CHAPTER-1
INTRODUCTION

An outline of the thesis is as follows: This thesis, consisting of five chapters, is primarily confined to a study on s-unitary matrices. In chapter I, review of literature, notations, preliminaries and summary of results obtained in the thesis are given.

In chapter II, s-unitary matrix is defined and characterizations of s-unitary matrices analogous to that of unitary matrices are obtained. The conditions for sums and products of s-unitary matrices to be s-unitary are discussed. Some equivalent conditions on s-unitary matrices are derived.

Chapter III contains the definition of s-eigen value of a matrix. s-eigen values of s-unitary matrices are discussed. Further, the secondary unitary similarity of matrices are introduced.

In chapter IV, the s-invariant partial ordering on matrices are discussed. The partial ordering of s-unitary matrices are introduced. The concept of ‘θ’ partial ordering of s-unitary matrices are introduced in this chapter.

In chapter V, the concept of s-g inverse of a given matrix are discussed and its characterizations are obtained. Some theorems on secondary generalized inverse of s-unitary matrices are introduced. Some theorems on the constructions of (1,2,3)-inverse, (1,2,4)-inverse, (1,2,3,4) inverses of s-unitary matrices are obtained.
1.1 REVIEW OF LITERATURE

The beginning of matrices and determinants dates back to the second century BC although traces can be seen back to the fourth century BC. However, it was not until the end of the 17th century that the ideas reappeared and developed real got underway. The first to use the term was Sylvester in 1850. Sylvester defined “a matrix to be an oblong arrangement of terms and saw it as something which lead to various determinants from square arrays contained within it”. After leaving America and returning to England in 1851, Sylvester became a lawyer to met Cayley, a fellow lawyer who shared his interest in Mathematics. Cayley quickly saw the significance of the matrix concept and in 1853, Cayley had published a not giving, for the first time, the inverse of matrix. In 1858 Cayley published “Memoir on the theory of matrices which is remarkable for containing the first definition of a matrix.

Every beginning student of Linear Algebra as a main class of algebra studied number of decades matrices over the real and complex fields. These are defined by the property of being a matrix about the main diagonal [9,11]. The theory of symmetric matrices [16,47] whose entries are symmetrical about the main diagonal, plays a vital role in almost all branches of Pure and Applied Mathematics.

In 1918, the concept of normal matrices over the complex field was introduced by Toeplitz [46] who gave necessary and sufficient conditions for a complex matrix to be normal. A matrix $A$ is normal if $AA^* = A^*A$. This concept of normal was introduced as a generalization of hermitian matrices. Sadkane.M [44] has studied about the normal matrices on singular values.
The concept of unitary matrix was introduced as a special case of normal matrix. A matrix $A$ is said to be unitary if $AA^* = A^*A = I$. It has been shown that the conjugate, transpose, conjugate transpose and inverse of a unitary matrix are also unitary and the product of two unitary matrices are also unitary. Unitary matrices have played an important role in the class of normal matrices. Moreover all unitary matrices are normal.

During 1976 Ann Lee [2] has introduced the study of secondary symmetric and secondary skew symmetric matrices that is, the matrices whose entries are symmetric and skew symmetric about the secondary diagonal. Later Ann Lee [3] defined secondary symmetric (s-symmetric), secondary skew symmetric (s-skew symmetric) matrices and secondary orthogonal (s-orthogonal) matrices to derive their properties.

The secondary transpose $A^s$ defined from $A = (a_{ij})$ by reflecting its entries through the secondary diagonal

$$(A^s)_{ij} = a_{n-j+1,n-i+1} \quad \text{for } i,j = 1,2,\ldots,n$$

and $A^s$ satisfying the following properties.

