Chapter – III

Mixed Integer Nonlinear Programming Approach to Fuzzy Retrial Queue
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MIXED INTEGER NONLINEAR PROGRAMMING APPROACH TO FUZZY RETRIAL QUEUE

This chapter develops a nonlinear programming approach to derive the membership functions of the steady-state performance measures in retrial queueing model with fuzzy arrival, retrial and service rates. The basic idea is based on Zadeh’s extension principle. Three pairs of mixed integer nonlinear programs (MINLP) with binary variables are formulated to calculate the upper and lower bounds of the system performance measure at possibility level $\alpha$. From different values of $\alpha$, the membership function of the system performance measure is constructed. For practical use, the defuzzification of performance measures is also provided via Yager ranking index.

3.1 PERFORMANCE MEASURE OF CRISP RETRIAL QUEUEING SYSTEM

Assume that the performance measure of interest is the expected waiting time and the expected number of customers in the orbit. From the knowledge of retrial queueing theory [28,41], if $x/y < 1$ we have the expected waiting time and the expected number of customers in the system for a crisp retrial queueing system, respectively, given by

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3 This part of the chapter has been published in Proceedings of the UGC Sponsored International Conference on Mathematical Methods and Computation, Allied Publishers, pp.356-365, July 2009.
\[ E[W_s] = \frac{x}{(y - x)} \left( \frac{1}{x} + \frac{1}{v} \right), \quad (3.1) \]

and

\[ E[N_s] = \frac{x^2}{(y - x)} \left( \frac{1}{x} + \frac{1}{v} \right) \quad (3.2) \]

### 3.2 PERFORMANCE MEASURE OF FUZZY RETRIAL QUEUEING SYSTEM

Following (2.9) and (3.1), the membership function for the expected waiting time \( E[W_s] \) in the system is

\[
\eta_{E[W_s]}(z) = \sup \min \left\{ \eta_{\hat{\lambda}}(x), \eta_{\hat{\theta}}(v), \eta_{\hat{\mu}}(y) \mid z = \frac{x}{(y - x)} \left( \frac{1}{x} + \frac{1}{v} \right) \right\} \quad (3.3)
\]

Likewise, the membership function for the expected number of customers in the system \( E[N_s] \) can be obtained from (2.9) and (3.2) by the following

\[
\eta_{E[N_s]}(z) = \sup \min \left\{ \eta_{\hat{\lambda}}(x), \eta_{\hat{\theta}}(v), \eta_{\hat{\mu}}(y) \mid z = \frac{x^2}{(y - x)} \left( \frac{1}{x} + \frac{1}{v} \right) \right\} \quad (3.4)
\]

Theoretically, these membership functions are correct; however, they are not in the usual forms for practical use moreover, it is even very difficult to imagine their shapes.

### 3.3 SOLUTION PROCEDURE

One approach to construct the membership function \( \eta_{E[W_s]}(z) \) of \( E[\tilde{W}_s] \) is on the basis of Zadeh’s approach, which relies on \( \alpha \)-cuts of \( E[\tilde{W}_s] \). Denote the \( \alpha \)-cuts of \( \tilde{\lambda}, \tilde{\theta}, \) and \( \tilde{\mu} \) as crisp intervals as follows:
\[ \lambda(\alpha) = \left[ x^L_\alpha, x^U_\alpha \right] = \left[ \min_{x \in X} \left\{ x / \eta_\lambda(x) \geq \alpha \right\}, \max_{x \in X} \left\{ x / \eta_\lambda(x) \geq \alpha \right\} \right] \quad (3.5a) \]

\[ \theta(\alpha) = \left[ v^L_\alpha, v^U_\alpha \right] = \left[ \min_{v \in V} \left\{ v / \eta_\theta(v) \geq \alpha \right\}, \max_{v \in V} \left\{ v / \eta_\theta(v) \geq \alpha \right\} \right] \quad (3.5b) \]

\[ \mu(\alpha) = \left[ y^L_\alpha, y^U_\alpha \right] = \left[ \min_{y \in Y} \left\{ \mu / \eta_\mu(y) \geq \alpha \right\}, \max_{y \in Y} \left\{ \mu / \eta_\mu(y) \geq \alpha \right\} \right] \quad (3.5c) \]

