Chapter – II

Parametric Nonlinear Approach on Fuzzy Retrial Queueing Model with Fuzzy Numbers
CHAPTER - II
PARAMETRIC NON LINEAR APPROACH ON FUZZY RETRIAL QUEUEING MODEL WITH FUZZY NUMBERS

In this chapter, parametric non-linear programming approach is dealt with trapezoidal fuzzy numbers and triangular fuzzy numbers to construct the membership functions of performance measures. In the first part we have considered the retrial queues with fuzzified arrival rates, fuzzified service rates and crisp retrial rate. In the later part all the three parameters are defined as fuzzy variables.

2.1 FUZZY RETRIAL QUEUEING MODEL WITH TRAPEZOIDAL FUZZY NUMBER1

2.1.1 PROBLEM FORMULATION

Consider a single server retrial queueing system in which the retrial rate \( \theta \) is being crisp, interarrival time \( \tilde{\lambda} \) and service time \( \tilde{\mu} \) which are approximately known and are represented by the following fuzzy sets:

\[
\tilde{\lambda} = \{x, \eta_{\tilde{\lambda}}(x) \mid x \in X\},
\]

\[
\tilde{\mu} = \{y, \eta_{\tilde{\mu}}(y) \mid y \in Y\}.
\]

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2 Section 2.2 of this chapter has been presented in National Seminar on Novel Approach in Graph Theory, Fuzzy Mathematics and Stochastic Models, Dec. 2008.
where \( X \) and \( Y \) are the crisp universal sets of the interarrival time and service time, and \( \eta_\lambda(x) \) and \( \eta_\mu(y) \) are the corresponding membership function. The \( \alpha \)-cuts or \( \alpha \)-level sets of \( \bar{\lambda} \) and \( \bar{\mu} \) are

\[
\lambda(\alpha) = \left\{ x \in X / \eta_\lambda(x) \geq \alpha \right\}, \quad (2.1a)
\]

\[
\mu(\alpha) = \left\{ y \in Y / \eta_\mu(y) \geq \alpha \right\} \quad (2.1b)
\]

Here \( \lambda(\alpha) \) and \( \mu(\alpha) \) are crisp sets. Using \( \alpha \)-cuts the interarrival time and service time can be represented by different levels of confidence intervals [64]. Consequently, a fuzzy retrial queue can be reduced to a family of crisp retrial queues with different \( \alpha \)-level sets \( \{ \lambda(\alpha) / 0 < \alpha \leq 1 \} \) and \( \{ \mu(\alpha) / 0 < \alpha \leq 1 \} \). These two sets represent sets of movable boundaries, and they form nested structures for expressing the relationship between ordinary sets and fuzzy sets [36]. Let \( \bar{\lambda} \) and \( \bar{\mu} \) are fuzzy numbers. Denote the intervals of confidence of \( \bar{\lambda} \) and \( \bar{\mu} \) as \( \big[ x^L_\alpha, x^U_\alpha \big] \) and \( \big[ y^L_\alpha, y^U_\alpha \big] \) respectively.

2.1.2 SOLUTION PROCEDURE

When both the interarrival time and service time are fuzzy numbers, based on Zadeh’s extension principle [57, 63], the membership function of the performance measure \( p(\bar{\lambda}, \bar{\mu}) \) is defined as:

\[
\eta_{p(\bar{\lambda}, \bar{\mu})}(z) = \sup_{x \in X, y \in Y} \min \left\{ \eta_\lambda(x), \eta_\mu(y) / z = p(x, y) \right\} \quad (2.2)
\]
To construct the membership function $\eta_{\rho(\lambda, \bar{\mu})}$ is to drive the $\alpha$-cuts of $\eta_{\rho(\lambda, \bar{\mu})}$.

