Appendices
ANNEXURE - I

Coding to find the inverse of a function for the construction of the membership functions using MATLAB® 7.0 Software.

Let the lower bound of the $\alpha$-cuts of any performance measure be represented as

$$f(x) = \frac{g(x)}{h(x)}$$

```matlab
syms x;
num1 = g(x);
denom1 = h(x);
f1 = num1/denom1;
L = finverse(f1)
```
Coding to find Yager Ranking index using MATLAB® 7.0 Software.

Yager ranking index is calculated for the convex fuzzy number from its \( \alpha \)-cuts (i.e.) \([\underline{p}_\alpha^L, \overline{p}_\alpha^U]\). Let \( \underline{p}_\alpha^L \) and \( \overline{p}_\alpha^U \) be denoted in terms of \( x \) as

\[
\underline{p}_\alpha^L = \frac{a_1(x)}{b_1(x)} \quad \text{and} \quad \overline{p}_\alpha^U = \frac{a_2(x)}{b_2(x)}. 
\]

```matlab
syms x;
g = [a_1(x)/b_1(x) + a_2(x)/b_2(x)];
h = simplify(g);
k = int(1/2 * h, 0, 1)
```
ANNEXURE - III

Coding to draw the graph for trapezoidal fuzzy number

Let the membership function $\eta(x)$ of trapezoidal fuzzy number be defined as

$$
\eta(x) = \begin{cases} 
L(x) & \text{if } z_1 \leq x \leq z_2 \\
1 & \text{if } z_2 \leq x \leq z_3 \\
R(x) & \text{if } z_3 \leq x \leq z_4
\end{cases}
$$

where $L(z_1) = R(z_4) = 0$

```matlab
syms x;

x = z_1:0.001:z_2;
y_1 = L(x);
plot (x, y_1);
hold on;

x = z_2:0.001:z_3;
y_2 = 1;
plot (x, y_2);
hold on;

x = z_3:0.001:z_4;
y_3 = R(x);
plot (x, y_3);
hold off
```
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EDITOR
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PARAMETRIC PROGRAMMING TO THE ANALYSIS OF FUZZY RETRIAL QUEUES

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Abstract: This paper proposes a general procedure to construct the membership functions of the performance measures in an M/M/1 retrial queue. When the inter arrival time and service time are fuzzy numbers and retrial rate being crisp. By applying the α-cut approach, a fuzzy retrial queue is reduced to a family of a crisp retrial queue. To describe that family of crisp retrial queues, a pair of parametric programs is formulated through which membership functions of the performance measures are derived. The procedure is illustrated for the FM/FM/1 fuzzy retrial queue.

Key words: Retrial Queue, Parametric Programming, Membership functions.

I. INTRODUCTION

Queuing model is the mathematical study of “queues” or “waiting lines”. A queue is formed whenever the demand for service exceeds the capacity to provide service at that point in time [1]. Queuing models have wide applications in service organizations as well as manufacturing firms, in that various types of customers are serviced by various types of servers according to specific queue disciplines [3].

If a telephone caller when dialing a number, gets a busy signal, the caller repeats his call after a random amount of time; other callers also do the same. While trying, a caller who finds the facility free immediately gets the facility. These considerations bring into focus the need for a new queuing system, which is called the retrial queuing system.

Queuing system with repeated attempts (trials) are characterized by the phenomenon that a customer finding all the servers busy upon arrival is obliged to leave the service area and repeat his request for service after some random time. Between trials, the blocked customer joins a pool of unsatisfied customers called “orbit”. Retrial queue have been widely used to model many problems a rising in telephone switching system, telecommunication network, computer networks and computer systems etc [2, 4, 14].

Li and Lee [6] have derived analytical results for two fuzzy queuing systems based on Zadeh’s extension principle [7] the possibility concept, and fuzzy Markov Chains [8]. However their approach is fairly complicated and is generally unsuitable for computational purpose. Therefore Negi and Lee [9] proposed two approaches the α-cut and two-variable simulation to analyze fuzzy queues. Unfortunately, their approach provides only possible numbers rather than intervals.

Aiming at the goal of deriving the membership functions of the performance measures for fuzzy retrial queues, this paper adopts the α-cut approach to decompose a fuzzy retrial queue to a crisp retrial queues. As
the $\alpha$ value varies, the parametric programming technique is applied to describe that family of crisp retrial queues. The solutions from the parametric programs derive the membership functions of the performance measures [10]. To demonstrate the validity of the proposed approach, typical fuzzy retrial queue, namely FM/FM/1 with retrial rate being crisp is exemplified. FM denotes fuzzified exponential time.

2. Model Description

We consider a single-serve retrial queue. New customers arrive from outside the system according to a poison process with rate $\lambda$. We assume that there is no waiting space and therefore if an arriving customer finds the server busy or down, the customer leaves the service area and enters a group of blocked customers called “orbit” in accordance with an FCFS discipline. That is, only the customer at the head of the orbit queue is allowed for access to server, after exponentially retrial time with constant rate $\theta$. We consider M/M/1 retrial queue in which the arrival stream is poisson (with rate $\lambda$), service time is exponential (with rate $\mu$), and retrial time is also exponential (with rate $\theta$).

3. Problem Formulation

Consider a single server retrial queuing system in which the retrial rate $\theta$ is being crisp, inter arrival time $\tilde{A}$ and service time $\tilde{S}$ are approximately known and are represented by the following fuzzy sets:

\[
\tilde{A} = \{(a, \mu_{\tilde{A}}(a) \mid a \in x)\} \quad (1a)
\]

\[
\tilde{S} = \{(s, \mu_{\tilde{S}}(s) \mid s \in y)\} \quad (1b)
\]

where $x$ and $y$ are the crisp universal sets of the inter arrival time and service time, and are the corresponding membership functions. The $\alpha$-cuts or $\alpha$-level sets of $\tilde{A}$ and $\tilde{S}$ are

\[
A(\alpha) = \{a \in x \mid \mu_{\tilde{A}}(a) \geq \alpha\} \quad (2a)
\]

\[
S(\alpha) = \{s \in y \mid \mu_{\tilde{S}}(s) \geq \alpha\} \quad (2b)
\]

Hence $A(\alpha)$ and $S(\alpha)$ are crisp sets using $\alpha$-cuts the inter arrival time and service time can be represented by different levels of confidence intervals [11]. Consequently, a fuzzy retrial queue can be reduced to a family of crisp retrial queues with different $\alpha$-level sets $A(\alpha) \mid 0 < \alpha \leq 1$ and $S(\alpha) \mid 0 < \alpha \leq 1$. These two sets represent sets of movable boundaries, and they form nested structures for expressing the relationship between ordinary sets and fuzzy sets [12].

Let $\tilde{A}$ and $\tilde{S}$ are fuzzy numbers. Denoting their intervals of confidence as $[l_{A(\alpha)}, u_{A(\alpha)}]$ and $[l_{S(\alpha)}, u_{S(\alpha)}]$, the idea of Negi and Lee [9] is to assume a uniform distribution in the interval of confidence and derive different performance measures analytically for different possibility level $\alpha$. In this paper we use the parametric programming technique to formulate the problem. From the solutions, the membership functions of interests can be constructed [10]. We discuss the model, for two fuzzy variables.

3.2. Two Fuzzy Variables

When both the inter arrival time and service time area fuzzy numbers, based on Zadeh’s extension principle, the membership function of the performance measure $p(\tilde{A}, \tilde{S})$ is defined as:

\[
\mu_{p(\tilde{A},\tilde{S})}(z) = \sup_{a \in x, s \in y} \min\{\mu_{\tilde{A}(a)}, \mu_{\tilde{S}}(s) \mid z = p(a, s)\} \quad (3)
\]

To construct the membership function $\mu_{p(\tilde{A},\tilde{S})}$ is to derive the $\alpha$-cuts of $\mu_{p(\tilde{A},\tilde{S})}$. According to (3), we need either $\mu_{\tilde{A}}(a) = \alpha$ and $\mu_{\tilde{S}}(s) \geq \alpha$ or $\mu_{\tilde{A}}(a) \geq \alpha$ and $\mu_{\tilde{S}}(s) = \alpha$ such that $Z = p(a, s)$ to satisfy $\mu_{p(\tilde{A},\tilde{S})}(z) = \alpha$. This is accomplished via the parametric programming technique. For the inter arrival
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In this context, the corresponding parametric programs for finding lower and upper bounds of the \( \alpha \)-cut of \( \mu_{p(A,S)}(z) \) are:

\[
\begin{align*}
l_p^{\alpha}(a) &= \min_p(p(a, s) \text{ s.t. } l_{A(\alpha)} \leq a \leq u_{A(\alpha)}) \quad (4a) \\
l_p^{\alpha}(a) &= \max_p(p(a, s) \text{ s.t. } l_{S(\alpha)} \leq s \leq u_{S(\alpha)}) \quad (4b)
\end{align*}
\]

For the service time:

\[
\begin{align*}
l_p^{\alpha}(s) &= \min_p(p(a, s) \text{ s.t. } l_{A(\alpha)} \leq a \leq u_{A(\alpha)}) \quad (4c) \\
l_p^{\alpha}(s) &= \max_p(p(a, s) \text{ s.t. } l_{S(\alpha)} \leq s \leq u_{S(\alpha)}) \quad (4d)
\end{align*}
\]