These intervals indicate where the arrival, retrial and service rates lie at possibility level \( \alpha \). These three sets cause nested structures for expressing the relationship between ordinary sets and fuzzy sets [36]; i.e., given \( 0 < \alpha_2 < \alpha_1 \leq 1 \), we have \([x^L_\alpha, x^U_\alpha] \subseteq [x^L_{\alpha_2}, x^U_{\alpha_2}]\), \([v^L_\alpha, v^U_\alpha] \subseteq [v^L_{\alpha_2}, v^U_{\alpha_2}]\) and \([y^L_\alpha, y^U_\alpha] \subseteq [y^L_{\alpha_2}, y^U_{\alpha_2}]\).

By the convexity of a fuzzy number [64], the bounds of these intervals are functions of \( \alpha \) and can be obtained as \( x^L_\alpha = \min \eta^{-1}_\lambda(\alpha) \), \( x^U_\alpha = \max \eta^{-1}_\lambda(\alpha) \), \( v^L_\alpha = \min \eta^{-1}_\theta(\alpha) \), \( v^U_\alpha = \max \eta^{-1}_\theta(\alpha) \), \( y^L_\alpha = \min \eta^{-1}_\mu(\alpha) \) and \( y^U_\alpha = \max \eta^{-1}_\mu(\alpha) \).

Clearly as defined in (3.3), the membership function of \( E[\hat{W}_s] \) is also parameterized by \( \alpha \). Consequently, we can use its \( \alpha \)-cut to construct the corresponding membership function.

Using Zadeh’s extension principle, \( \eta_{E[\hat{W}_s]}(z) \) is the minimum of \( \eta_\lambda(x) \), \( \eta_\theta(v) \) and \( \eta_\mu(y) \). We need at least one of the following cases to hold such that:

\[ Z = \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{y} \right) \]

satisfies \( \eta_{E[\hat{W}_s]}(z) = \alpha \).
Case (i) : \( (\eta_x(x) = \alpha, \eta_v(v) \geq \alpha, \eta_y(y) \geq \alpha) \),

Case (ii) : \( (\eta_x(x) \geq \alpha, \eta_v(v) = \alpha, \eta_y(y) \geq \alpha) \),

Case (iii) : \( (\eta_x(x) \geq \alpha, \eta_v(v) \geq \alpha, \eta_y(y) = \alpha) \).

To find the membership function \( \eta_{E[W_s]}(z) \), it suffices to find the left shape function and the right shape function of \( \eta_{E[W_s]}(z) \), which is equivalent to finding the lower bound \( \left( E[W_s] \right)_a^L \) and the upper bound \( \left( E[W_s] \right)_a^U \) of the \( \alpha \)-cuts of \( E[W_s] \). Since the requirement of \( \eta_x(x) = \alpha \) can be represented by \( x = \lambda_\alpha \) or \( x = \lambda_\alpha^U \), this can be formulated as the constraint of \( x = \beta_1 \lambda_\alpha^L + (1 - \beta_1) \lambda_\alpha^U \), where \( \beta_1 = 0 \) or \( 1 \); similarly, \( \eta_v(v) = \alpha \) can be formulated as the constraint of \( v = \beta_2 \theta_\alpha^L + (1 - \beta_2) \theta_\alpha^U \), where \( \beta_2 = 0 \) or \( 1 \) and \( \eta_y(y) = \alpha \) can be formulated as the constraint of \( y = \beta_3 \mu_\alpha^L + (1 - \beta_3) \mu_\alpha^U \), where \( \beta_3 = 0 \) or \( 1 \). Moreover, from the definition of \( \lambda(\alpha) \), \( \theta(\alpha) \) and \( \mu(\alpha) \) in (3.5a), (3.5b) and (3.5c), \( x \in \lambda(\alpha) \), \( v \in \theta(\alpha) \) and \( y \in \mu(\alpha) \) can be respectively replaced by \( x \in \left[ \lambda_\alpha^L, \lambda_\alpha^U \right] \), \( v \in \left[ \theta_\alpha^L, \theta_\alpha^U \right] \) and \( y \in \left[ \mu_\alpha^L, \mu_\alpha^U \right] \). Consequently, considering all of these three cases above, the membership function \( \eta_{E[W_s]}(z) \) can be constructed via finding the lower bound \( \left( E[W_s] \right)_a^L \) and upper bound \( \left( E[W_s] \right)_a^U \) of the \( \alpha \)-cuts of \( E[W_s] \), in that we set \( \left( E[W_s] \right)_a^L = \min \{ E[W_s]_a^L, E[W_s]_a^U \} \) and \( \left( E[W_s] \right)_a^U = \max \{ E[W_s]_a^L, E[W_s]_a^U \} \) respectively, where
\[ E[W_s]_{a}^{t_1} = \min_{s.t. \ x, y \leq \text{c}} \left[ \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{v} \right) \right] \]