According to (2.2) we need either

\begin{enumerate}
  \item case (i) $\eta_{\lambda}(x) = \alpha$ and $\eta_{\mu}(y) \geq \alpha$
  \item or
  \item case (ii) $\eta_{\lambda}(x) \geq \alpha$ and $\eta_{\mu}(y) = \alpha$
\end{enumerate}

such that $z = \rho(\hat{\lambda}, \bar{\mu})$ to satisfy $\eta_{\rho(\lambda, \bar{\mu})}(z) = \alpha$. This is accomplished via the parametric programming technique. For the interarrival time, the corresponding parametric programs for finding the lower and upper bounds of the $\alpha$ - cuts of $\eta_{\rho(\lambda, \bar{\mu})}(z)$ for case (i) are

\begin{align}
L_{p(\alpha)} &= \min p(x, y) \\
\text{s.t} & \quad L_{\lambda(\alpha)} \leq x \leq U_{\lambda(\alpha)}, \\
& \quad y \in \mu(\alpha), \\
U_{p(\alpha)} &= \max p(x, y), \\
\text{s.t} & \quad L_{\lambda(\alpha)} \leq x \leq U_{\lambda(\alpha)}, \\
& \quad y \in \mu(\alpha)
\end{align}

and for the case (ii) are

\begin{align}
L_{p(\alpha)}^\prime &= \min p(x, y) \\
\text{s.t} & \quad L_{\mu(\alpha)} \leq y \leq U_{\mu(\alpha)}, \\
& \quad x \in \lambda(\alpha)
\end{align}
\[ U^*_p(\alpha) = \max p(x, y) \quad (2.3d) \]

s.t \[ L_\mu(\alpha) \leq y \leq U_\mu(\alpha), \]

\[ x \in \lambda(\alpha) \]

As per the definition of \( \lambda(\alpha) \) and \( \mu(\alpha) \) in (2.1a) and (2.1b), \( x \in \lambda(\alpha) \) and \( y \in \mu(\alpha) \) can be replaced by \( x \in [L_\lambda(\alpha), U_\lambda(\alpha)] \) and \( y \in [L_\mu(\alpha), U_\mu(\alpha)] \). Therefore (2.3a) and (2.3c), and (2.3b) and (2.3d) are the same, respectively, which can be rewritten as:

\[ L^*_p(\alpha) = \min p(x, y) \quad (2.4a) \]

s.t \[ L_\lambda(\alpha) \leq x \leq U_\lambda(\alpha) \]

\[ L_\mu(\alpha) \leq y \leq U_\mu(\alpha) \]

\[ U^*_p(\alpha) = \max p(x, y) \quad (2.4b) \]

s.t \[ L_\lambda(\alpha) \leq x \leq U_\lambda(\alpha) \]

\[ L_\mu(\alpha) \leq y \leq U_\mu(\alpha) \]

This pair of mathematical programs involve the systematic study of how the optimal solutions change as the bounds \( L_\lambda(\alpha) \) and \( U_\lambda(\alpha) \) vary over the interval \( \alpha \in (0, 1] \), they fall into the category of parametric programming [30].

If both \( L^*_p(\alpha) \) and \( U^*_p(\alpha) \) are invertible with respect to \( \alpha \), then a left shape function \( L(z) = L^{-1}_p(\alpha) \) and a right shape function \( R(z) = U^{-1}_p(\alpha) \) can be obtained, from which the membership function \( \eta_{p(\lambda, \mu)}(z) \) is constructed:
\( \eta_{\tilde{\lambda}, \tilde{\mu}}(z) = \begin{cases} 
L(z), & z_1 \leq z \leq z_2 \\
1, & z_2 \leq z \leq z_3 \\
R(z), & z_3 \leq z \leq z_4 
\end{cases} \) \hspace{1cm} (2.5)

where \( z_1 \leq z_2 \leq z_3 \leq z_4 \) and \( L(z_1) = R(z_4) = 0 \).