(i) $$(A^s)^s = A$$

(ii) $$(A^s)^t = A^s$$

Ann Lee has explained for a complex square matrix $A$, the usual transpose $A^T$ and secondary transpose $A^s$ are related as $A^s = VA^TV$ where $V$ is the permutation matrix with its units in the secondary diagonal. Also
\[ A^S = VA^T V \]
\[ = V A^T V \]
\[ (\overline{A})^S = VA^* V \]

\[ \Rightarrow A^\theta = VA^* V \text{ where } \overline{A}^S = A^\theta \]

The conjugate secondary transpose \( A^\theta \) satisfies the following properties:

(i) \( (A^\theta)^\theta = A \)

(ii) \( (A+B)^\theta = A^\theta + B^\theta \)

(iii) \( (AB)^\theta = B^\theta A^\theta \)

(iv) \( (A^{-1})^\theta = (A^\theta)^{-1} \) if \( A \) is nonsingular.

In addition Ann Lee has established that there exists a secondary symmetric matrix \( C \) such that \( A^S = CA^{-1}V \) [3] which is similar to the result of Tuassky and Zassawntans [47] for secondary symmetric matrix. Meenakshi.A.R has derived some special decomposition of matrices into product of hermitian matrices and symmetric matrices [34]. Moore [38,39] and Penrose [40] extended this concept to singular and nonsingular matrices.

Hill.R.D and Waters. S.R [17] have developed the theory of k-real and k-hermitian matrices where k is the fixed product of disjoint transpositions in \( S_n \) - the set of all permutations on \( \{1,2,\ldots n\} \). A complex matrix \( A = \{a_{ij}\} \) of order n is said to
be k-hermitian if and only if \( a_{ij} = \overline{a_{k(i)k(j)}} \) for \( i, j = 1, 2, \ldots, n \). ‘\( A^* \)’ is said to be skew hermitian if and only if \( a_{ij} = -\overline{a_{k(i)k(j)}} \) for \( i, j = 1, 2, \ldots, n \).

The author discussed the problem of characterizing k-real and k-hermitian preserving linear transformations. It is also illustrated in [17] that k-real and k-hermitian are not normal in general. Subsequently Meenakshi. A.R. and Krishnamoorthy. S [35] introduced the concept of range k-hermitian (k-EP) as a generalization of k-hermitian matrices. A theory of k-EP matrix was developed which reduces to that of EP matrices as a special case when \( k \) is the identity permutation. Also they have studied on Schur complement in k-EP matrices [36]. After that Krishnamoorthy. S brought the concept of s-EP matrices by introducing it as a generalization of s-hermitian and EP matrices. Necessary and sufficient conditions are determined for a matrix to be s-EP. The concept and characterizations of s-normal matrices is introduced by Krishnamoorthy. S and Vijayakumar. R. Some equivalent conditions on s-normal matrices are also derived [33].

The importance of normal matrices explains the appearance of the survey [12]. As putforth by Grone. R, Johnson. C. R., Sa. E. M, Wolkowicz. H, it is hoped that it will be useful to a wide audience, a long list of conditions on an \( nxn \) complex matrix \( A \), equivalent to its being normal, is presented. In most cases, the description of why the condition is equivalent to normality is given [12]. Elsner. L and Ikramov. Kh. D. [8] have given a list of 70 conditions on an \( nxn \) complex matrix \( A \) equivalent to its being normal, published nearly 10 years ago by Grone, Sa, and Wolkowicz has been proved to be very useful. Semigroups of normal complex square matrices having finite spectrum were earlier reported by Bojana Zalar [7]. It is shown that every member \( A \) of such a semigroup satisfies the condition \( A^* = A^n \) where \( n \) does not
depend on the matrix \( A \). It is shown that such a semigroup is completely reducible and that each irreducible contraction consists of a group of unitary matrices and eventually the zero matrix.