\[ s.t. 
\begin{align*}
x &= t_1 x_a^{L} + (1 - t_1) x_a^{U} \\
y_a^{L} &\leq y \leq y_a^{U}, \\
y_a^{L} &\leq y \leq y_a^{U}, \\
t_1 &= 0 \text{ or } 1
\end{align*} \tag{3.6a} \]

\[ E[W_s]_{a}^{t_2} = \min_{s.t. \ x, y \leq \text{c}} \left[ \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{v} \right) \right] \]

\[ s.t. 
\begin{align*}
v &= t_2 v_a^{L} + (1 - t_2) v_a^{U}, \\
x_a^{L} &\leq x \leq x_a^{U}, \\
y_a^{L} &\leq y \leq y_a^{U}, \\
t_2 &= 0 \text{ or } 1
\end{align*} \tag{3.6b} \]

\[ E[W_s]_{a}^{t_3} = \min_{s.t. \ x, y \leq \text{c}} \left[ \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{v} \right) \right] \]

\[ s.t. 
\begin{align*}
y &= t_3 y_a^{L} + (1 - t_3) y_a^{U}, \\
x_a^{L} &\leq x \leq x_a^{U}, \\
v_a^{L} &\leq v \leq v_a^{U}, \\
t_3 &= 0 \text{ or } 1
\end{align*} \tag{3.6c} \]

\[ 44 \]
$$E[W_s]_{a}^{\mu} = \max_{x/y \in \Omega} \left[ \frac{x}{y-x} \left( \frac{1}{x} + \frac{1}{v} \right) \right]$$

\[(3.6d)\]

s.t \hspace{1em} x = t_4 x_a^L + (1 - t_4) x_a^U , \hspace{1em}

\[v_a^L \leq v \leq v_a^U , \]

\[y_a^L \leq y \leq y_a^U , \]

\[t_4 = 0 \text{ or } 1 \]

$$E[W_s]_{a}^{\nu} = \max_{x/y \in \Omega} \left[ \frac{x}{y-x} \left( \frac{1}{x} + \frac{1}{v} \right) \right]$$

\[(3.6e)\]

s.t \hspace{1em} v = t_5 v_a^L + (1 - t_5) v_a^U , \hspace{1em}

\[x_a^L \leq x \leq x_a^U , \]

\[y_a^L \leq y \leq y_a^U , \]

\[t_5 = 0 \text{ or } 1 \]

$$E[W_s]_{a}^{\lambda} = \max_{x/y \in \Omega} \left[ \frac{x}{y-x} \left( \frac{1}{x} + \frac{1}{v} \right) \right]$$

\[(3.6f)\]

s.t \hspace{1em} y = t_6 y_a^L + (1 - t_6) y_a^U , \hspace{1em}

\[x_a^L \leq x \leq x_a^U , \]