As per the definition of \( A(\alpha) \) and \( S(\alpha) \) in (2), \( a \in A(\alpha) \) and \( s \in S(\alpha) \) can be replaced by \( a \in [l_{A(\alpha)}, u_{A(\alpha)}] \) and \( s \in [l_{S(\alpha)}, u_{S(\alpha)}] \). Therefore (4a) and (4c), and (4b) and (4d) are the same, respectively, which can be rewritten as:

\[
\begin{align*}
l_{p(\alpha)} &= \min_p(p(a, s) \\
\text{s.t. } l_{A(\alpha)} \leq a \leq u_{A(\alpha)} \quad (5a) \\
l_{S(\alpha)} \leq s \leq u_{S(\alpha)} \\
U_{I(\alpha)} &= \max_p(p(a, s) \\
\text{s.t. } l_{A(\alpha)} \leq a \leq u_{A(\alpha)} \quad (5b) \\
l_{S(\alpha)} \leq s \leq u_{S(\alpha)}
\end{align*}
\]

This pair of mathematical programs involve the systematic study of how the optimal solutions change as the bounds \( l_{A(\alpha)} \) and \( u_{A(\alpha)} \) vary over the interval \( \alpha \in (0,1] \), they fall into the category of parametric programming [13]. If both \( l_{p(\alpha)} \) and \( u_{p(\alpha)} \) are invertible with respect to \( \alpha \), then a left shape function \( L(z) = l_{p(\alpha)}^{-1} \) and a right shape function \( R(z) = u_{p(\alpha)}^{-1} \) can be obtained, from which the membership function \( \mu_{p(A,S)}(z) \) is constructed:

\[
\mu_{p(A,S)}(z) = \begin{cases} 
L(z) & z_1 \leq z \leq z_2 \\
1 & z_2 \leq z \leq z_3 \\
R(z) & z_3 \leq z \leq z_4 
\end{cases}
\]

where \( z_1, z_2, z_3, z_4 \) and \( L(z) = R(z) = 0 \).

4. THE FM/FM/1 RETRIAL QUEUE, WITH RETRIVAL RATE BEING CRISP

Li and Lee [6] have studied a one-server queuing system denoted by FM/FM/1, in which arrivals and departures are both poison processes with fuzzy rates \( \bar{\Lambda} \) and \( \bar{\mu} \). The membership function for the expected number of persons in the system \( \bar{L} \) and the expected waiting time in the system \( \bar{W} \) are:

\[
\mu_{\bar{L}}(z) = \sup_{x,y \in \mathbb{R}^+} \min \{ \mu_{\bar{\Lambda}}(x), \mu_{\bar{\mu}}(y) \mid z = [x(x + \theta)]/(y - x) \theta \} \quad (7a)
\]

\[
\mu_{\bar{W}}(z) = \sup_{x,y \in \mathbb{R}^+} \min \{ \mu_{\bar{\Lambda}}(x), \mu_{\bar{\mu}}(y) \mid z = [y(y - x)][(\theta + x)/y\theta] \} \quad (7b)
\]

where \( \mu_{\bar{\Lambda}} \) and \( \mu_{\bar{\mu}} \) are the membership functions of \( \bar{\Lambda} \) and \( \bar{\mu} \) and \( \theta \) the retrial rate being crisp. Although the membership function in (7) are theoretically correct, they are not in the usual form for practical use. By applying the concept of \( \alpha \)-cut, the FM/FM/1 retrial queue reduces to a family of M/M/1
retrial queues. From the knowledge of retrial queuing theory, we have \( W = (\mu/(\mu - \lambda))/((\theta + \lambda)/\mu\theta) \) and \( L = [\lambda(\lambda + \theta)]/[(\mu - \lambda)\theta] \) where \( \lambda \) and \( \mu \) are the crisp arrival and service rates and \( \theta \) being crisp retrial rate. The procedure described in section 3.2 can then be applied to find the membership functions of the performance measures.

Illustration

Consider an FM/FM/1 retrial queue, where both the arrival rate \( \bar{\lambda} \), and service rate \( \bar{\mu} \), are fuzzy numbers and retrial rate, \( \theta \) is 17. Let \( \bar{\lambda} = [15, 16, 17, 18] \) and \( \bar{\mu} = [19, 20, 21, 22] \). From these we can derive the interval of confidence at possibility level \( \alpha \) as \([15 + \alpha, 18 - \alpha]\) and \([19 + \alpha, 20 - \alpha]\). Following (5), the parametric programs to derive the membership function for \( \bar{L} \) are

\[
\begin{align*}
I_{L(\alpha)}(x) &= \min\{x(x + \theta)/(\mu - \theta)\} \\
\text{s.t.} & \quad (15 + \alpha) \leq x \leq (18 - \alpha) \\
& \quad (19 + \alpha) \leq y \leq (22 - \alpha), \theta = 17, \\
& \quad u_{L(\alpha)}(x) = \max\{x(x + \theta)/(\mu - \theta)\} \\
\text{s.t.} & \quad (15 + \alpha) \leq x \leq (18 - \alpha) \\
& \quad (19 + \alpha) \leq y \leq (22 - \alpha), \theta = 17.
\end{align*}
\]

When \( x \) reaches its lower bound and \( y \) reaches its upper bound, \( x(x + \theta)/(\mu - \theta) \) attains its minimum. Consequently, the optimal solution for (8a) is\( I_{L(x)} = [(15 + \alpha)(32 + \alpha)]/17(7 - 2\alpha) \). On the contrary, to maximize \( x(x + \theta)/(\mu - \theta) \) it is desired that \( x \) increases to its upper bound and \( y \) decreases to lower bound. In this case (8b) has the optimal solution \( u_{L(\alpha)} = [(18 + \alpha)(35 + \alpha)]/17 \).

The inverse function of \( I_{L(\alpha)} \) and \( u_{L(\alpha)} \) exists, which give the membership function \( \mu_{L(\alpha)} \) as:

\[
\begin{align*}
\mu_{L(\alpha)}(x) = \begin{cases} 
\frac{1}{2} & \frac{450}{17} \leq x \leq \frac{536}{17} \\
\frac{1}{2} & \frac{536}{17} \leq x \leq \frac{639}{17} \\
\frac{1}{2} & \frac{639}{17} \leq x \leq \frac{742}{17} \\
\end{cases}
\end{align*}
\]

The membership function for \( \bar{L} \) in illustration is constructed for \( \alpha \) values as shown in figure 4.1.
5. CONCLUSION

Fuzzy sets theory has been applied to some classical queuing systems to provide wider applications in some previous studies [6, 9]. When the inter arrival time and service time are fuzzy variables, according to Zadeh’s extension principle, the performance measures such as the average system length, the average waiting time, etc., will be fuzzy as well. This paper applies the concept of \( \alpha \)-cut to reduce a fuzzy retrial queue into a family of crisp retrial queue which can be described by a pair of parametric programs to find the \( \alpha \)-cuts of the membership functions of the performance measures. With the \( \alpha \)-cuts, the corresponding membership function is derived consequently. Since the performance measures are expressed by membership function rather than possible values as derived in other studies, they provide more information.

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Mixed Integer Nonlinear Programming Approach to Fuzzy Retrial Queue

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Abstract—This paper develops a nonlinear programming approach to derive the membership functions of the steady-state performance measures in retrial queuing model with fuzzy arrival, retrial and service rates. The basic idea is based on Zadeh’s extension principle. Three pairs of mixed integer nonlinear programs (MINLP) with binary variables are formulated to calculate the upper and lower bounds of the system performance measure at possibility level \(\alpha\). From different values of \(\alpha\), the membership function of the system performance measure is constructed. For practice use, the defuzzification of performance measures is also provided via Yager ranking index. To demonstrate the validity of the proposed method, a numerical example is solved successfully. Since the system characteristics are expressed and governed by the membership functions, more information is provide for use by management. By extending this model to the fuzzy environment, fuzzy retrial-queue is represented more accurately and analytic results are more useful for system designers and practitioners.

Keywords—Fuzzy parameters, Mixed integer nonlinear programming, Retrial queue.

1. INTRODUCTION

Retrial queuing models are characterized by the feature that arriving customer who finds the server busy is obliged to leave the service area and join a pool of unsatisfied customers called the ‘orbit’. From the orbit, each customer reappears for service after a random amount of time. Retrial queues are widely used as mathematical models of several computer systems; packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol and collision avoidance in local area networks. Review of retrial queue literature could be found in Ref [1-4]. A number of applications of retrial queues in science and engineering can be found in Ref [5].

In crisp environments, many articles on retrial system have been published [6-8]. Most of the related studies are based on traditional queuing theory, in that the inter-arrival times, inter-retrial times and service times of customers are assumed to follow certain probability distributions with fixed parameters. However in practice there are cases that these parameters may be obtained subjectively. Thus fuzzy retrial queues would be potentially much more useful and realistic than the commonly used crisp retrial queues [9-10]. If the usual crisp retrial queues can be extended to fuzzy retrial queues, these retrial queuing models would have wider applications.

Relatively few articles have been published on the topic of fuzzy queues. Based on Zadeh’s extension principle [11-13] and fuzzy Markov chain [14], Li and Lee [9] proposed a general approach for analyzing fuzzy queues. A useful modeling and inferential technique would be to apply their approach to general fuzzy queuing problems. However, their approach is complicated and not suitable for computational purposes; moreover, it cannot easily be used to derive analytical results for other complicated queuing systems. Negi and Lee [15] proposed a procedure using two-variable simulation [16] and the \(\alpha\)-cut concept to analyze fuzzy queues. Unfortunately, their approach provides only crisp solutions, i.e. it does not fully describe the membership functions of the system characteristics Kao et al [17] constructed the membership functions of the system characteristics for fuzzy queues and successfully applied them to four simple fuzzy queue models: M [F] 1 \(\alpha\), F [M] 1 \(\alpha\), F [F] 1 \(\alpha\) and FM [FM] 1 \(\alpha\). Recently, Chen [18, 19] developed FM [FM] 1 \(L\) and FM [FM] 1 \(\alpha\) fuzzy systems using the same approach.