The relationship between the star and minus partial ordering was settled by Baksalary [4] Mitra [37] and Hartwig [13]. Several characterizations of \( A \preceq_* B \) are given with particular emphasis on what extra conditions must be added in order that rank subtractivity becomes the strong star order; a key tool is new canonical form for rank subtractivity. Connection with simultaneous singular value decomposition, Schur complements and idempotent matrices were also mentioned by Hartwig .R.E and Styan.G.P.H [14]. Xifu Liu and Yang [51] have shown results relating to block matrix partial ordering and submatrix the partial ordering. Special attention was given to star partial ordering of a sum of two matrices and minus ordering of matrix product. Several equivalent conditions for the minus orderings are established. In [22] Jorma.K.Merikoski and Xiaog Li have developed star partial ordering on normal matrices. Baksalary.J.K., Pukelsheim.F and Styan G.P.H [5] discussed the problem of inheriting certain property under a given ordering, preserving ordering under some matrix multiplication, relationship between an ordering direct and Hadamard products and the corresponding ordering between factors involved, ordering between generalized inverse of given matrix and preserving or reversing a given ordering under generalized inversion. Jurgen Grob[24] observed some remarks on partial ordering on hermitian matrices. Jorma K.Merikoski and Xiu.Liu characterize the rank subtractivity between normal matrices and have remarked that some results follow from well known results on EP matrices [21]. Julio Benitez, Xiaoji Liu and Jin Zhong have introduced their results relating to
different matrix partial ordering and the reverse order law for the Moore-Penrose inverse and the group inverse. Special attention was paid when at least one of the two involved matrices is EP [23]. Hauke Jan and Markiewcz, Augustin elaborated on the partial ordering on the set of rectangular matrices [15].

Any square matrix over a field is similar to its transpose and any square complex matrix is similar to a symmetric complex matrix. Similarity of a matrix and its conjugate transpose was settled by John. Duke. W [20]. Horn. R. A, Johnson. C. R discussed the conditions for two matrices \( A \) and \( B \) to be unitarily similar [18, 19]. As reported by Vermeer. J, any square matrix over a field is similar to its transpose and any square complex matrix is similar to a symmetric complex matrix. He investigated the situation for real orthogonal similarity [50]. Khakim and Ikramov settled some results on complex matrices with simple spectrum that are unitarily similar to real matrices [25]. As quoted by Alphin. Yu. A and Ikramov. Kh. D, the unitary similarity between two complex \( nxn \) matrices is extended to the unitary similarity between two normal matrix sets of cardinality \( m \). This property means that the algebra generated by a set is closed with respect to conjugate transpose operation [1]. Some extended results of Vermeer in the context of secondary orthogonal matrices were highlighted by Krishnamoorthy. S and Jaikumar. K [31].

Moore actually introduced the concept of the reciprocal of singular and rectangular matrices but Penrose without knowing the work of Moore established that for any \( mxn \) matrix which has a unique solution. The unique solution of four equations is as follows.
\[ AXA = A, \quad XAX = X, \quad (AX)^* = AX, (XA)^* = XA \] called Moore-Penrose inverse of \( A \).

Penrose describes a generalization of the inverse of a nonsingular matrix, as the unique solution of certain set of equations. This generalized inverse exists for any matrix whatsoever with complex elements. It is used here for solving linear matrix equations and among other applications for finding an expression for the principle idempotent elements of a matrix. Also a new type of spectral decomposition is given[40].

In the present work, the concept of s-unitary matrices are introduced as a generalization of unitary matrices. Characterization of s-unitary matrices analogous to that of an unitary matrix are determined. The conditions for sums and product of s-unitary matrices to be s-unitary matrices are investigated. Further, some equivalent conditions on s-unitary matrices are also derived.

Secondary eigen value of a matrix is defined as a special case of generalized eigen value problem \( Ax = \lambda Bx \) [10,42] for some matrices \( A \) and \( B \). The concept of s-unitarily similar matrices are introduced and established some theorems on s-unitary matrices.

Partial ordering on matrices which are invariant under \( V \) are established. Some theorems relating to partial ordering on s-unitary matrices are derived. The concept of ‘\( \theta \)’ partial ordering are introduced in this work and some theorems related to s-unitary matrices are also established.
The concept of secondary generalized inverse of a given matrix is discussed and its characterizations are obtained. Theorems involving (1)-inverse, (2)-inverse, (1,3)-inverse, (1,4) inverses of s-unitary matrices are obtained. Some results on the constructions of, (1,2,3)-inverse, (1,2,4)-inverse, (1,2,3,4) inverses of s-unitary matrices are also obtained in the present investigation.
1.2 NOTATIONS AND PRELIMINARIES

In this section, notations, definitions and theorems which are used in the thesis are given. Throughout, it is concerned with complex square matrices unless otherwise specified. Let \( A \) denotes a complex square matrix of order \( n \).