\[v_a^L \leq v \leq v_a^U , \]

\[t_6 = 0 \text{ or } 1 \]
where $x_\alpha^U < y_\alpha^L$. At least one of $x$, $v$ or $y$ must hit the boundaries of their $\alpha$-cuts to satisfy $\eta_{E[W_s]}(z) = \alpha$. From the knowledge of calculus, a unique minimum and a unique maximum of the objective function of models (3.6a), (3.6b), (3.6c), (3.6d), (3.6e) or (3.6f) are assured, which shows that the lower bound $E[W_s]_\alpha^L$ and upper bound $E[W_s]_\alpha^U$ of the $\alpha$-cuts of $E[W_s]$ can be found by solving these six models. In fact, these six models are MINLP with 0-1 variables. There are several effective and efficient methods for solving these problems [34]. Moreover, they involve the systematic study of how the optimal solutions change as $x_\alpha^L$, $x_\alpha^U$, $v_\alpha^L$, $v_\alpha^U$, $y_\alpha^L$ and $y_\alpha^U$ vary over the interval $\alpha \in [0,1]$; they fall into the category of parametric programming [30].

The crisp interval $[E[W_s]_\alpha^L, E[W_s]_\alpha^U]$ obtained by solving models (3.6a), (3.6b), (3.6c), (3.6d), (3.6e) and (3.6f) represents the $\alpha$-cut of $E[W_s]$. An attractive feature of the $\alpha$-cut approach is that all $\alpha$-cuts form a nested structure with respect to $\alpha$ [36]. According to Zadeh’s extension principle, $E[W_s]$ defined in (3.3) is a fuzzy number that possesses convexity [36,64]. Therefore, for two values $\alpha_1$ and $\alpha_2$ such that $0 < \alpha_2 < \alpha_1 \leq 1$, we have $E[W_s]_{\alpha_1}^L \geq E[W_s]_{\alpha_2}^L$ and $E[W_s]_{\alpha_1}^U \leq E[W_s]_{\alpha_2}^U$, in other words, $E[W_s]_\alpha^L$ is non decreasing with respect to $\alpha$ and $E[W_s]_\alpha^U$ is non increasing with respect to $\alpha$. This property assures the convexity of $E[W_s]$. Consequently, the membership function $\eta_{E[W_s]}(z)$ can be obtained from the solutions of models (3.6a), (3.6b), (3.6c), (3.6d), (3.6e) and (3.6f).
If both $E[W_s]_\alpha^L$ and $E[W_s]_\alpha^U$ are invertible with respect to $\alpha$, then a left shape function $L(z) = \left[\left( E[W_s]_\alpha^L \right) \right]^{-1}$ and a right shape function $R(z) = \left[\left( E[W_s]_\alpha^U \right) \right]^{-1}$ can be derived, from which the membership function $\eta_{E[W_s]}(z)$ is constructed:

$$
\eta_{E[W_s]}(z) = \begin{cases} 
L(z), & \text{if } (E[W_s])_\alpha^L \leq z \leq (E[W_s])_\alpha^L, \\
R(z), & \text{if } (E[W_s])_\alpha^U \leq z \leq (E[W_s])_\alpha^U.
\end{cases} 
$$

(3.7)

In most cases, the values of $(E[W_s])_\alpha^L$ and $(E[W_s])_\alpha^U$ cannot be solved analytically, the numerical solutions for $(E[W_s])_\alpha^L$ and $(E[W_s])_\alpha^U$ at different possibility level $\alpha$ can be collected to approximate the shapes of $L(z)$ and $R(z)$. That is, the set of intervals $\{(E[W_s])_\alpha^L, (E[W_s])_\alpha^U | \alpha \in [0, 1]\}$ reveals the shape of $\eta_{E[W_s]}$ although the exact function is not known explicitly. Note that the membership functions of the expected number of customers in the system and expected waiting time and expected number of customers in the orbit for this retrial system can be derived in a similar manner, using Little’s Formula [45].