### 2.1.3 THE FM/FM/1 RETRIAL QUEUE WITH RETRIAL RATE BEING CRISP

Li and Lee [43] have studied a one server fuzzy queueing system denoted by FM/FM/1 in which arrivals and services are both poisson processes with fuzzy rates \( \tilde{\lambda} \) and \( \tilde{\mu} \). The membership function of expected number of customers in system for fuzzy retrial queue with crisp retrial rate is

\[
\eta_{E[\tilde{\lambda}], \tilde{\mu}}(z) = \sup_{x, y \in R} \min_{x/y < 1} \left\{ \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y) / z = \frac{x(x + \theta)}{(y - x)\theta} \right\} \hspace{1cm} (2.6)
\]

where \( \eta_{\tilde{\lambda}}, \eta_{\tilde{\mu}} \) are the membership functions of \( \tilde{\lambda}, \tilde{\mu} \) respectively and \( \theta \) the retrial rate being crisp. Although the membership functions in (2.6) are theoretically correct, they are not in the usual form for practical use. By applying the concept of \( \alpha \)-cut, the FM/FM/1 retrial queue reduces to the family of M/M/1 retrial queues. From the knowledge of retrial queueing theory, \( E[N_s] = [\lambda(\lambda + \theta)] / [\theta(\mu - \lambda)] \) is the expected number of customers in system, where \( \lambda, \mu \) and \( \theta \) are the arrival rate, service rate and retrial rate respectively.

The procedure described in section 2.1.2 can then be applied to find the membership functions of the performance measures.
2.1.4 NUMERICAL EXAMPLE

Consider an FM/FM/1 retrial queue, where both the arrival rate $\tilde{\lambda}$ and service rate $\tilde{\mu}$ are trapezoidal fuzzy numbers and retrial rate, $\theta$ is 17. Let $\tilde{\lambda} = [15, 16, 17, 18]$ and $\tilde{\mu} = [19, 20, 21, 22]$. The interval of confidence at possibility level $\alpha$ for $\tilde{\lambda}$ and $\tilde{\mu}$ is $[15+\alpha, 18-\alpha]$ and $[19+\alpha, 22-\alpha]$ respectively. Following (2.4a) and (2.4b), the parametric programs to derive the membership function for $E[\tilde{N}_j]$ are

\[
(E[N_j])^L_\alpha = \min \left[ \frac{x (x + \theta)}{\theta (y - x)} \right] \quad (2.7a)
\]

s.t \quad (15 + \alpha) \leq x \leq (18 - \alpha),

(19 + \alpha) \leq y \leq (22 - \alpha),

$\theta = 17$

\[
(E[N_j])^U_\alpha = \max \left[ \frac{x (x + \theta)}{\theta (y - x)} \right] \quad (2.7b)
\]

s.t \quad (15 + \alpha) \leq x \leq (18 - \alpha),

(19 + \alpha) \leq y \leq (22 - \alpha),

$\theta = 17$

When $x$ reaches its lower bound and $y$ reaches its upper bound, $\left[ \frac{x (x + \theta)}{\theta (y - x)} \right]$ attains its minimum. Consequently, the solution for (2.7a) is

\[
(E[N_j])^L_\alpha = \frac{[(15 + \alpha)(32 + \alpha)]}{17(7 - 2\alpha)}
\]
On the contrary, to maximize \( [x (x + \theta) / [\theta (y - x)] \) it is desired that \( x \) increases to its upper bound and \( y \) decreases to lower bound. In this case (2.7b) has the optimal solution.

\[
(E[N_s])^U = \frac{(18 + \alpha)(35 + \alpha)}{17}
\]