\[ C_{n \times n} : \] The space of complex square matrices of order \( n \)

\[ C_n : \] The space of complex \( n \)-tuples

\[ I_n : \] Identity matrix of order \( n \)

\[ S_n : \] Symmetric group of of order \( n \)

\[ V : \] Permutations matrix with units in the secondary diagonal and all other elements are zero.

\[ \overline{A} : \] The conjugate of \( A \)

\[ A^T : \] The transpose of \( A \)

\[ A^* : \] The conjugate transpose of \( A \)

\[ A^S : \] The secondary transpose of \( A \)

\[ A^\theta : \] The conjugate secondary transpose of \( A \)

\[ A^{-1} : \] The inverse of \( A \)

\[ A^\dagger : \] Moore-Penrose inverse of \( A \)

\[ A^\# : \] Group inverse of \( A \)

\[ A^{(1)} : \] Secondary (1) -inverse of \( A \)

\[ A^{(2)} : \] Secondary (2) -inverse of \( A \)

\[ A^{s} : \] s-g inverse of \( A \)
\( R(A) \) : Range space of \( A = \{ y \in \mathbb{C}^n / y = Ax \text{ for some } x \in \mathbb{C}^n \} \)

\( N(A) \) : Null space of \( A = \{ Ax = 0 \} \)

\( \text{rk} (A) \) : Rank of \( A \)

\( \text{tr} (A) \) : trace of \( A \)

\( \det (A) \) : determinant of \( A \)

\( \geq_L \) : Lowener partial order

\( \geq_* \) : Star order

\( \geq_{rs} \) : Minus order (or) rank subtractivity order

s-transpose : secondary transpose

s-hermitian : secondary hermitian

s-orthogonal : secondary orthogonal

s-normal : secondary normal

s-similar : secondary similar

s-unitarily similar : secondary unitarily similar

s-g inverse : secondary generalized inverse
Preliminary definitions and theorems:

Definition 1.2.1

A matrix $A \in C_{n\times n}$ is said to be symmetric if $A^T = A$

Definition 1.2.2

A matrix $A \in C_{n\times n}$ is said to be skew symmetric if $A^T = -A$

Definition 1.2.3

A matrix $A \in C_{n\times n}$ is said to be hermitian if $A^* = A$

Definition 1.2.4

A matrix $A \in C_{n\times n}$ is said to be skew hermitian if $A^* = -A$

Definition 1.2.5

A permutation matrix is one which is obtained from the identity matrix by performing elementary row or column operations.

Definition 1.2.6

A matrix $A \in C_{n\times n}$ is said to be orthogonal if $AA^T = A^T A = I$

Definition 1.2.7

A matrix $A \in C_{n\times n}$ is said to be normal if $AA^* = A^* A$

Definition 1.2.8 [45]

A matrix $A \in C_{n\times n}$ is said to be unitary if $AA^* = A^* A = I$

Definition 1.2.9

A matrix $A \in C_{n\times n}$ is said to be involutary if $A^2 = I$
Definition 1.2.10

A matrix \( A \in \mathbb{C}^{n \times n} \) is said to be idempotent if \( A^2 = A \)

Definition 1.2.11

Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \). Then the secondary transpose of \( A \), denoted by \( A^S \) is defined as \( A = (b_{ij}) \) where \( b_{ij} = a_{n-j+1,n-i+1} \)

Definition 1.2.12

Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \). Then the conjugate secondary transpose of \( A \), denoted \( A^\theta = \overline{A^S} = (c_{ij}) \) where \( c_{ij} = a_{n-j+1,n-i+1} \)

Example 1.2.13

Let \( A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) then \( A^S = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \)

Remark 1.2.14

The secondary transpose of a given matrix \( A \in \mathbb{C}^{n \times n} \) plays an important role what transpose of \( A \) does. \( A^S \) satisfies the following properties:

(i) \((A^S)^S = A\)

(ii) \((\overline{A^S}) = \overline{A^S}\)
The conjugate secondary transpose $A^\theta$ satisfies the following properties:

1. $(A^\theta)^\theta = A$
2. $(A + B)^\theta = A^\theta + B^\theta$
3. $(AB)^\theta = B^\theta A^\theta$
4. $(A^{-1})^\theta = (A^\theta)^{-1}$ if $A$ is nonsingular.