Since the performance measures are described by membership function, the values conserve completely all of fuzziness of arrival rate, service rate and retrial rate. However, in practical point of view, the management would prefer one crisp value for each system characteristics rather than a fuzzy set. In order to overcome this problem, we defuzzify the fuzzy values of system characteristics by Yager’s ranking index method [56]. Since Yager’s ranking index method possesses the property of area compensation, we adopt this
method for transforming the fuzzy values of performance measure, crisp one to provide suitable values for performance measure. The recommended suitable values of performance measure are calculated by,

\[ O(\tilde{\Lambda}) = \frac{1}{2} \int_{0}^{1} \lambda^L_{\alpha} + \lambda^U_{\alpha} d\alpha \]  

(3.8)

where \( \tilde{\Lambda} \) is a convex fuzzy number and \( (\lambda^L_{\alpha}, \lambda^U_{\alpha}) \) is the \( \alpha \) - cut of \( \tilde{\Lambda} \). Note that this method is a robust ranking technique that possesses the properties of compensation, linearity and additively [29].

### 3.4 NUMERICAL EXAMPLE

To demonstrate the practical use of the proposed approach, an example inspired by Artalejo and Lopez-Herrero [37] is solved.

In a packet-switching network, we considered a computer network in which there are a group of host computers connected to interface message processors. Message arrives at the host computer following a Poisson stream. If the host computer wishes to transmit the message to another host computer, it must send the message and the final address to the interface message processor with which it is associated. If the processor is free the message is accepted; otherwise, the message comes back to the host computer and is stored in a buffer to be retransmitted some time later. The buffer in the host computer, the interface processor and the retransmission policy correspond to the orbit, the server and the retrial discipline, respectively, in the queueing terminology. Clearly, a FM/FM/1/1-(FR) queueing model can model the above
system. Concerned with system efficiency, the management wants to obtain the performance measure, including the expected waiting time and the number of customers in the system and queue.

3.4.1 THE FUZZY EXPECTED WAITING TIME IN THE SYSTEM ($E[\tilde{W}_s]$)

Suppose the arrival, retrial and service rates are triangular fuzzy numbers represented by $\tilde{\lambda} = [3, 4, 5]$, $\tilde{\theta} = [1, 4, 7]$ and $\tilde{\mu} = [6, 7, 8]$. It is easy to find that $[x^L_a, x^U_a] = [3 + \alpha, 5 - \alpha]$, $[v^L_a, v^U_a] = [1 + 3\alpha, 7 - 3\alpha]$ and $[y^L_a, y^U_a] = [6 + \alpha, 8 - \alpha]$. It is clear that in this example the steady-state condition $\rho = x/y < 1$ is satisfied, thus the performance measure of interest can be constructed by using the proposed approach stated in section 3.3. Following (3.6a), (3.6b), (3.6c), (3.6d), (3.6e) and (3.6f), three pairs of MINLP models for deriving the membership function of $E[\tilde{W}_s]$ can be formulated, whose solutions are as follows:

$$E[W_s]_a^L = \frac{10 - 2\alpha}{6\alpha^2 - 29\alpha + 35}$$

$$E[W_s]_a^U = \frac{6 + 2\alpha}{6\alpha^2 + 5\alpha + 1}$$

With the help of MATLAB$^\text{®}$ 7.0 the inverse functions of $E[W_s]_a^L$ and $E[W_s]_a^U$ exist, which give the membership function,

$$\eta_{E[\tilde{W}_s]}(z) = \begin{cases} 
\frac{(29z - 2) - (z^2 + 124z + 4)^{1/2}}{12z}; \frac{2}{7} \leq z \leq \frac{2}{3} \\
\frac{-(5z - 2) + (z^2 + 124z + 4)^{1/2}}{12z}; \frac{2}{3} \leq z \leq 6
\end{cases}$$
Applying Yager ranking index method stated (2.8), the suitable expected waiting time in the system for this example is given by

\[
O\left(E[\tilde{W}_s]\right) = \frac{1}{2} \int_0^1 \left[ \frac{10 - 2\alpha}{6\alpha^2 - 29\alpha + 35} + \frac{6 + 2\alpha}{6\alpha^2 + 5\alpha + 1} \right] d\alpha = 1.1656
\]