The inverse function of \((E[N_s])_L^U\) and \((E[N_s])_L^U\) exists, which gives the membership function \( \eta_{E[N_s]}(z) \) as

\[
\eta_{E[N_s]}(z) = \begin{cases} 
-(34z + 47) + (1156z^2 + 3672z + 289)^{1/2} & ; \frac{480}{119} \leq z \leq \frac{528}{85} \\
2 & ; \frac{528}{85} \leq z \leq \frac{228}{17} \\
1 - (34z - 53) - (1156z^2 - 3536z + 289)^{1/2} & ; \frac{228}{17} \leq z \leq \frac{630}{17} \\
2 & 
\end{cases}
\]

The above membership function is shown in the following Fig. 2.1

*Fig. 2.1 The membership function of the expected number of customers in the system*
2.2 FUZZY RETRIAL QUEUEING MODEL WITH TRIANGULAR FUZZY NUMBER

2.2.1 MODEL DESCRIPTION

We consider an FM/FM/1/1-(FR) [37] queueing system, where FR represents the fuzzified exponential retrial rate and the first 1 represents the single server and the second 1 represents the system capacity. In this queueing system the customers arrive at a service facility from outside at rate $\tilde{\lambda}$, where $\tilde{\lambda}$ is a fuzzy number. On arriving at customer’s service facility if the facility is not occupied, the customer enters the orbit and attempts service after an uncertain amount of time, called retrial time. Unless otherwise mentioned, the orbit capacity is assumed to be infinite. The successive retrial times are independent and identically distributed according to an exponential distribution with fuzzy retrial rate $\tilde{\theta}$. Arriving customers at the server form a single waiting line and are served in the order of their arrivals. The service time provided by a single server is exponentially distributed with fuzzy rate $\tilde{\mu}$.

In this model the arrival rate $\tilde{\lambda}$, retrial rate $\tilde{\theta}$, service rate $\tilde{\mu}$ are approximately known and can be represented by convex fuzzy sets. Note that a fuzzy set $\tilde{A}$ in its universal set $Z$ is convex if $\eta_{\tilde{A}} (\phi z_1 + (1-\phi) z_2) \geq \min \{ \eta_{\tilde{A}} (z_1), \eta_{\tilde{A}} (z_2) \}$, where $\eta_{\tilde{A}}$ is its membership function, $\phi \in [0,1]$, and $z_1, z_2 \in Z$. Let $\eta_{\tilde{\lambda}} (x)$, $\eta_{\tilde{\theta}} (v)$ and $\eta_{\tilde{\mu}} (y)$ denote the membership functions of $\tilde{\lambda}$, $\tilde{\theta}$ and $\tilde{\mu}$ respectively. We have,
\[ \tilde{\lambda} = \{ (x, \eta_\lambda(x)) / x \in X \}, \quad (2.8a) \]
\[ \tilde{\theta} = \{ (v, \eta_\theta(v)) / v \in V \}, \quad (2.8b) \]
\[ \tilde{\mu} = \{ (y, \eta_\mu(y)) / y \in Y \}, \quad (2.8c) \]

where \( X, V \) and \( Y \) are the crisp universal sets of the arrival, retrial and service rates respectively. Let \( p(x,v,y) \) denote the system characteristic of interest. When \( \tilde{\lambda}, \tilde{\theta}, \tilde{\mu} \) are fuzzy numbers, \( p(\tilde{\lambda}, \tilde{\theta}, \tilde{\mu}) \) is also a fuzzy number.

According to Zadeh’s extension principle [57,63], the membership function of the system characteristic \( p(\tilde{\lambda}, \tilde{\theta}, \tilde{\mu}) \) is defined as

\[
\eta_{p(\tilde{\lambda}, \tilde{\theta}, \tilde{\mu})}(z) = \sup_{x \in X, v \in V, y \in Y} \min \left\{ \eta_\lambda(x), \eta_\theta(v), \eta_\mu(y) / z = p(x,v,y) \right\} \quad (2.9)
\]

Assume that the system characteristic of interest is the expected waiting time in the system. From the knowledge of retrial queueing theory [19,28], if \( x/y < 1 \) we have the expected waiting time in the system for a crisp retrial queueing system, given by

\[
E[W_s] = \frac{x}{(y-x)} \left( 1 + \frac{1}{x} \right) \quad (2.10)
\]