Ann Lee [2] has initiated the study of secondary symmetric matrices. Also, she has shown that for a complex matrix $A$, the usual transpose $A^T$ are related as $A^S = VA^T V$ where $V$ permutations matrix with units in the secondary diagonal and all other elements are zero [3]. Thus the relation between conjugate transpose and conjugate secondary transpose is given by $A^\theta = VA^\theta V$

Also $V$ satisfies the following properties

(i) $\bar{V} = V$
(ii) $V^2 = I$
(iii) $V^T = V$
(iv) $V^S = V$
(v) $V^* = V$
(vi) $V^\theta = V$
(vii) $VV^* = V^*V = I$
(viii) $V\theta V = V^\theta V = I$
Definition 1.2.15

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be secondary symmetric (s-symmetric) if $A = A^S$

Definition 1.2.16

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be secondary skew-symmetric (s-skew-symmetric) if $A = -A^S$

Definition 1.2.17

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be secondary hermitian (s-hermitian) if $A = A^\theta$

Definition 1.2.18

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be secondary skew Hermitian (s-skew hermitian) if $A = -A^\theta$

Definition 1.2.19

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be secondary orthogonal (s-orthogonal) if $AA^S = A^S A = I$

Definition 1.2.20 [32]

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be secondary normal(s-normal) if $AA^\theta = A^\theta A$

Definition 1.2.21

Two matrices $A$ and $B$ are said to be similar if there exists a nonsingular matrix $S$ such that $A = S^{-1}BS$.

Definition 1.2.22

Two matrices $A$ and $B$ are said to be unitarily similar if there exists a unitary matrix $U$ such that $A = U^*BU$. 

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Definition 1.2.23

A matrix $A \in \mathbb{C}^{3 \times 3}$ is said to be Block diagonal if $A$ is of the form

$$
\begin{pmatrix}
A_{11} & 0 & 0 \\
0 & A_{22} & 0 \\
0 & 0 & A_{33}
\end{pmatrix}
$$

Definition 1.2.24 [19]

The spectrum of a matrix $A \in \mathbb{C}^{n \times n}$ is the set of all eigen values of $A$.

Theorem 1.2.25 [12]

Let $X, Y \in \mathbb{C}^{n \times n}$. If $X$ and $Y$ are normal matrices and $X$ commutes with $Y$ then $X + Y$ and $X - Y$ are normal matrices.

Theorem 1.2.26 [45]

The product of two unitary matrices is unitary.

Theorem 1.2.27 [49]

The absolute value of eigen value of a unitary matrix is always 1.

Theorem 1.2.28 [12]

A matrix $A \in \mathbb{C}^{n \times n}$ is normal iff every matrix unitarily equivalent to $A$ is normal.

Theorem 1.2.29 [12]

If $A \in \mathbb{C}^{n \times n}$ unitarily equivalent to a diagonal matrix then $A$ is normal.

Theorem 1.2.30 [7]

If the semigroup $H$ of normal complex matrices has a finite spectrum $H$ consists of power hermitian matrices of the form $A^n = A^* A^n$ for some $n \geq 1$. 

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Theorem 1.2.31 [43]

Let $A, B \in C_{nxn}$ .

Then

(i) $N(A) \subseteq N(B) \iff R(B^*) \subseteq R(A^*)$

$\iff B = BA^*A$ for all $A^* \in A^\{l\}$

(ii) $N(A^*) \subseteq N(B^*) \iff R(B) \subseteq R(A)$

$\iff B = AA^*B$ for all $A^* \in A^\{l\}$

Theorem 1.2.32 [6]

Let $A \in C_{nxn}$ . Then $A$ is $EP \iff A^\| = A^\dagger$ when $A^\|$ exists.

Theorem 1.2.33 [40]

If $P$ and $Q$ are unitary then $(PAQ)^\dagger = Q^*A^\dagger P^*$

Remark 1.2.34 [48]

For any $A, B \in C_{nxn}$.