### 3.4.2 FUZZY EXPECTED NUMBER OF CUSTOMERS IN THE SYSTEM \(E[\tilde{N}_s]\)

By the same argument, the \(\alpha\)-cuts of \(E[\tilde{N}_s]\) are

\[
\left(E[\tilde{N}_s]\right)_\alpha^L = \frac{-2\alpha^2 + 4\alpha + 30}{6\alpha^2 - 29\alpha + 35},
\]

\[
\left(E[\tilde{N}_s]\right)_\alpha^U = \frac{-2\alpha^2 + 4\alpha + 30}{6\alpha^2 - 5\alpha + 1}.
\]

The membership function is

\[
\eta_{E[\tilde{N}_s]}(z) = \begin{cases} 
\frac{(4 + 29z) - (z^2 + 672z + 256)^{1/2}}{2(2 + 6z)}, & 30 \leq z \leq \frac{32}{12} \\
\frac{(4 - 5z) + (z^2 + 672z + 256)^{1/2}}{2(2 + 6z)}, & \frac{32}{12} \leq z \leq 30
\end{cases}
\]

We obtain Yager ranking index of the fuzzy expected number of customers in the system,

\[
O\left(E[\tilde{N}_s]\right) = \frac{1}{2} \int_0^1 \left[ \frac{-2\alpha^2 + 4\alpha + 30}{6\alpha^2 - 29\alpha + 35} + \frac{-2\alpha^2 + 4\alpha + 30}{6\alpha^2 + 5\alpha + 1} \right] d\alpha = 5.2124
\]
3.4.3. FUZZY EXPECTED WAITING TIME IN THE QUEUE \( E[\tilde{W}_q] \)

By the same argument, the \( \alpha \)-cuts of \( E[\tilde{W}_q] \) are

\[
(E[\tilde{W}_q])^L_\alpha = \frac{-4\alpha^2 + 3\alpha + 45}{-6\alpha^3 + 77\alpha^2 - 267\alpha + 280},
\]

\[
(E[\tilde{W}_q])^U_\alpha = \frac{-4\alpha^2 + 13\alpha + 35}{6\alpha^3 + 41\alpha^2 + 31\alpha + 6}
\]

The membership function is

\[
\eta_{E[\tilde{W}_q]}(z) = \begin{cases} 
L(z), & \frac{45}{280} \leq z \leq \frac{44}{84} \\
R(z), & \frac{44}{84} \leq z \leq \frac{35}{6} 
\end{cases}
\]

where

\[
L(z) = \frac{1}{18} z(3372z + 14205z^2 + 37520z^3 + 64 + 9(-1697424z - 8269083z^2 - 34992 - 13090374z^3 - 104907z^4)^{1/3} + \]

\[
\frac{1}{18}(562z + 1123z^2 + 16)/(3372z + 14205z^2 + 37520z^3 + 64 - 9(-1697424z - 8269083z^2 - 34992 - 13090374z^3 - 104907z^4)^{1/3} + \]

and

\[
R(z) = \frac{1}{18} z(-3372z - 14205z^2 - 37520z^3 - 64 + 9(-1697424z - 8269083z^2 - 34992 - 13090374z^3 - 104907z^4)^{1/3} + \]

\[
\frac{1}{18}(562z + 1123z^2 + 16)/(z(-3372z - 14205z^2 - 37520z^3 - 64 + 9(-1697424z - 8269083z^2 - 34992 - 13090374z^3 - 104907z^4)^{1/3} - \]
Yager ranking index of the fuzzy expected waiting time in the queue is

\[ O \left( E[\tilde{W}_q] \right) = \frac{1}{2} \left[ \frac{-4\alpha^2 + 3\alpha + 45}{-6\alpha^3 + 77\alpha^2 - 267\alpha + 280} + \frac{-4\alpha^2 + 13\alpha + 35}{6\alpha^3 + 41\alpha^2 + 31\alpha + 6} \right] d\alpha = 1.0218 \]