In this paper, we develop an approach that provides system characteristics for the retrial queues with three fuzzy variables: fuzzified exponential arrival, retrial and service rates. The membership functions of system characteristics can be derived completely. The basic idea is to apply Zadeh’s extension principle [11-13]. Three pairs of mixed integer non-linear programming (MINLP) models [20] are formulated to calculate the lower and upper bounds of the \(\alpha\)-cut of the system characteristic. The membership function of the system characteristic is derived analytically.
The remainder of this paper is organized as follows. Section 2 presents the system characteristics of standard and fuzzy retrial queuing models. In section 3, a mathematical programming approach is developed to derive the membership functions of the system characteristics. To demonstrate the validity of the proposed approach, a realistic numerical example is described and solved. Discussion is provided in section 4 and conclusions are drawn in section 5. For notational convenience, our model in this paper is hereafter denoted as FM| FM| 1| 1-(FR), [21] where FR represents the fuzzified exponential retrial rate, and the first 1 represents the single server and the second 1 represents the system capacity.

2. FUZZY RETRIAL QUEUES

Consider an FM| FM| 1| 1-(FR) queuing system in which customers arrive at a single-server facility as a poisson process with arrival rate \( \tilde{\lambda} \), where \( \tilde{\lambda} \) is a fuzzy number. An arriving customer enters the service facility if the server is not occupied. Otherwise the customer enters the orbit and attempts service after an uncertain amount of time, called retrial time. Unless otherwise mentioned, the orbit capacity is assumed to be infinite. The successive retrial times are independent and identically distributed according to an exponential distribution with fuzzy retrial rate \( \tilde{\theta} \). Arriving customers are served in accordance with an FCFS discipline. The service time provided by a single server is exponentially distributed with fuzzy rate \( \tilde{\mu} \).

In this model the arrival rate \( \tilde{\lambda} \), retrial rate \( \tilde{\theta} \), and service rate \( \tilde{\mu} \) are approximately known and can be represented by convex fuzzy sets. Note that a fuzzy set \( \tilde{A} \) in its universal set \( Z \) is convex if \( \mu_{\tilde{A}}(\phi Z_1 + (1-\phi)Z_2) \geq \min\{\mu_{\tilde{A}}(Z_1), \mu_{\tilde{A}}(Z_2)\} \), where \( \mu_{\tilde{A}} \) is its membership function, \( \phi \in [0,1] \) and \( Z_1, Z_2 \in Z \). Let \( \eta_x(x) \), \( \eta_v(v) \) and \( \eta_y(y) \) denote the membership functions of \( \tilde{\lambda} \), \( \tilde{\theta} \) and \( \tilde{\mu} \) respectively. We have

\[
\tilde{\lambda} = \{(x, \eta_x(x)) | x \in X\}
\]

\[
\tilde{\theta} = \{(v, \eta_v(v)) | v \in V\}
\]

\[
\tilde{\mu} = \{(y, \eta_y(y)) | y \in Y\}
\]

(1a) \hspace{2cm} (1b) \hspace{2cm} (1c)

where \( x, v \) and \( y \) are the crisp universal sets of the arrival, retrial and service rates respectively. Let \( p(x, v, y) \) denote the system characteristic of interest. Clearly, when \( \tilde{\lambda}, \tilde{\theta} \) and \( \tilde{\mu} \) are fuzzy numbers, \( p(\tilde{\lambda}, \tilde{\theta}, \tilde{\mu}) \) will be fuzzy number as well. On the basis of Zadeh's extension principle [11, 12], the membership function of the system characteristic \( p(\tilde{\lambda}, \tilde{\theta}, \tilde{\mu}) \) is defined as

\[
\eta_{p(\tilde{\lambda}, \tilde{\theta}, \tilde{\mu})}(z) = \sup_{x \in X, v \in V, y \in Y} \min\{\eta_x(x), \eta_v(v), \eta_y(y)/z = p(x, v, y)\}
\]

(2)

If the \( \alpha \)-cuts of \( p(\tilde{\lambda}, \tilde{\theta}, \tilde{\mu}) \) at all \( \alpha \) values degenerate to the same point, then the value of the system characteristic is a crisp number. Otherwise, it is a fuzzy number. Assume that the system characteristic of interest is the expected waiting time and the expected number of customers in the system as well as queue. From the knowledge of retrial queuing theory [2, 5], if \( x/y < 1 \) we have the expected waiting time and the expected number of customers in the queue and the system for a crisp retrial queuing system, respectively, given by

\[
E[\tilde{W}_q] = \frac{(x)}{(y-x)} \left[ 1 + \frac{1}{x} \right],
\]

(3a)

and

\[
E[\tilde{N}_q] = \frac{x^2}{(y-x)} \left[ 1 + \frac{1}{x} \right]
\]

(3b)

\[
E[\tilde{W}_s] = \frac{1}{y-x} \left[ 1 + \frac{x}{y} \right],
\]

(3c)

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\[ E[ \bar{N}_a] = \frac{x}{y - x} \left( 1 + \frac{x}{v} \right) \]  

(3d)

Following (2) and (3a), the membership function for the expected waiting time \( E[W] \) in the queue is

\[ \eta_{\text{EW}_{a}}(z) = \text{Sup}_{x/y < 1} \min \left\{ \eta_{i}(x), \eta_{b}(v), \eta_{e}(y)/z = \frac{x}{(y - x)} \left( 1 + \frac{1}{x/v} \right) \right\} \]  

(4a)

Likewise, the membership function for the expected number of customers in the queue \( E[N] \) can be obtained from (2) and (3b) by the following

\[ \eta_{\text{EN}_{a}}(z) = \text{Sup}_{x/y < 1} \min \left\{ \eta_{i}(x), \eta_{b}(v), \eta_{e}(y)/z = \frac{x^2}{(y - x)} \left( 1 + \frac{1}{x/v} \right) \right\} \]  

(4b)

Similarly, the membership function for the expected waiting time and expected number of customers in the system are as follows,

\[ \eta_{\text{EWS}_{a}}(z) = \text{Sup}_{x/y < 1} \min \left\{ \eta_{i}(x), \eta_{b}(v), \eta_{e}(y)/z = \frac{1}{y - x} \left( 1 + \frac{x}{v} \right) \right\} \]  

(4c)

\[ \eta_{\text{ENS}_{a}}(z) = \text{Sup}_{x/y < 1} \min \left\{ \eta_{i}(x), \eta_{b}(v), \eta_{e}(y)/z = \frac{x}{(y - x)} \left( 1 + \frac{x}{v} \right) \right\} \]  

(4d)

Theoretically, these membership functions are correct; however, they are not in the usual forms for practical use. Moreover, it is even very difficult to imagine their shapes.

3. THE SOLUTION PROCEDURE

One approach to construct the membership function \( \eta_{\text{EW}_{a}}(z) \) of \( E[\bar{W}] \) is on the basis of deriving Zadeh's approach, which relies on \( \alpha \)-cuts of \( E[\bar{W}] \). Denote the \( \alpha \)-cuts of \( \bar{\lambda}, \bar{\theta}, \bar{\mu} \) as crisp intervals are as follows:

\[ \lambda(\alpha) = \left[ x^L_a, x^U_a \right] = \left[ \min_{x \in X} \{ x | \eta_{i}(x) \geq \alpha \}, \max_{x \in X} \{ x | \eta_{i}(x) \geq \alpha \} \right] \]  

(5a)

\[ \theta(\alpha) = \left[ v^L_b, v^U_b \right] = \left[ \min_{v \in Y} \{ v | \eta_{b}(v) \geq \alpha \}, \max_{v \in Y} \{ v | \eta_{b}(v) \geq \alpha \} \right] \]  

(5b)

\[ \mu(\alpha) = \left[ y^L_a, y^U_a \right] = \left[ \min_{y \in Y} \{ y | \eta_{e}(y) \geq \alpha \}, \max_{y \in Y} \{ y | \eta_{e}(y) \geq \alpha \} \right] \]  

(5c)

These intervals indicate where the arrival, retrial and service rates lie at possibility level \( \alpha \). These three sets cause nested structures for expressing the relationship between ordinary sets and fuzzy sets [22] i.e. given \( 0 < \alpha_2 < \alpha_1 \leq 1 \), we have

\[ \left[ x^L_a, x^U_a \right] \subseteq \left[ x^L_i, x^U_i \right], \left[ v^L_b, v^U_b \right] \subseteq \left[ v^L_{i}, v^U_{i} \right] \text{ and } \left[ y^L_a, y^U_a \right] \subseteq \left[ y^L_{i}, y^U_{i} \right]. \]

By the convexity of a fuzzy number [13], the bounds of these intervals are functions of \( \alpha \) and can be obtained as

\[ x^L_a = \min \eta_{i}^{-1}(\alpha), x^U_a = \max \eta_{i}^{-1}(\alpha) \]

\[ v^L_b = \min \eta_{b}^{-1}(\alpha), v^U_b = \max \eta_{b}^{-1}(\alpha) \]

\[ y^L_a = \min \eta_{e}^{-1}(\alpha), y^U_a = \max \eta_{e}^{-1}(\alpha) \]

Clearly, as defined in (4a), the membership function of \( E[\bar{W}] \) is also parametrized by \( \alpha \). Consequently, we can use it's \( \alpha \)-cut to construct the corresponding membership function.

Using Zadeh's Extension Principle, \( \eta_{\text{EW}_{a}}(z) \) is the minimum of \( \eta_{i}(x), \eta_{b}(v) \) and \( \eta_{e}(y) \). We need atleast one of the following cases to hold such that:

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\[ Z = \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{v} \right) \]

satisfies \( \eta_{\text{left}}(z) = \alpha \).