Following (2.9) and (2.10), the membership function for the expected waiting time \( E[W_s] \) in the system is

\[
\eta_{E[W_s]}(z) = \sup_{x/y < 1} \min \left\{ \eta_\lambda(x), \eta_\theta(v), \eta_\mu(y) / z = \frac{x}{(y-x)} \left( 1 + \frac{1}{x} \right) \right\} \quad (2.11)
\]

Although the membership function is theoretically correct, it is not in usual form for practical use and it is very difficult to imagine its shape. In this
paper we approach the problem via mathematical programming technique. Parametric NLPs are developed to find the $\alpha$-cuts of $p(\bar{\lambda}, \bar{\theta}, \bar{\mu})$ based on the extension principle.

2.2.2 SOLUTION PROCEDURE

To re-express the membership function $\eta_{E[\tilde{W}_s]}(z)$ of $E[\tilde{W}_s]$ in an understandable and useable form, we adopt Zadeh’s approach, which relies on $\alpha$-cuts of $E[\tilde{W}_s]$. Definitions for the $\alpha$-cuts of $\bar{\lambda}, \bar{\theta},$ and $\bar{\mu}$ as crisp intervals are as follows:

$$
\bar{\lambda}(\alpha) = \left[ x^L_a, x^U_a \right] = \left[ \min_{x \in X} \{ x / \eta_{\bar{\lambda}}(x) \geq \alpha \}, \max_{x \in X} \{ x / \eta_{\bar{\lambda}}(x) \geq \alpha \} \right] \tag{2.12a}
$$

$$
\bar{\theta}(\alpha) = \left[ v^L_a, v^U_a \right] = \left[ \min_{v \in V} \{ v / \eta_{\bar{\theta}}(v) \geq \alpha \}, \max_{v \in V} \{ v / \eta_{\bar{\theta}}(v) \geq \alpha \} \right] \tag{2.12b}
$$

$$
\bar{\mu}(\alpha) = \left[ y^L_a, y^U_a \right] = \left[ \min_{y \in Y} \{ y / \eta_{\bar{\mu}}(y) \geq \alpha \}, \max_{y \in Y} \{ y / \eta_{\bar{\mu}}(y) \geq \alpha \} \right] \tag{2.12c}
$$

The constant arrival, retrial and service rates are shown as intervals when the membership functions are no less than a given possibility level for $\alpha$. As a result, the bounds of these intervals can be described as functions of $\alpha$ and can be obtained as $x^L_a = \min \eta^{-1}_{\bar{\lambda}}(\alpha), \ x^U_a = \max \eta^{-1}_{\bar{\lambda}}(\alpha), \ v^L_a = \min \eta^{-1}_{\bar{\theta}}(\alpha), \ v^U_a = \max \eta^{-1}_{\bar{\theta}}(\alpha), \ y^L_a = \min \eta^{-1}_{\bar{\mu}}(\alpha), \ y^U_a = \max \eta^{-1}_{\bar{\mu}}(\alpha)$. Therefore, we can use the $\alpha$-cuts of $E[\tilde{W}_s]$ to construct its membership function since the membership function defined in (2.11) is parameterized by $\alpha$. 

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From the extension principle stated in (2.11), $\eta_{E[\hat{w}_1]}(z)$ is the minimum of $\eta_{\lambda}(x)$, $\eta_{\theta}(v)$, and $\eta_{\mu}(y)$. To derive the membership function $\eta_{E[\hat{w}_1]}(z)$, one needs at least one of the following cases to hold such that:

$$Z = \frac{x}{y - x \left( \frac{1}{x} + \frac{1}{v} \right)}$$

satisfied $\eta_{E[\hat{w}_1]}(z) = \alpha$.