3.4.4 FUZZY EXPECTED NUMBER OF CUSTOMERS IN THE QUEUE \( E[\tilde{N}_q] \)

By the same argument, the \( \alpha \)-cuts of \( E[\tilde{N}_q] \) are

\[ (E[\tilde{N}_q])^L_\alpha = \frac{-4\alpha^3 - 9\alpha^2 + 54\alpha + 135}{-6\alpha^3 + 77\alpha^2 - 267\alpha + 280} \]

\[ (E[\tilde{N}_q])^U_\alpha = \frac{4\alpha^3 - 33\alpha^2 + 30\alpha + 175}{6\alpha^3 + 41\alpha^2 + 31\alpha + 6} \]

The inverse function of \( (E[\tilde{N}_q])^L_\alpha \) and \( (E[\tilde{N}_q])^U_\alpha \) exists, and its membership function is,

\[ \eta_{E[\tilde{N}_q]}(z) = \begin{cases} 
L(z), & 135 \leq z \leq 176 \\
\frac{280}{84} & \\
R(z), & \frac{176}{84} \leq z \leq \frac{175}{6} \end{cases} \]

where,

\[ L(z) = \frac{1}{6} (-2 + 3z) (-19683 + 271647z + 105948z^2 + 37520z^3 \]

\[ - 66(3z(289z^3 + 195534z^2 + 779193z + 1259712))^{1/2} \]

\[ + 99(-3z(289z^3 + 195534z^2 + 779193z + 1259712)^{1/2}z)^{1/3} \]

\[ + \frac{1}{6} (729 + 3619z + 1123z^2) /(-2 + 3z)/(-19683 + 2716473 \]

\[ + 105948z^2 + 37520z^3 - 66(-3z(289z^3 + 195534z^2 \]

\[ + 779193z + 1259712))^{1/2} + 99(-3z(289z^3 + 195534z^2 \]

\[ + 779193z + 1259712)^{1/2}z)^{1/3} + \frac{1}{6} (9 + 77z)/(-2 + 3z) \]

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\[ R(z) = \frac{1}{6} (-2 + 3z) \left( +19683 - 271647z - 105948z^2 - 37520z^3 - 66(3z(289z^3 + 195534z^2 + 779193z + 1259712))^{1/2} + 99(-3z(289z^3 + 195534z^2 + 779193z + 1259712))^{1/2} + 779193z + 1259712) \right)^{1/3} + \frac{1}{6} (729 + 3619z + 1123z^2) / (-2 + 3z) \]

\[ \left( 19683 - 271647z - 105948z^2 - 37520z^3 \right) \left( -2 + 3z \right) / (19683 - 271647z - 105948z^2 - 37520z^3) \]

\[ - 66(-3z(289z^3 + 195534z^2 + 779193z + 1259712))^{1/2} \]

\[ + 99(-3z(289z^3 + 195534z^2 + 779193z + 1259712))^{1/2} \]

\[ - \frac{1}{6} (31 + 41z) / (-2 + 3z) \]

Yager ranking index of fuzzy expected number of customers in the queue is,

\[ O\left(E[\tilde{N}_q]\right) = \int_0^1 \frac{-4\alpha^3 - 9\alpha^2 + 54\alpha + 135}{-6\alpha^2 + 77\alpha^2 - 267\alpha + 280} \left[ \frac{4\alpha^3 - 33\alpha^2 + 30\alpha + 175}{6\alpha^2 + 41\alpha^2 + 31\alpha + 6} \right] d\alpha = 5.1215 \]