Case (i): \( \eta_L(x) = \alpha, \eta_B(\nu) \geq \alpha, \eta_R(\nu) \geq \alpha \),

Case (ii): \( \eta_L(x) \geq \alpha, \eta_B(\nu) = \alpha, \eta_R(\nu) \geq \alpha \),

Case (iii): \( \eta_L(x) \geq \alpha, \eta_B(\nu) \geq \alpha, \eta_R(\nu) = \alpha \).

To find the membership function \( \eta_{\text{right}}(z) \), it suffices to find the left shape function and the right shape function of \( \eta_{\text{left}}(z) \), which is equivalent to finding the lower bound \( (E[W_i])_a^L \) and the upper bound \( (E[W_i])_a^U \) of the \( \alpha \)-cuts of \( E[\tilde{W}_a] \).

Since the requirement of \( \eta_L(x) = \alpha \) can be represented by \( x = x_a^L \) or \( x = x_a^U \), this can be formulated as the constraint of \( x = \beta_1 x_a^L + (1 - \beta_1) x_a^U \), where \( \beta_1 \geq 0 \) or 1. Similarly, \( \eta_B(\nu) = \alpha \) can be formulated as the constraint of \( \nu = \beta_2 \nu_a^L + (1 - \beta_2) \nu_a^U \), where \( \beta_2 = 0 \) or 1 and \( \eta_R(\nu) = \alpha \) can be formulated as the constraint of \( \nu = \beta_3 \nu_a^L + (1 - \beta_3) \nu_a^U \), \( \beta_3 \geq 0 \) or 1. Moreover, from the definition of \( \lambda(\alpha), \theta(\alpha) \) and \( \mu(\alpha) \) in (5a), (5b) and (5c), \( x \in \lambda(\alpha), \nu \in \theta(\alpha) \) and \( y \in \mu(\alpha) \) can be respectively replaced by \( x \in [x_a^L, x_a^U], \nu \in [\nu_a^L, \nu_a^U] \) and \( y \in [y_a^L, y_a^U] \). Consequently, considering all of these three cases above, the membership function \( \eta_{\text{right}}(z) \) can be constructed via finding the lower bound \( (E[\tilde{W}_a])_a^L \) and upper bound \( (E[\tilde{W}_a])_a^U \) of the \( \alpha \)-cuts of \( E[\tilde{W}]_a \), in that we set \( (E[\tilde{W}]_a)^L = \min \{ (E[\tilde{W}_a]^L_a), (E[\tilde{W}_a]^L_a) \} \) and \( (E[\tilde{W}]_a)^U = \max \{ (E[\tilde{W}_a]^L_a), (E[\tilde{W}_a]^L_a) \} \) respectively, where

\[
(E[\tilde{W}_a])_a^L = \min_{x, v \leq 1} \left[ \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{v} \right) \right]
\]

s.t \( x = t_x x_a^L + (1 - t_x) x_a^U \) \hbox{(6a)}  
\( v_a^L \leq \nu \leq v_a^U \),  
\( y_a^L \leq y \leq y_a^U \),  
\( t_x = 0 \) or 1.

\[
(E[\tilde{W}_a])_a^U = \min_{x, v \leq 1} \left[ \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{v} \right) \right]
\]

s.t \( \gamma = t_x y_a^L + (1 - t_x) y_a^U \) \hbox{(6b)}  
\( x_a^L \leq x \leq x_a^U \),  
\( y_a^L \leq y \leq y_a^U \),  
\( t_x = 0 \) or 1.

\[
(E[W])_a^L = \min_{x, v \leq 1} \left[ \frac{x}{y - x} \left( \frac{1}{x} + \frac{1}{v} \right) \right]
\]

s.t \( y = t_x y_a^L + (1 - t_x) y_a^U \) \hbox{(6c)}
\[ x_a^L \leq x \leq x_a^U, \]
\[ v_a^L \leq v \leq v_a^U, \]
\[ t_3 = 0 \text{ or } 1. \]

\[
\begin{align*}
(E[W_a])^u_{t_3} &= \max_{x/y < 1} \left[ \frac{x}{(y - x)} \left( \frac{1 + 1}{x + y} \right) \right] \\
\text{s.t.} & \quad t_3 x_a^L + (1 - t_3) x_a^U \\
& \quad v_a^L \leq v \leq v_a^U, \\
& \quad y_a^L \leq y \leq y_a^U, \\
& \quad t_4 = 0 \text{ or } 1. 
\end{align*}
\]  \hfill (6d)

\[
\begin{align*}
(E[W_a])^l_{t_4} &= \max_{x/y < 1} \left[ \frac{x}{(y - x)} \left( \frac{1 + 1}{x + y} \right) \right] \\
\text{s.t.} & \quad t_4 y_a^L + (1 - t_4) y_a^U \\
& \quad x_a^L \leq x \leq x_a^U, \\
& \quad v_a^L \leq v \leq v_a^U, \\
& \quad t_5 = 0 \text{ or } 1. 
\end{align*}
\]  \hfill (6d)

where \( x_a^U < y_a^L \). At least one of \( x, v \) or \( y \) must hit the boundaries of their \( \alpha \)-cuts to satisfy \( E[W_a](x) = \alpha \). From the knowledge of calculus, a unique minimum and a unique maximum of the objective function of models (6a), (6b), (6c), (6d), (6e) or (6f) are assured, which shows that the lower bound \( (E[W_a])^l_{t_3} \) and upper bound \( (E[W_a])^u_{t_3} \) of the \( \alpha \)-cuts of \( E[W] \) can be found by solving these six models. In fact, these six models are MINLP with 0-1 variables. There are several effective and efficient methods for solving these problems [23]. Moreover, they involve the systematic study of how the optimal solutions change as \( x_a^L, x_a^U, v_a^L, v_a^U, y_a^L \) and \( y_a^U \) vary over the interval \( \alpha \in [0, 1] \); they fall into the category of parametric programming [24].

The crisp interval \( \left[ (E[W_a])^l_{t_3}, (E[W_a])^u_{t_3} \right] \) obtained by solving models (6a), (6b), (6c), (6d), (6e) and (6f) represents the \( \alpha \)-cut of \( E[W_a] \). An attractive feature of the \( \alpha \)-cut approach is that all \( \alpha \)-cut approach is that all \( \alpha \)-cuts form a nested structure with respect to \( \alpha \) [22]. According to Zadeh's Extension Principle, \( E[W_a] \) defined in (4a) is a fuzzy number that possesses convexity [22, 13]. Therefore, for two values \( \alpha_1 \) and \( \alpha_2 \) such that \( 0 < \alpha_2 < \alpha_1 \leq 1 \) we have \( (E[W_a])^l_{\alpha_1} \geq (E[W_a])^l_{\alpha_2} \) and \( (E[W_a])^u_{\alpha_1} \leq (E[W_a])^u_{\alpha_2} \); in other words, \( (E[W_a])^l_{\alpha_1} \) is non-decreasing with respect to \( \alpha \) and \( (E[W_a])^u_{\alpha} \) is non-increasing.
with respect to \( \alpha \). This property assures the convexity of \( E[\tilde{W}] \). Consequently, the membership function \( \eta_{\tilde{W}_i}(z) \) can be obtained from the solutions of models (6a), (6b), (6c), (6d), (6e) and (6f).

If both \( \left( E[W_i^L] \right)^L \) and \( \left( E[W_i^U] \right)^U \) are invertible with respect to \( \alpha \), then a left shape function \( L(z) = \left( \left( E[W_i^L] \right)^L \right)^{-1} \) and a right shape function \( R(z) = \left( \left( E[W_i^U] \right)^U \right)^{-1} \) can be derived, from which the membership function \( \eta_{\tilde{W}_i}(z) \) is constructed:

\[
\eta_{\tilde{W}_i}(z) = \begin{cases} 
L(z), & \left( E[W_i^L] \right)^L_{\alpha=0} \leq z \leq \left( E[W_i^L] \right)^L_{\alpha=1} \\
R(z), & \left( E[W_i^U] \right)^U_{\alpha=0} \leq z \leq \left( E[W_i^U] \right)^U_{\alpha=1}
\end{cases}
\]

In most cases, the values of \( \left( E[W_i^L] \right)^L \) and \( \left( E[W_i^U] \right)^U \) cannot be solved analytically, the numerical solutions for \( \left( E[W_i^L] \right)^L \) and \( \left( E[W_i^U] \right)^U \) at different possibility level \( \alpha \) can be collected to approximate the shapes of \( L(z) \) and \( R(z) \). That is, the set of intervals \( \left( \left( E[W_i^L] \right)^L, \left( E[W_i^U] \right)^U \right) \mid \alpha \in [0, 1] \) reveals the shape of \( \eta_{\tilde{W}_i}(z) \) although the exact function is not known explicitly. Note that the membership functions of the expected number of customers in the orbit and waiting time in the system and expected number of customers in the system for this retrial system can be derived in a similar manner.

Since the system characteristics are described by membership function, the values conserve completely all of fuzziness of arrival rate, service rate and retrial rate. However, in practical point of view, the management would prefer one crisp value for each system characteristics rather than a fuzzy set. In order to overcome this problem, we defuzzify the fuzzy values of system characteristics by Yager’s ranking index method [25]. Since the Yager’s ranking index method possesses the property of area compensation, we adopt this method for transforming the fuzzy value of system characteristics into a crisp one to provide suitable values for system characteristics. The recommended suitable values of system characteristics are calculated by

\[
\tilde{O}(\tilde{\lambda}) = \int_{0}^{1} \frac{\lambda^L + \lambda^U}{2} \, d\alpha
\]

(8)

where \( \tilde{\lambda} \) is a convex fuzzy number and \( (\lambda^L, \lambda^U) \) is the \( \alpha \)-cut of \( \tilde{\lambda} \). Note that this method is a robust ranking technique that possesses the properties of compensation, linearity and additivity [26].

