V = A^{-1} \]

\[ VA^{-1}V = A^{-1} \]  \hspace{1cm} \text{[Since \( A \) is unitary ]}

Taking inverse on both sides

\[ (VA^{-1}V)^{-1} = (A^{-1})^{-1} \]

\[ VA^*V = A \]

\[ \Rightarrow AV = VA \]
(ii) and (iii) ⇒ (i)

\[ A \text{ is s-unitary} \Rightarrow VA^*V = A^{-1} \]

\[ VVA^* = A^{-1} \quad \text{[By Remark 2.3.5]} \]

\[ A^* = A^{-1} \]

Hence \( A \) is unitary

(iii) and (i) ⇒ (ii)

\[ A^* = A^{-1} \quad \text{[Since \( A \) is unitary]} \]

\[ VVA^* = A^{-1} \]

\[ VA^*V = A^{-1} \quad \text{[By Remark 2.3.5]} \]

Hence \( A \) is s-unitary

Remark 2.3.7

If a matrix \( A \) is nonsingular then by Cayley-Hamilton theorem we can find a polynomial \( P(t) \) such that \( A^{-1} = P(A) \).

Theorem 2.3.8

Let \( A \in \mathbb{C}^{n \times n} \). Any two of the following imply the other:

(i) \( A \) is s-unitary (ii) \( A^{-1} = P(A) \) where \( P(A) \) is a polynomial \( A \)

(iii) \( A^* = P(VAV) \)
Proof:

(i) and (ii) ⇒ (iii)

Since \( A \) is s-unitary and nonsingular, by remark (2.3.7), it is possible to find a polynomial \( P(t) \) such that \( A^{-1} = P(A) \).

Let \( P(A) = \alpha_0 A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \ldots + \alpha_n I, \alpha_0 \neq 0, \)

But \( VA^*V = A^{-1} = P(A) \) [By Definition 2.1.1]

\[
A^* = VA^{-1}V = VP(A)V
\]

\[
A^* = V(\alpha_0 A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \ldots + \alpha_n I)V
\]

\[
= \alpha_0 VA^nV + \alpha_1 VA^{n-1}V + \alpha_2 VA^{n-2}V + \ldots + \alpha_n VV
\]

\[
= \alpha_0 (VAV)^n + \alpha_1 (VAV)^{n-1} + \alpha_2 (VAV)^{n-2} + \ldots + \alpha_n (I)
\]

\( A^* = P(VAV) \)

(ii) and (iii) ⇒ (i)

\( A^* = P(VAV) \)

Premultiplying and postmultiplying by \( V \), we get

\[
VA^*V = V(\alpha_0 (VAV)^n + \alpha_1 (VAV)^{n-1} + \alpha_2 (VAV)^{n-2} + \ldots + \alpha_n (I))V
\]

\[
= V(\alpha_0 VA^nV + \alpha_1 VA^{n-1}V + \alpha_2 VA^{n-2}V + \ldots + \alpha_n I)
\]

\[
= \alpha_0 A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \ldots + \alpha_n I = P(A) = A^{-1}
\]

Therefore \( VA^*V = A^{-1} \). Hence \( A \) is s-unitary.
(iii) and (i) \Rightarrow (ii)

Since $A$ is s-unitary and nonsingular, by remark (2.3.7) it is possible to find a polynomial $q(t)$ such that $A^{-1} = q(A)$.

Let $q(A) = \beta_0 A^m + \beta_1 A^{m-1} + \beta_2 A^{m-2} + \ldots + \beta_m I$, $\beta_0 \neq 0$

$A$ is s-unitary $\Rightarrow VA^*V = A^{-1}$

Put $A^* = P(VAV)$

$$= VP(VAV)V$$

$$= A^{-1}$$

$$= q(A)$$

$$V[\alpha_0 (VAV)^n + \alpha_1 (VAV)^n-1 + \alpha_2 (VAV)^n-2 + \ldots + \alpha_n (I)]V$$

$$= \beta_0 A^m + \beta_1 A^{m-1} + \beta_2 A^{m-2} + \ldots + \beta_m I$$

$$V[\alpha_0 VA^nV + \alpha_1 VA^{n-1}V + \alpha_2 VA^{n-2}V + \ldots + \alpha_n I]V$$

$$= \beta_0 A^m + \beta_1 A^{m-1} + \beta_2 A^{m-2} + \ldots + \beta_m I$$

Therefore $\alpha_0 A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \ldots + \alpha_n I$

$$= \beta_0 A^m + \beta_1 A^{m-1} + \beta_2 A^{m-2} + \ldots + \beta_m I$$

Since two polynomials in $A$ are equal we must have $m = n$ and $\alpha_i = \beta_i$ for all $i$.

Therefore $P(A) = q(A)$.