**Case (i):** $(\eta_{\lambda}(x) = \alpha, \eta_{\theta}(v) \geq \alpha, \eta_{\mu}(y) \geq \alpha)$,

**Case (ii):** $(\eta_{\lambda}(x) \geq \alpha, \eta_{\theta}(v) = \alpha, \eta_{\mu}(y) \geq \alpha)$,

**Case (iii):** $(\eta_{\lambda}(x) \geq \alpha, \eta_{\theta}(v) \geq \alpha, \eta_{\mu}(y) = \alpha)$.

This can be accomplished via the parametric NLP techniques. The NLP to find the lower and upper bounds of the $\alpha$-cut of $\eta_{E[\hat{w}_1]}$ for case (i) are

\begin{align*}
(E[W_j])_{a}^{L_1} &= \min_{x/y < 1} \left[ \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{v} \right) \right], \\
(E[W_j])_{a}^{U_1} &= \max_{x/y < 1} \left[ \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{v} \right) \right],
\end{align*}

for case (ii) are

\begin{align*}
(E[W_j])_{a}^{L_2} &= \min_{x/y < 1} \left[ \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{v} \right) \right], \\
(E[W_j])_{a}^{U_2} &= \max_{x/y < 1} \left[ \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{v} \right) \right],
\end{align*}
and for case (iii) are

\[
(E[W_j])_{\alpha}^{L} = \min_{x/y \in \mathbb{C}} \left[ \frac{x}{(y - x)} \left( \frac{1}{x} + \frac{1}{v} \right) \right],
\]

(2.13e)

\[
(E[W_j])_{\alpha}^{U} = \max_{x/y \in \mathbb{C}} \left[ \frac{x}{(y - x)} \left( \frac{1}{x} + \frac{1}{v} \right) \right].
\]

(2.13f)

For the definitions of \( \lambda(\alpha) \), \( \theta(\alpha) \) and \( \mu(\alpha) \) in (2.12a, 2.12b, 2.12c), \( x \in \lambda(\alpha) \), \( v \in \theta(\alpha) \) and \( y \in \mu(\alpha) \) can be replaced by \( x \in [x_{a1}^L, x_{a2}^U] \), \( \theta \in [v_{a1}^L, v_{a2}^U] \) and \( y \in [y_{a1}^L, y_{a2}^U] \). The \( \alpha \) - cuts form a nested structure with respect to \( \alpha \) [36, 64]; i.e., given \( 0 < \alpha_0 < \alpha_1 \leq 1 \), we have \( [x_{a1}^L, x_{a1}^U] \subseteq [x_{a2}^L, x_{a2}^U] \), \( [v_{a1}^L, v_{a1}^U] \subseteq [v_{a2}^L, v_{a2}^U] \) and \( [y_{a1}^L, y_{a1}^U] \subseteq [y_{a2}^L, y_{a2}^U] \). Therefore (2.13a), (2.13c) and (2.13e) have the same smallest element and (2.13b), (2.13d) and (2.13f) have the same largest element. To find the membership function \( \eta_{E[W_j]} \) it suffices to find the left and right shape functions of \( \eta_{E[W_j]} \), which is equivalent to finding the lower bound \( (E[W_j])_{\alpha}^{L} \) and upper bound \( (E[W_j])_{\alpha}^{U} \) of the \( \alpha \) - cuts of \( E[\tilde{W}_j] \), which, based on (2.10), can be rewritten as

\[
(E[W_j])_{\alpha}^{L} = \min_{x/y \in \mathbb{C}} \left[ \frac{x}{(y - x)} \left( \frac{1}{x} + \frac{1}{v} \right) \right] \quad \text{s.t.} \quad x_{a}^L \leq x \leq x_{a}^U, \quad v_{a}^L \leq v \leq v_{a}^U \quad \text{and} \quad y_{a}^L \leq y \leq y_{a}^U,
\]