4. NUMERICAL EXAMPLE

To demonstrate the practical use of the proposed approach, an example inspired by Artalejo and Lopez-Herrero [21] is solved.

Example: In a packet switching network, we considered a computer network in which there are a group of host computers connected to interface message processors. Message arrive at the host computer following a Poisson stream. If the host computer wishes to transmit the message to another host computer, it must send the message and the final address to the interface message processor to which it is associated. If the processor is free the message is accepted. Otherwise, the message comes back to the host computer and is stored in a buffer to be retransmitted some time later. The buffer in the host computer, the interface processor and the retransmission policy correspond to the orbit, the server and the retrial discipline, respectively, in the queuing terminology. Clearly, aP/FM/1/1-(FR) queuing model can model the above system. Concerned with system efficiency, the management wants to obtain the system characteristics, including the expected waiting time and the number of customers in the system and queue.

Suppose the arrival, retrial and service rates are triangular fuzzy numbers represented by \( \tilde{\lambda} = [3, 4, 5], \tilde{\theta} = [1, 4, 7] \) and \( \tilde{\mu} = [6, 7, 8] \). It is easy to find that \( (x^L, x^U) = [3 + \alpha, 5 - \alpha], (v^L, v^U) = [1 + 3\alpha, 7 - 3\alpha] \) and \( (v^L, v^U) = [6 + \alpha, 8 - \alpha] \).
Mixed Integer Nonlinear Programming Approach to Fuzzy Retrial Queue

It is clear that in this example the steady-state condition $P = r/y < 1$ is satisfied, thus the system characteristics of interest can be constructed by using the proposed approach stated in Section 3. Following (6a), (6b), (6c), (6d) and (6f), three pairs of MINLP models for deriving the membership function of $(E[W, I])$ can be formulated, whose solutions are as follows:

\[
\begin{align*}
(E[W, I])_L^1 &= \frac{10 - 2a}{6a^2 - 29a + 35}; \\
(E[W, I])_U^1 &= \frac{6 + 2a}{6a^2 + 5a + 1}
\end{align*}
\]

with the help of MATLAB\textsuperscript{\textcopyright} 7.0.4, the inverse functions of $(E[W, I])_L^1$ and $(E[W, I])_U^1$ exist, which give the membership function

\[
\eta_{E[W, I]}(z) = \begin{cases}
\frac{(29z - 2) - (2z^2 + 124z + 4)^{1/2}}{12z}, & \frac{2}{7} \leq z \leq \frac{2}{3} \\
\frac{-5z - 2 + (2z^2 + 124z + 4)^{1/2}}{12z}, & \frac{2}{3} \leq z \leq 6
\end{cases}
\]

Applying the Yager ranking index method stated (8), the suitable expected waiting time in the system for this example is given by

\[
O(E[W, I]) = \frac{1}{2} \int_0^{1} \frac{10 - 2a}{6a^2 - 29a + 35} + \frac{6 + 2a}{6a^2 + 5a + 1} \, da = 1.1656
\]

Similarly, by the same argument the membership functions of other system characteristics are as follows:

\[
\begin{align*}
\mu_{E[W, I]}(z) &= \begin{cases}
\frac{(4 + 29z) - (2z^2 + 672z + 256)^{1/2}}{2(2 + 6z)}, & \frac{30}{35} \leq z \leq \frac{32}{12} \\
\frac{(4 - 5z) + (2z^2 + 672z + 256)^{1/2}}{2(2 + 6z)}, & \frac{32}{12} \leq z \leq 30
\end{cases} \\
\mu_{R[\lambda]}(z) &= \begin{cases}
\frac{45}{280}, & \frac{45}{84} \leq z \leq \frac{44}{84} \\
\frac{44}{84}, & \frac{44}{84} \leq z \leq \frac{35}{6}
\end{cases}
\]

where

\[
L(z) = \frac{1}{18} z(3372z + 14205z^2 + 37520z^3 + 64 + 9(-1697424z - 8269083z^2 - 34992 - 13090374z^3 - 104907z^4)^{1/2}z)^{1/3} + \frac{1}{18} (562z + 1123z^2 + 16)/z(3372z + 14205z^2 + 37520z^3 + 64 + 9(-1697424z - 8269083z^2 - 34992 - 13090374z^3 - 104907z^4)^{1/2}z)^{1/3} + \frac{1}{18} (4 + 77z)/z
\]

and

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\[ R(z) = \frac{1}{18} \cdot z(-3372z - 14205z^2 - 37520z^3 - 64 + 9(-1697424z - 8269083z^2 - 34992 - 13090374z^3 - 104907z^4 - 12z)^{1/3} + \]
\[ \frac{1}{18} \cdot (562z + 1123z^2 + 16)z(-3372z - 14205z^2 - 37520z^3 - 64 + \]
\[ 9(-1697424z - 8269083z^2 - 34992 - 13090374z^3 - 104907z^4 - 12z)^{1/3} - \frac{1}{18} \cdot (4 + 41z)z \]

\[ \mu_{E^{\bar{N}}_1}(z) = \begin{cases} 
L(z), & \frac{135}{280} \leq z \leq \frac{176}{84} \\
R(z), & \frac{176}{84} \leq z \leq \frac{175}{6} 
\end{cases} \]

where \( z = \frac{1}{6}(-2 + 3z)(-19683 + 271647z + 105948z^2 + 37520z^3 - 66(3z(289z^2) \]
\[ + 195534z^2 + 779193z + 1259712)^{1/2} + 99(-3z(289z^2 + 195534z^2) \]
\[ + 779193z + 1259712)^{1/2} z)^{1/3} + 1 \]
\[ 6(729 + 3619z + 1123z^2)(-2 + 3z) \]
\[ (-19683 - 19683 + 271647z + 105948z^2 - 37520z^3 - 66(-3z(289z^2 + 195534z^2) \]
\[ + 779193z + 1259712)^{1/2} + 99(-3z(289z^2 + 195534z^2 + 779193z + 1259712)^{1/2} z)^{1/3} + \]
\[ 1 \]
\[ 6(9 + 77z)(-2 + 3z) \]

\[ R(z) = \frac{1}{6}(-2 + 3z)(+19683 - 271647z - 105948z^2 - 37520z^3 - 66(3z(289z^3) \]
\[ + 195534z^2 + 779193z + 1259712)^{1/2} + 99(-3z(289z^3 + 195534z^2) \]
\[ + 779193z + 1259712)^{1/2} z)^{1/3} + 1 \]
\[ 6(729 + 3619z + 1123z^2)(-2 + 3z) \]
\[ (19683 - 271647z - 105948z^2 - 37520z^3 - 66(-3z(289z^2 + 195534z^2) \]
\[ + 779193z + 1259712)^{1/2} + 99(-3z(289z^3 + 195534z^2 + 779193z + 1259712)^{1/2} z)^{1/3} - \]
\[ 1 \]
\[ 6(31 + 41z)(-2 + 3z) \]

Also, we obtain the Yager ranking index of the system characteristic measures:

\[ O(E[\bar{N}_1]) = \int_0^1 \frac{(-2a^2 + 4a + 30)}{6a^2 + 29a + 35} + \frac{-2a^2 + 4a + 30}{6a^2 + 5a + 1} \, da = 5.2124 \]

\[ O(E[W_2]) = \int_0^1 \frac{(-4a^2 + 3a + 45)}{-6a^3 + 77a^2 - 267a + 280} + \frac{-4a^2 + 13a + 35}{6a^3 + 41a^2 - 31a + 6} \, da = 1.0218 \]

\[ O(E[N_4]) = \int_0^1 \frac{(-4a^3 - 9a^2 + 54a + 135)}{-6a^3 + 77a^2 - 267a + 280} + \frac{4a^3 - 33a^2 + 30a + 175}{6a^3 + 41a^2 + 31a + 6} \, da = 5.1215 \]

Figure 1 depicts the shape of the membership functions of expected waiting time in the system, expected number of customer in the system.
Mixed Integer Nonlinear Programming Approach to Fuzzy Retrial Queue

Fig. 1: membership functions of waiting time in the system and system length

Table 1: The \( \alpha \)-cuts of the system characteristics at 11\( \alpha \) values

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( (E[W_s])^L )</th>
<th>( (E[W_s])^U )</th>
<th>( (E[N_s])^L )</th>
<th>( (E[N_s])^U )</th>
<th>( (E[W_q])^L )</th>
<th>( (E[W_q])^U )</th>
<th>( (E[N_q])^L )</th>
<th>( (E[N_q])^U )</th>
</tr>
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<tr>
<td>0.0</td>
<td>0.2857</td>
<td>6.0000</td>
<td>0.8571</td>
<td>30</td>
<td>0.1607</td>
<td>5.8333</td>
<td>0.4821</td>
<td>29.1667</td>
</tr>
<tr>
<td>0.1</td>
<td>0.3047</td>
<td>3.9744</td>
<td>0.9447</td>
<td>19.4744</td>
<td>0.1781</td>
<td>3.8104</td>
<td>0.5522</td>
<td>18.6710</td>
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<tr>
<td>0.2</td>
<td>0.3261</td>
<td>2.8571</td>
<td>1.0435</td>
<td>13.7143</td>
<td>0.1979</td>
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</tr>
<tr>
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<td>0.2204</td>
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</tr>
<tr>
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<td>0.5238</td>
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<td>2.0952</td>
</tr>
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</table>

Table 1 lists the \( \alpha \)-cuts of these four system characteristics at 11 distinct \( \alpha \) values: 0, 0.1, 0.2, …, 1.0. These \( \alpha \)-cuts represent the possibility that these four system characteristic measures will appear in the associated range. Specially, the \( \alpha = 0 \) cut show the range that these four system characteristic measures could appear and the \( \alpha = 1 \) shows these four system characteristic measures that are most likely to be. For example, while these four system characteristics are fuzzy, the most likely value of the expected system length \( (E[N_s])_a \) falls at 2.667 and its value is impossible to fall outside the range of 0.8571 and 30.0000; it is definitely possible that the expected queue waiting time \( E[W_q] \) falls at 0.5238 min (or 31.428 sec) approximately, and it will never fall below 0.1607 min (or 9.642 sec) or exceed 5.833 min approximately. The above information will be very useful for designing a retial queueing system.