Figure 3.1 depicts the membership function of the system length, the expected waiting time of the customer in the system, the queue length and expected waiting time of the customers in the queue.
Table 3.1 lists the $\alpha$-cuts of these four performance measures at 11 distinct $\alpha$ values; 0, 0.1, 0.2, ..., 1.0. These $\alpha$-cuts represent the possibility that these four performance measures will appear in the associated range. Specially, the $\alpha = 0$ cut shows the range that these four performance measures could appear and the $\alpha = 1$ cut shows these four performance measures that are most likely to be. For example, while these four performance measures are fuzzy, the most likely value of the expected system length $\left(E[\bar{N}_q]\right)_\alpha$ falls at 2.6667 and
Table 3.1

The $\alpha$-cuts of the performance measures at 11 $\alpha$ values

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$(E[W])_a^L$</th>
<th>$(E[W])_a^U$</th>
<th>$(E[N])_a^L$</th>
<th>$(E[N])_a^U$</th>
<th>$(E[W])_a^L$</th>
<th>$(E[W])_a^U$</th>
<th>$(E[N])_a^L$</th>
<th>$(E[N])_a^U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.2857</td>
<td>6.000</td>
<td>0.8571</td>
<td>30</td>
<td>0.1607</td>
<td>5.8333</td>
<td>0.4821</td>
<td>29.1667</td>
</tr>
<tr>
<td>0.1</td>
<td>0.3047</td>
<td>3.9744</td>
<td>0.9447</td>
<td>19.4744</td>
<td>0.1781</td>
<td>3.8104</td>
<td>0.5522</td>
<td>18.6710</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3261</td>
<td>2.8571</td>
<td>1.0435</td>
<td>13.7143</td>
<td>0.1979</td>
<td>2.6959</td>
<td>0.6332</td>
<td>12.9401</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3502</td>
<td>2.1711</td>
<td>1.1557</td>
<td>10.2040</td>
<td>0.2204</td>
<td>2.0123</td>
<td>0.7271</td>
<td>9.4579</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3777</td>
<td>1.7172</td>
<td>1.2841</td>
<td>7.8990</td>
<td>0.2461</td>
<td>1.5609</td>
<td>0.8367</td>
<td>7.1802</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4091</td>
<td>1.4000</td>
<td>1.4318</td>
<td>6.3000</td>
<td>0.2758</td>
<td>1.2462</td>
<td>0.9652</td>
<td>5.6077</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4453</td>
<td>1.1689</td>
<td>1.6032</td>
<td>5.1429</td>
<td>0.3102</td>
<td>1.0173</td>
<td>1.1168</td>
<td>4.4762</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4875</td>
<td>0.9946</td>
<td>1.8039</td>
<td>4.2768</td>
<td>0.3505</td>
<td>0.8454</td>
<td>1.2970</td>
<td>3.6351</td>
</tr>
<tr>
<td>0.8</td>
<td>0.5371</td>
<td>0.8591</td>
<td>2.0409</td>
<td>3.6109</td>
<td>0.3982</td>
<td>0.7127</td>
<td>1.5131</td>
<td>2.9932</td>
</tr>
<tr>
<td>0.9</td>
<td>0.5959</td>
<td>0.7529</td>
<td>2.3241</td>
<td>3.0869</td>
<td>0.4551</td>
<td>0.6080</td>
<td>1.7748</td>
<td>2.4927</td>
</tr>
<tr>
<td>1.0</td>
<td>0.6667</td>
<td>0.6667</td>
<td>2.6677</td>
<td>2.6667</td>
<td>0.5238</td>
<td>0.5238</td>
<td>2.0952</td>
<td>2.0952</td>
</tr>
</tbody>
</table>

its value is impossible to fall outside the range of 0.8571 and 30.000; it is definitely possible that the expected queue waiting time $E[\hat{W}_q]$ falls at 0.5238 min (or 31.428 sec) approximately, and it will never fall below 0.1607 min (or 9.642 sec) or exceed 5.833 min approximately. The above information will be very useful for designing a retrial queueing system.
Chapter – IV

Parametric Nonlinear Programming Approach to Fuzzy Retrial Queues with Batch Arrivals