(2.14a)

\[
(E[W_j])_{\alpha}^{U} = \max_{x/y \in \mathbb{C}} \left[ \frac{x}{(y - x)} \left( \frac{1}{x} + \frac{1}{v} \right) \right] \quad \text{s.t.} \quad x_{a}^L \leq x \leq x_{a}^U, \quad v_{a}^L \leq v \leq v_{a}^U \quad \text{and} \quad y_{a}^L \leq y \leq y_{a}^U.
\]
Atleast one of $x$, $v$ or $y$ must hit the boundaries of their $\alpha$-cuts to satisfy $\eta_{E[W_s]}(z) = \alpha$. This model is a set of mathematical programs with boundary constraints and lends itself to the systematic study of how the optimal solutions change with $x^L_\alpha$, $x^U_\alpha$, $v^L_\alpha$, $v^U_\alpha$, $y^L_\alpha$ and $y^U_\alpha$ as $\alpha$ varies over $(0,1]$. The model is a special case of parametric NLPs [30].

The crisp interval $[(E[W_s])^L_\alpha, (E[W_s])^U_\alpha]$, obtained from (2.14a), (2.14b) represents the $\alpha$-cuts of $E[\hat{W}_s]$. Again by applying the results of Zimmermann [64] and Kaufmann [26] and convexity properties to $E[\hat{W}_s]$, we have $(E[W_s])^L_{\alpha_1} \geq (E[W_s])^L_{\alpha_2}$ and $(E[W_s])^U_{\alpha_1} \leq (E[W_s])^U_{\alpha_2}$ where $0 < \alpha_2 < \alpha_1 \leq 1$. In other words, $(E[W_s])^L_\alpha$ increases and $(E[W_s])^U_\alpha$ decreases as $\alpha$ increases. Consequently, the membership function $\eta_{E[\hat{W}_s]}(z)$ can be found from (2.14a) and (2.14b).

If both $(E[W_s])^L_\alpha$ and $(E[W_s])^U_\alpha$ in Eq. (2.14a) and (2.14b) are invertible with respect to $\alpha$, then a left shape function $L(z) = [(E[W_s])^L_\alpha]^{-1}$ and a right shape function $R(z) = [(E[W_s])^U_\alpha]^{-1}$ can be derived, from which the membership function $\eta_{E[\hat{W}_s]}$ is constructed:

$$\eta_{E[\hat{W}_s]}(z) = \begin{cases} 
L(z) & (E[W_s])^L_{\alpha=0} \leq z \leq (E[W_s])^L_{\alpha=1} \\
R(z) & (E[W_s])^U_{\alpha=1} \leq z \leq (E[W_s])^U_{\alpha=0} \end{cases}$$ (2.15)

In most cases, the values of $(E[W_s])^L_\alpha$ and $(E[W_s])^U_\alpha$ cannot be solved analytically. Consequently, a closed–form membership function for $E[\hat{W}_s]$ cannot be
obtained. However, the numerical solutions for \((E[W_s])_\alpha^L\) and \((E[W_s])_\alpha^U\) at different possibility levels can be collected to approximate the shape of \(L(z)\) and \(R(z)\). That is, the set of interval \(\{ (E[W_s])_\alpha^L, (E[W_s])_\alpha^U / \alpha \in [0,1] \}\) shows the shape of \(\eta_{E[W_s]}\) although the exact function is not known explicitly.

Since the performance measures are described by membership function, the values conserve completely all of fuzziness of arrival rate, service rate and retrial rate. However, the managers or practitioners would prefer only crisp value for one of performance measures rather than a fuzzy set. In order to overcome this problem, we defuzzify the fuzzy values of performance measures. The recommended suitable values of performance measures are calculated by

\[
O(\tilde{A}) = \frac{a_1 + 2a_2 + a_3}{4}
\]

where \(\tilde{A} = [a_1, a_2, a_3]\) is a triangular fuzzy number.