5. CONCLUSION

This paper develops a method to find the membership function of the system characteristic measures when the arrival rate, service rate and retrial rate are fuzzy numbers. The concept of \( \alpha \)-cuts and Zadeh’s extension principle to construct membership function of the system characteristic are applied. Three pairs of MINLP models are described to transform the fuzzy retrial queue to the family of crisp retrial queues. Then from the obtained solution, the \( \alpha \)-cuts of the membership functions of the system characteristic measures can be found. Since the system characteristic measures are expressed by the membership function rather than by a crisp value, it maintains the fuzziness of input information, and the results can be used to represent the fuzzy system more accurately. In numerical example section, the arrival rate, service rate and retrial rate are assumed to be triangular fuzzy numbers.
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PARAMETRIC NONLINEAR PROGRAMMING APPROACH TO FUZZY RETRIAL QUEUES WITH BATCH ARRIVAL

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ABSTRACT

This paper proposes a procedure to construct the membership functions of the steady-state performance measures in a retrial queueing model with batch arrival where the arrival rate, service rate and retrial rate are fuzzy numbers. By using Zadeh’s extension principle and α-cut approach, a pair of parametric nonlinear programs is developed through which the membership functions of performance measures are derived. From different values of α, the membership function of the performance measures is constructed. For practical use, the defuzzification of performance measures is also provided via Yager ranking index. Numerical example is solved successfully to illustrate the validity of the proposed approach. Since the performance measures are expressed and governed by the membership functions, more information is provided for use by management. By extending to fuzzy environment, the retrial queues with batch arrival would have wider applications for system designers and practitioners.

Key words: Retrial Queue, Fuzzy parameters, Batch arrival, Membership functions, Parametric nonlinear programming.

1. INTRODUCTION

Retrial queues are widely used as mathematical models of several computer systems, packet switching networks, shared bus local area networks, computer networks operating under the carrier-sense multiple access protocol and collision avoidance star local area networks. Recent bibliographies on retrial queues can be found in [1, 2], the surveys [10, 24] and the book [11]. Artabjo and Falin [3] carried out a comparative study between the standard queues and the retrial queues. A number of applications of retrial queues in science and engineering can be found in [17].

In crisp environments, many articles on retrial systems have been published [4, 8, 9]. Most of the related studies are based on traditional queueing theory, in that the inter-arrival times, inter-retrial times and service times of customers are assumed to follow certain probability distributions with fixed parameters. However in practice these parameters may be obtained subjectively.
(i.e.) the arrival pattern, retrial pattern and service pattern are described by linguistic terms such as fast, slow or moderate, rather than by probability distributions. Thus fuzzy retrial queues would be potentially much more useful and realistic than the commonly used crisp retrial queues [20]. If the usual crisp retrial queues with batch arrival can be extended to fuzzy retrial queues with batch arrival, queueing models would have even wider applications.

Relatively few articles have been published on the topic of fuzzy queues. Based on Zadeh’s extension principle [23,25,26] and fuzzy Markov chain [21], Li and Lee [18] proposed a general approach for analyzing two fuzzy queues, namely M/F/1 and FM/FM/1, where F denotes fuzzy time and FM denotes fuzzified exponential time. Though their approach can be applied to general fuzzy queueing problem but it is not suitable for deriving analytical results of other complicated queueing systems. The procedure proposed by Negi and Lee [19] using α-cuts and two-variable simulation to analyze fuzzy queues provides only crisp solutions. Kao et al. [18] adopt parametric programming to construct the membership functions of the performance measures for four simple fuzzy queues, namely, M/F/1, F/M/1, F/F/1 and FM/FM/1. Recently, Chen [5,6] developed FM/FM/1/L and FM/FM[K]/1/∞ fuzzy systems using the same approach. Also, Ke et al. [16] applied the same approach on fuzzy retrial queues.

In this paper, we develop an approach that provides performance characteristics for the retrial queue with batch arrival with fuzzified exponential arrival, retrial and service rates. By employing the α-cuts and Zadeh's extension principle, the fuzzy retrial queue with batch arrival is transformed to a family of crisp retrial queue with batch arrival. As the α value varies, the family of crisp retrial queues with batch arrival is described and solved using parametric non-linear programming (NLP). The NLP solutions completely and successfully yield the membership functions of the performance characteristics.

In section 2, the performance characteristics of standard and fuzzy retrial queueing models with batch arrival is presented. In section 3, we develop a consolidated solution method for deriving the membership functions of these performance measures. To demonstrate the validity of the proposed approach, a numerical illustration is described and solved. Discussion is provided in section 4 and conclusions are drawn from the discussion in section 5. For notational convenience, our model in this paper is hereafter denoted as FM/FM/1/1-(FR), where FR represents the fuzzified exponential retrial rate and the first 1 represents the single server and the second 1 represents the system capacity.

2. FUZZY RETRIAL QUEUES WITH BATCH ARRIVALS


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2. FUZZY RETRIAL QUEUES WITH BATCH ARRIVALS

Consider a retrial queueing system in which customers arrive at a single-server facility in batches as a Poisson process with group arrival rate \( \lambda \) of all batches, where \( \tilde{\lambda} \) is a fuzzy number, and all the service times are independent and identically distributed and follows an exponential distribution with fuzzy service rate \( \tilde{\mu} \). Further assume that the batch size K customers arrives with probability \( f(K) \). If the server is busy at the arrival epoch then all these customers join the orbit whereas if the server is free then one of the arriving customers begins his / her service and the others join orbit. The customers in the orbit attempts service after an uncertain amount of time, called retrial time. The orbit capacity is assumed to be infinite. The successive retrial times are independent and identically distributed according to an exponential distribution with fuzzy retrial rate \( \tilde{\theta} \). If an arriving customer finds the server free, he / she is served and leaves the system after service. This model will hereafter be denoted by FM[K]/FM/1/1-(FR).

In this model the arrival rate \( \tilde{\lambda} \), retrial rate \( \tilde{\theta} \), and service rate \( \tilde{\mu} \) are approximately known and can be represented by convex fuzzy sets. Let \( \eta_{\tilde{\lambda}}(x) \), \( \eta_{\tilde{\theta}}(v) \) and \( \eta_{\tilde{\mu}}(y) \) denote the membership functions of \( \tilde{\lambda} \), \( \tilde{\theta} \) and \( \tilde{\mu} \), respectively. We then have the following fuzzy sets:

\[
\tilde{\lambda} = \{(x, \eta_{\tilde{\lambda}}(x) / x \in X) \quad \ldots \quad (1a)\}
\]

\[
\tilde{\theta} = \{(v, \eta_{\tilde{\theta}}(v) / v \in V) \quad \ldots \quad (1b)\}
\]

\[
\tilde{\mu} = \{(y, \eta_{\tilde{\mu}}(y) / y \in Y) \quad \ldots \quad (1c)\}
\]

where \( X, V \) and \( Y \) are the crisp universal sets of the arrival, retrial and service rates, respectively. Let \( p(x,v,y) \) denote the system performance measure of interest. Clearly, when \( \tilde{\lambda}, \tilde{\theta} \) and \( \tilde{\mu} \) are fuzzy numbers, \( p(\tilde{\lambda}, \tilde{\theta}, \tilde{\mu}) \) will be fuzzy number as well.

On the basis of Zadeh's extension principle [23,25,26], the membership function of the performance measure \( p(\tilde{\lambda}, \tilde{\theta}, \tilde{\mu}) \) is defined as

\[
\eta_{p(\tilde{\lambda}, \tilde{\theta}, \tilde{\mu})}(z) = \sup_{x \in X, v \in V, y \in Y} \min\{\eta_{\tilde{\lambda}}(x), \eta_{\tilde{\theta}}(v), \eta_{\tilde{\mu}}(y) | Z = p(x, v, y)\} \quad \ldots \quad (2)
\]

Assume that the performance measure of interest is the expected number of customers in the queue \( L_q \). From the knowledge of retrial queueing theory [11,17], under the steady – state condition \( p = xE[K]/\gamma < 1 \), where \( E[K] \) denotes Bulletin of Pure and Applied Sciences. Vol. 28E(No. 2) 2009

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the expectation of \( K \), the expected number of customers in the queue of a crisp retrial queueing system with batch arrival is

\[
L_q = \frac{\frac{x}{2y[E(K)^2] + y[E(K^2)] + \frac{2y}{v}[E[K](x + y) - y]}}{2y(y - xE[K])}
\]  

... (3)

Following (2), the membership function of \( \tilde{L}_q \) is

\[
\eta_{\tilde{L}_q}(Z) = \sup_{x,X,v,Y,E[K],y} \min\{\eta_{\tilde{L}_q}(x), \eta_{\tilde{L}_q}(v), \eta_{\tilde{L}_q}(y) \mid Z = \frac{x}{2y[E(K)^2] + y[E(K^2)] + \frac{2y}{v}[E[K](x + y) - y]} \}
\]

... (4)

Membership functions for other performance measures can be obtained according to Little’s formula [13] in the same manner. Unfortunately, the above membership function is not expressed in the usual forms, making it very difficult to imagine their shapes. In this paper we approach the representation problem using parametric non linear programming technique. Parametric NLPs are developed to find the \( \alpha \)-cuts of \( p(\lambda, \theta, \mu) \) based on the extension principle.