Hence $A^{-1} = q(A)$. 

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Theorem 2.3.9

Let $A \in C_{n \times n}$. Let $A$ be s-unitary matrix then $A$ is normal iff $(AV)^2 = (VA)^2$

Proof:

Assume that $A$ is normal.

$$AA^* = A^*A \quad \text{[By Definition 1.2.7]}$$

$$AVA^{-1}V = VA^{-1}VA \quad \text{------------------(2.3.10)}$$

Premultiplying and postmultiplying (2.3.10) by $A^{-1}$

$$VA^{-1}V = A^{-1}VA^{-1}V$$

Taking inverse on both sides

$$(VA^{-1}V)^{-1} = (A^{-1}VA^{-1}V)^{-1}$$

$$AVA = VAVA$$

$$(AV)^2 = (VA)^2$$

Conversely $(AV)^2 = (VA)^2$

$$AVA = VAVA \quad \text{--------------------------(2.3.11)}$$

Premultiplying (2.3.11) by $(AV)^{-1}$ and postmultiplying (2.3.11) by $(VA)^{-1}$ we get,

$$(AV)(VA)^{-1} = (AV)^{-1}(VA)$$

$$AVA^{-1}V = VA^{-1}VA \quad \text{[Since $(AV)^{-1} = VA^{-1}$]}$$

$$AA^* = A^*A$$

Therefore $A$ is normal.
Theorem 2.3.12

Let $A \in C_{nn}$. Given $U$ is s-unitary then $A$ is s-unitary iff $U^\theta AU$ is s-unitary.

**Proof:**

Given $A$ is s-unitary $\Rightarrow AA^\theta = A^\theta A = I$

$U$ is s-unitary $\Rightarrow UU^\theta = U^\theta U = I$

$(U^\theta AU)^\theta(U^\theta AU) = U^\theta A^\theta UU^\theta AU$

$= U^\theta A^\theta AU$  \hspace{1cm} [Since $U^\theta U = I$]

$= U^\theta IU$  \hspace{1cm} [Since $A^\theta A = I$]

$= U^\theta U = I$

$(U^\theta AU)(U^\theta AU)^\theta = U^\theta AUAU^\theta A^\theta U$

$= U^\theta AIA^\theta U$  \hspace{1cm} [Since $U^\theta U = I$]

$= U^\theta AA^\theta U$

$= U^\theta IU = U^\theta U = I$  \hspace{1cm} [Since $A^\theta A = I$]

Therefore $U^\theta AU$ is s-unitary.

Conversely, Assume that $U^\theta AU$ is s-unitary where $U$ is s-unitary.

i.e $(U^\theta AU)(U^\theta AU)^\theta = (U^\theta AU)^\theta(U^\theta AU) = I$

$\Rightarrow U^\theta AUU^\theta A^\theta U = U^\theta A^\theta AUU^\theta AU = I$

$\Rightarrow U^\theta AA^\theta U = U^\theta A^\theta AU = I$  \hspace{1cm} [Since $U^\theta U = I$]

Premultiplying by $U$ and postmultiplying by $U^\theta$ we get,
\[ \Rightarrow UU^\theta AA^\theta UU^\theta = UU^\theta A^\theta A UU^\theta = UIU^\theta \]

\[ \Rightarrow AA^\theta = A^\theta A = I \]

Hence \( A \) is s-unitary.

**Theorem 2.3.13:** Let \( A \in C_{n \times n} \) be s-unitary. Let \( B \) is similar to \( A \) such that \( B = C^{-1} AC \).

If a matrix \( C \) is s-unitary then \( B \) is s-unitary.

**Proof:**

\[ VB^*V = V(C^{-1} AC)^*V = VC^*A^*(C^{-1})^*V \]

\[ = VC^*A^*(VC^*V)^*V \]

\[ = VC^*A^*VCVV \]

\[ = VC^*A^*VC \]

[Since \( V^2 = I \)]

\[ = V(C^{-1} V)(VA^{-1} V)VC \]

\[ = C^{-1} A^{-1} C \]

\[ = (C^{-1} AC)^{-1} = B^{-1} \]

[By Definition 1.2.21]

Hence \( B \) is s-unitary.
CHAPTER-3

S-EIGEN VALUES AND S-UNITARY SIMILARITY OF MATRICES

In this chapter, eigen value and s-eigen value of a matrix are found. Relation between s-eigen value of a matrix $A$ and eigen value of a matrix $VA$ are analysed. A s-eigen value of a matrix is defined as a special case of generalized eigen value problem $Ax = \lambda Bx$ for some matrices $A$ and $B$ [42].

In this chapter, theorems related to s-unitary matrices are derived. The concept of s-unitaly similar matrices are introduced and established some theorems on this. The concept of families of matrices to be s-unitarily similar are also introduced and derived some theorems.