### 2.2.3 Numerical Example

Consider an FM/FM/1/1-(FR) queueing system, where the arrival, retrial and service rates are triangular fuzzy numbers represented by \(\lambda = [8, 9, 10]\), \(\theta = [1, 5, 9]\) and \(\mu = [12, 13, 14]\). First it is easy to find that \([x_\alpha^L, x_\alpha^U] = [8 + \alpha, 10 - \alpha]\), \([v_\alpha^L, v_\alpha^U] = [1 + 4\alpha, 9 - 4\alpha]\) and \([y_\alpha^L, y_\alpha^U] = [12 + \alpha, 14 - \alpha]\).

Next, it is obvious that when \(x = x_\alpha^U\), \(v = v_\alpha^L\) and \(y = y_\alpha^U\), the expected waiting time in the system attains its maximum values, and when \(x = x_\alpha^L\), \(v = v_\alpha^U\) and
\[ y = y^U_\alpha \], the expected waiting time in the system attains its minimum value.

According to (2.14a) and (2.14b) the \( \alpha \)-cuts of \( E[W_s] \) are

\[
(E[W_s])^L_\alpha = \frac{-3\alpha + 17}{8\alpha^2 - 42\alpha + 54}
\]

\[
(E[W_s])^U_\alpha = \frac{3\alpha + 11}{8\alpha^2 + 10\alpha + 2}
\]

With the help of MATLAB\textsuperscript{®} 7.0, the inverse functions of \((E[W_s])^L_\alpha\) and \((E[W_s])^U_\alpha\) exist, yielding the membership function

\[
\eta_{E[W_s]}(z) = \begin{cases} 
\frac{(-3 + 42z) - (36z^2 + 292z + 9)^{1/2}}{16z} & \text{if } \frac{17}{54} \leq z \leq \frac{7}{10} \\
\frac{(-10z - 3) + (36z^2 + 292z + 9)^{1/2}}{16z} & \text{if } \frac{7}{10} \leq z \leq \frac{11}{2}
\end{cases}
\]

The above membership function is shown in Fig. 2.2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig22.png}
\caption{Membership function of \( E[W_s] \)}
\end{figure}
Applying defuzzification, the suitable expected waiting time in the system for this example is given by $\lambda = 10$, $\mu = 13$, $\theta = 5$ and $E[\hat{W}_s] = 0.7$.

Next, we perform $\alpha$ - cuts of arrival, retrial and service rates and fuzzy expected waiting time in the system at eleven distinct $\alpha$ values: 0.0, 0.1, 0.2, ..., 1.0

Crisp intervals for fuzzy waiting time at different possibilistic $\alpha$ levels are presented in Table 2.1.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$x_a^L$</th>
<th>$x_a^U$</th>
<th>$v_a^L$</th>
<th>$v_a^U$</th>
<th>$y_a^L$</th>
<th>$y_a^U$</th>
<th>$(E[W_s])_a^L$</th>
<th>$(E[W_s])_a^U$</th>
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<tr>
<td>0.00</td>
<td>8</td>
<td>10</td>
<td>1</td>
<td>9</td>
<td>12</td>
<td>14</td>
<td>0.3148</td>
<td>5.5</td>
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<td>9.9</td>
<td>1.4</td>
<td>8.6</td>
<td>12.1</td>
<td>13.9</td>
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<td>3.6688</td>
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<td>6.6</td>
<td>12.6</td>
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</table>

The fuzzy expected waiting time $E[\hat{W}_s]$ has two characteristics to be noted. First, the support of $E[\hat{W}_s]$ ranges from 0.3148 to 5.5; this indicates that, though the expected waiting time is fuzzy, it is impossible for its values to fall below 0.3148 or exceed 5.5. Second, the $\alpha$ - cut at $\alpha = 1$ contains the value 0.7, which is the most possible value for the expected waiting time in the system.