3. SOLUTION PROCEDURE

One approach to construct the membership function \( \eta_{p(\lambda, \theta, \mu)} \) is on the basis of deriving the \( \alpha \)-cuts of \( \eta_{p(\lambda, \theta, \mu)} \). Denote the \( \alpha \)-cuts of \( \lambda, \theta \) and \( \mu \) as crisp intervals as follows:

\[
\lambda(\alpha) = \left[ x_A^L, x_A^U \right] = \left[ \min_{x \in X} \{ x/\eta_{\lambda}(x) \geq \alpha \}, \max_{x \in X} \{ x/\eta_{\lambda}(x) \geq \alpha \} \right]
\]  

... (5a)

\[
\theta(\alpha) = \left[ v_A^L, v_A^U \right] = \left[ \min_{v \in V} \{ v/\eta_{\theta}(v) \geq \alpha \}, \max_{v \in V} \{ v/\eta_{\theta}(v) \geq \alpha \} \right]
\]  

... (5b)

\[
\mu(\alpha) = \left[ y_A^L, y_A^U \right] = \left[ \min_{y \in Y} \{ y/\eta_{\mu}(y) \geq \alpha \}, \max_{y \in Y} \{ y/\eta_{\mu}(y) \geq \alpha \} \right]
\]  

... (5c)

The constant arrival, retrial and service rates are shown as intervals when the membership functions are no less than a given possibility level for \( \alpha \). As a result, the bounds of these intervals can be described as functions of \( \alpha \) and can be obtained as \( x_A^L = \min \eta_{\lambda}^{-1}(\alpha), x_A^U = \max \eta_{\lambda}^{-1}(\alpha), v_A^L = \min \eta_{\theta}^{-1}(\alpha), v_A^U = \max \eta_{\theta}^{-1}(\alpha), y_A^L = \min \eta_{\mu}^{-1}(\alpha) \) and \( y_A^U = \max \eta_{\mu}^{-1}(\alpha) \). Therefore, we can use the \( \alpha \)-cuts of \( \tilde{L}_q \) to construct its membership since the membership function defined in (4) is parametrized by \( \alpha \).


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Using Zadeh’s extension principle, \( \eta_{\text{Lq}}(z) \) is the minimum of \( \eta_{\text{L}}(x) \), \( \eta_{\text{B}}(v) \) and \( \eta_{\text{H}}(y) \). To derive the membership function \( \eta_{\text{Lq}}(z) \), we need at least one of the following cases to hold such that:

\[
Z = \frac{x[2x(E(K))^2 + yE[K^2] + \frac{2y}{v}\{E[K](x + y) - y\}]}{2y\{y - xE[K]\}}
\]

Satisfies \( \eta_{\text{Lq}}(z) = \alpha \).

Case (i) : \( \eta_{\text{L}}(x) = \alpha, \eta_{\text{B}}(v) \geq \alpha, \eta_{\text{H}}(y) \geq \alpha \)

Case (ii) : \( \eta_{\text{L}}(x) \geq \alpha, \eta_{\text{B}}(v) = \alpha, \eta_{\text{H}}(y) \geq \alpha \)

Case (iii) : \( \eta_{\text{L}}(x) \geq \alpha, \eta_{\text{B}}(v) \geq \alpha, \eta_{\text{H}}(y) = \alpha \)

This can be accomplished via the parametric NLP technique. The NLP to find the lower and upper bounds of the \( \alpha \)-cuts of \( \eta_{\text{Lq}} \) for case (i) are

\[
L^1_a = \min_{x \in [E[K], y]} \left[ \frac{x[2x(E(K))^2 + yE[K^2] + \frac{2y}{v}\{E[K](x + y) - y\}]}{2y\{y - xE[K]\}} \right] \quad \ldots (6a)
\]

\[
U^1_a = \max_{x \in [E[K], y]} \left[ \frac{x[2x(E(K))^2 + yE[K^2] + \frac{2y}{v}\{E[K](x + y) - y\}]}{2y\{y - xE[K]\}} \right] \quad \ldots (6b)
\]

for case (ii) are

\[
L^2_a = \min_{x \in [E[K], y]} \left[ \frac{x[2x(E(K))^2 + yE[K^2] + \frac{2y}{v}\{E[K](x + y) - y\}]}{2y\{y - xE[K]\}} \right] \quad \ldots (6c)
\]

\[
U^2_a = \max_{x \in [E[K], y]} \left[ \frac{x[2x(E(K))^2 + yE[K^2] + \frac{2y}{v}\{E[K](x + y) - y\}]}{2y\{y - xE[K]\}} \right] \quad \ldots (6d)
\]

and for case (iii) are

\[
B u l l e t i n \ o f \ P u r e \ a n d \ A p p l i e d \ S c i e n c e s, \text{Vol.}\text{28E(No.}\text{2)2009}  \]

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\[
I_{\alpha}^{L} = \min_{x \in [E(K), y]} \left[ \frac{x[2x[E(K)]^2 + yE[K^2] + \frac{2y}{v} E[K](x + y) - y]}{2y(y - xE[K])} \right] \quad \ldots \ (6e)
\]

\[
I_{\alpha}^{U} = \max_{x \in [E(K), y]} \left[ \frac{x[2x[E(K)]^2 + yE[K^2] + \frac{2y}{v} E[K](x + y) - y]}{2y(y - xE[K])} \right] \quad \ldots \ (6f)
\]

From the definitions of \(\lambda(\alpha), \theta(\alpha)\) and \(\mu(\alpha)\) in (5), \(x \in \lambda(\alpha), v \in \theta(\alpha)\) and \(y \in \mu(\alpha)\) can be replaced by \(x \in [x_{a}^{L}, x_{a}^{U}], v \in [v_{a}^{L}, v_{a}^{U}]\) and \(y \in [y_{a}^{L}, y_{a}^{U}]\). Note that all \(\alpha\)-cuts form a nested structure with respect to \(\alpha [14, 23] ; i.e., \) given \(0 < \alpha_2 < \alpha_1 \leq 1\), we have \([x_{a_{1}}^{L}, x_{a_{1}}^{U}] \subseteq [x_{a_{2}}^{L}, x_{a_{2}}^{U}], [v_{a_{1}}^{L}, v_{a_{1}}^{U}] \subseteq [v_{a_{2}}^{L}, v_{a_{2}}^{U}]\) and \([y_{a_{1}}^{L}, y_{a_{1}}^{U}] \subseteq [y_{a_{2}}^{L}, y_{a_{2}}^{U}]\). Therefore (6a), (6c) and (6e) have the same smallest element and (6b), (6d) and (6f) have the same largest element. Thus, based on (3), to find the membership function of \(\eta_{q}^{\alpha}\), it suffices to find the lower bound \(I_{\alpha}^{L}\) and upper bound \(I_{\alpha}^{U}\) of the \(\alpha\)-cuts of \(\eta_{q}^{\alpha}\), which can be rewritten as:

\[
I_{\alpha}^{L} = \min_{x \in [E(K), y]} \left[ \frac{x[2x[E(K)]^2 + yE[K^2] + \frac{2y}{v} E[K](x + y) - y]}{2y(y - xE[K])} \right] \quad \ldots \ (7a)
\]

subject to \(x_{a}^{L} \leq x \leq x_{a}^{U}, v_{a}^{L} \leq v \leq v_{a}^{U}\) and \(y_{a}^{L} \leq y \leq y_{a}^{U}\),

\[
I_{\alpha}^{U} = \max_{x \in [E(K), y]} \left[ \frac{x[2x[E(K)]^2 + yE[K^2] + \frac{2y}{v} E[K](x + y) - y]}{2y(y - xE[K])} \right] \quad \ldots \ (7b)
\]

subject to \(x_{a}^{L} \leq x \leq x_{a}^{U}, v_{a}^{L} \leq v \leq v_{a}^{U}\) and \(y_{a}^{L} \leq y \leq y_{a}^{U}\).

This model is a set of mathematical programs with boundary constraints and lends itself to the systematic study of how the optimal solutions change with \(x_{a}^{L}, x_{a}^{U}, v_{a}, v_{a}^{L}, v_{a}^{U}, y_{a}^{L}\) and \(y_{a}^{U}\) as \(\alpha\) varies over \((0, 1)\). The model is a special case of parametric NLPS [12].

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If both \( L^L_q \) and \( L^U_q \) in (7) are invertible with respect to \( \alpha \), then a left shape function \( L(z) = (L^L_q)^{-1} \) and a right shape function \( R(z) = (L^U_q)^{-1} \) can be obtained, from which the membership function \( \eta_{L_q} \) is constructed:

\[
\eta_{L_q}(z) = \begin{cases} 
L(z), & (L^L_q)^{q_0} \leq z \leq (L^L_q)^{q_1} \\
R(z), & (L^U_q)^{q_1} \leq z \leq (L^U_q)^{q_0} 
\end{cases} \quad \ldots (8)
\]

In most cases, the values of \( L^L_q \) and \( L^U_q \) cannot be solved analytically. However, the numerical solution for \( L^L_q \) and \( L^U_q \) at different possibility levels can be collected to approximate the shapes of \( L(z) \) and \( R(z) \). That is, the set of intervals \( \{L^L_q, L^U_q\} / \alpha \in (0,1) \) show the shape of \( \eta_{L_q} \), although the exact function is not known explicitly.