Also $(VA)^\dagger = A^\dagger V$ and $(AV)^\dagger = VA^\dagger$
1.3 SUMMARY OF RESULTS

In this section, a short account of the results obtained in the thesis are given.

Characterizations of s-unitary matrices:

The concept of s-unitary matrices is introduced for complex matrices and exhibited as generalization of unitary matrices. A matrix $A \in C_{nxn}$ is said to be secondary unitary (s-unitary) if $AA^\theta = A^\theta A = I$ where $A^\theta = \overline{A}^S$ denotes the conjugate secondary transpose of $A$. Here all s-unitary matrices are s-normal. We proved that $A^T, \overline{A}, A^*, A^{-1}$ and $iA$ are s-unitary.

Sums and products of s-unitary matrices:

Necessary and sufficient conditions for sum and difference of two s-unitary matrices $A$ and $B$ to be s-unitary is proved.

Let $A, B \in C_{nxn}$ and $A, B$ are s-unitary matrices and $A\overline{B}^S = \overline{B}^S A, B\overline{A}^S = \overline{A}^S B$. If $A\overline{B}^S + \overline{B}^S A = -I$ then $A + B$ is s-unitary. If $A\overline{B}^S + \overline{B}^S A = I$ then $A - B$ is s-unitary. It is shown that product of two s-unitary matrices is a s-unitary matrix.

Equivalent conditions on s-unitary matrices:

A list of conditions on an $nxn$ complex matrix $A$ each of which is equivalent to $A$ being s-unitary, is given and their proof are also given.

Some equivalent conditions are given below.

1. Any two of the following imply the other:
   (i) $A$ is s-unitary (ii) $A$ is hermitian (iii) $VA$ or $AV$ is involutary

2. Any two of the following imply the other:
   (i) $A$ is unitary (ii) $A$ is s-unitary (iii) $AV = VA$

3. Any two of the following imply the other:
   (i) $A$ is s-unitary (ii) $A$ is s-hermitian (iii) $A$ is involutary
4. Any two of the following imply the other:

(i) \( A \) is s-unitary    (ii) \( A^{-1} = P(A) \) where \( P(A) \) is a polynomial in \( A \)

(iii) \( A^* = P(VAV) \)

Secondary unitary similarity of matrices:

The concept of s-unitarily similar matrices are introduced. Let \( A, B \in C_{nxn} \). \( A \) is said to be s-unitarily similar to \( B \) if there is s-unitary matrix \( U \) such that \( A = U^\theta BU \)

The concept of families of matrices of s-unitarily similar matrices are introduced.

Two families of \( nxn \) matrices \( \{ A_1, A_2, \ldots, A_m \} \) and \( \{ B_1, B_2, \ldots, B_m \} \) are said to be s-unitarily similar if there exists a s-unitary matrix \( U \) such that \( U^\theta A_i U = B_i \) \( i = 1, 2, \ldots, n \). Using this concept some theorems on s-unitary matrices are derived.

Partial ordering of s-unitary matrices:

Lowener partial order, star order and minus order or rank subtractivity order are defined. It is proved that all standard partial orderings are preserved under \( V \), a permutation matrix. Some theorems on partial ordering of s-unitary matrices are derived. The concept of \( \theta \) partial ordering on s-unitary matrices are introduced. Some results relating to partial ordering are derived.

(i) Let \( A \geq L B \) and \( A \geq \ast B \). If \( A \) and \( B \) are s-unitary matrices then \( A = B \).

(ii) Let \( A \) and \( B \) be s-unitary matrices. Then \( A \geq rs B \) iff \( A^{-1} \geq rs B^{-1} \)

(iii) Let \( A \) and \( B \) are s-unitary matrices. If \( A \leq L B \) and \( A \leq \theta B \) then \( A = B \).
Secondary generalized inverse of $s$-unitary matrices:

The concept of $s$-g inverse (secondary generalized inverse) of a given matrix is discussed and its characterizations are obtained. Some theorems on (1)-inverse, (2)-inverse, (1,3)-inverse, (1,4) inverse are established. Some results on the constructions of (1,2,3)-inverse, (1,2,4)-inverse, (1,2,3,4)-inverses of $s$-unitary matrices are obtained.