Since the performance measures are described by membership function, the values conserve completely all of fuzziness of arrival rate, service rate and retrial rate. However the practitioners would prefer one crisp value for one performance measures rather than a fuzzy set. In order to overcome this problem, we defuzzify the fuzzy values of system performance measures by Yager’s ranking index method [25]. Since the Yager’s ranking index method possesses the property of area compensation, we adopt this method for transforming the fuzzy values of performance measures into a crisp one to provide suitable values for performance measures. The recommended suitable values of system characteristics are calculated by

\[
O(\tilde{\alpha}) = \int_0^{\tilde{\alpha}} \frac{\hat{L}_a^U + \hat{L}_a^L}{2} d\alpha, \quad \ldots (9)
\]

where \( \tilde{\alpha} \) is a convex fuzzy number and \( (\hat{L}_a^L, \hat{L}_a^U) \) is the \( \alpha \)-cut of \( \tilde{\alpha} \). Note that this method is a robust ranking technique that possesses the properties of compensation, linearity and additivity.

4. NUMERICAL EXAMPLE

Consider the performance evaluation of Local Area Networks (LAN) operating under transmission protocols like the CSMA / CD (Carrier Sense Multiple Access with Collision Detection) [22]. In this context, messages which are transmitted could consist of a random number of packets. If, upon arrival,
the channel is free, one packet is randomly chosen to be transmitted and rest of
them are stored in the orbit. The probability mass function of the batch size
random variable $K$ is a geometrical distribution [13], with expected value of 2 i.e.
the probability mass function is $P_r (K = k) = 0.5 (1 - 0.5)^{k-1}, k = 1, 2, \ldots$. Messages
arrive at this system in accordance with a Poisson process, and the service times
follow an exponential distribution. Both the group arrival rate and service rate
are triangular fuzzy numbers represented by $\bar{A} = [4, 5, 6]$ and $\bar{B} = [14, 15, 16]$ per
minute respectively. Let the retrial rate be represented by $\bar{\delta} = [1, 3, 5]$.

We have $E[K] = 2$ and it is easy to find $E[K^2] = Var [K] + [E[K]]^2 = 6.$
Following (3), we have

$$L_q = \frac{x[2x(E[K])^2 + yE[K]^2 + \frac{2y}{v} (E[K](x+y) - y)]}{2y(y - xE[K])} = \frac{x}{y - 2x}\left[\frac{4x + 3y}{y} + \frac{2x + y}{v}\right]$$

It is easy to find that $[x_L \equiv x_u] = [4 + \alpha, 6 - \alpha], \quad [v_L \equiv v_u] = [1 + 2\alpha, 5 - 2\alpha]$ and $[y_L \equiv y_u] = [14 + \alpha, 16 - \alpha]$. It is obvious that when $x = x_u, \quad v = v_u$ and $y = y_u$, the
expected queue length attains its maximum value and when $x = x_L, \quad v = v_u$ and
$y = y_u$, the expected queue length attains its minimum value. According to (7),
the $\alpha$-cuts of $I_q^L$ are

$$L_q^L = -\frac{3\alpha^3 - 143\alpha^2 + 180\alpha + 2816}{-6\alpha^3 + 127\alpha^2 - 536\alpha + 640},$$

$$L_q^U = \frac{3\alpha^3 - 161\alpha^2 + 428\alpha + 2580}{6\alpha^3 + 91\alpha^2 + 100\alpha + 28}$$

With the help of MATLAB®, the inverse functions of $L_q^L$ and $L_q^U$ exist, yield

the membership function $\eta_{L_q}(z) = \begin{cases} L(z); & \frac{2816}{640} \leq z \leq \frac{38}{3} \\ R(z); & \frac{38}{3} \leq z \leq \frac{2580}{28} \end{cases}$

where $L(z) = \frac{1}{9(-1 + 2x)} \{[(2929553 + 9885783x + 3471981x^2 + 521479x^3)
- 90 (-8748x^4 - 35669484x^3 - 293468871x^2 - 756584922x - 267434787)^{1/2} +
180 (-8748x^4 - 35669484x^3 - 293468871x^2 - 756584922x - 267434787)^{1/2}(2x^3)^{1/3}] +
[(22069 + 37906x + 6481x^2)(2929553 + 9885783x + 3471981x^2 + 521479x^3) - 90
}}$
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\[
R(z) = \frac{1}{9(-1+2x)} \left[ \left( -2929553 - 9885783x - 3471981x^2 - 521479x^3 \right) + 90 \left( -8748x^4 - 35669484x^3 - 293468871x^2 - 756584922x - 267434787x^{1/2} \right) + 180 \left( -8748x^4 - 35669484x^3 - 293468871x^2 - 756584922x - 267434787x^{1/2} \right) + \left( 22069 + 37906x + 6481x^2 \right) \left( -2929553 - 9885783x - 3471981x^2 - 521479x^3 \right) + 90 \left( -8748x^4 - 35669484x^3 - 293468871x^2 - 756584922x - 267434787x^{1/2} \right) + 180 \left( -8748x^4 - 35669484x^3 - 293468871x^2 - 756584922x - 267434787x^{1/2} \right) \right]^{1/3} + [7 (23 + 13x)]
\]

Applying the Yager ranking index method stated (9), the suitable expected queue length for this example is given by

\[
Q(L_q) = \frac{1}{2} \left[ \left( -3a^3 - 143a^2 + 180a + 281 \right) + \left( 3a^3 - 161a^2 + 428a + 2580 \right) \right] \cdot 2.9177
\]

Next, we perform \( \alpha \)-cuts of arrival, retrial and service rates and fuzzy expected queue length at eleven distinct \( \alpha \) values : 0.0, 0.1, 0.2, ..., 1.0. Crisp intervals for fuzzy expected queue length at different possibilistic \( \alpha \) levels are presented in Table 1.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( x_a^L )</th>
<th>( x_a^U )</th>
<th>( v_a^L )</th>
<th>( v_a^U )</th>
<th>( y_a^L )</th>
<th>( y_a^U )</th>
<th>( L_q^L )</th>
<th>( L_q^U )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>4.0</td>
<td>6.0</td>
<td>1.0</td>
<td>5.0</td>
<td>4.0</td>
<td>16.0</td>
<td>4.4</td>
<td>92.1</td>
</tr>
<tr>
<td>0.10</td>
<td>4.1</td>
<td>5.9</td>
<td>1.2</td>
<td>4.8</td>
<td>14.1</td>
<td>15.9</td>
<td>4.8</td>
<td>67.4</td>
</tr>
<tr>
<td>0.20</td>
<td>4.2</td>
<td>5.8</td>
<td>1.4</td>
<td>4.6</td>
<td>14.2</td>
<td>15.8</td>
<td>5.3</td>
<td>51.5</td>
</tr>
<tr>
<td>0.30</td>
<td>4.3</td>
<td>5.7</td>
<td>1.6</td>
<td>4.4</td>
<td>14.3</td>
<td>15.7</td>
<td>5.8</td>
<td>40.6</td>
</tr>
<tr>
<td>0.40</td>
<td>4.4</td>
<td>5.6</td>
<td>1.8</td>
<td>4.2</td>
<td>14.4</td>
<td>15.6</td>
<td>6.4</td>
<td>32.9</td>
</tr>
<tr>
<td>0.50</td>
<td>4.5</td>
<td>5.5</td>
<td>2.0</td>
<td>4.0</td>
<td>14.5</td>
<td>15.5</td>
<td>7.1</td>
<td>27.1</td>
</tr>
<tr>
<td>0.60</td>
<td>4.6</td>
<td>5.4</td>
<td>2.2</td>
<td>3.8</td>
<td>14.6</td>
<td>15.4</td>
<td>7.9</td>
<td>22.8</td>
</tr>
</tbody>
</table>
Table 1 reports $\alpha$-cuts of arrival, retrial and service rates and the fuzzy expected number of customers in the orbit for the 11 selected $\alpha$ values. For the fuzzy expected queue length $\hat{L}_q$, the value of $\hat{L}_q$ at $\alpha = 1$ is 12.7, indicating that it is definitely possible that the number of customers in the queue is 12.7. Moreover, the range of $\hat{L}_q$ at $\alpha = 0$ is [4.4, 92.1], indicating that the number of customers in the queue will never exceed 92.1 or fall below 4.4.

The following figure depicts the membership function of expected queue length.

![Membership function of expected queue length](image)

The membership function of expected number of customers in the queue

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PARAMETRIC NONLINEAR PROGRAMMING APPROACH TO FUZZY RETRIAL QUEUES WITH BATCH ARRIVAL

5. CONCLUSIONS

This paper applies the concepts of $\alpha$-cuts and Zadeh's extension principle to construct membership function of the expected queue length using paired non-linear programming models. Following the proposed approach, $\alpha$-cuts of the membership functions are found and their interval limits inverted to attain closed-form expressions for the performance measures. This paper proposed the approach which provides practical information for system manager and practitioners to develop or improve system processes.

REFERENCES


On Fuzzy Retrial Queue with Priority Subscribers

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Abstract

This work constructs the membership functions of the system characteristics in retrial queuing model with priority subscribers where the arrival rate, service rate and retrial rate are fuzzy. The basic idea is to transform a fuzzy retrial queue into a family of conventional crisp retrial queues by applying a cut approach. By using Zadeh's extension principle and a cut approach, a set of parametric nonlinear programs is developed to describe the family of crisp retrial queues. From different values of α, the membership function of the system characteristics is constructed. For practical use, the defuzzification of system characteristics is also provided via Yager ranking index. To demonstrate the validity of the proposed approach a real world example is solved successfully. Since the system characteristics are expressed by membership functions rather than by crisp values, the fuzziness of input information is completely conserved. By extending this model to the fuzzy environment, fuzzy retrial-queue with priority subscribers are represented more accurately and the analytic results are more useful for system designers and practitioners.

Keywords  Fuzzy parameters, Retrial queue, α-cut, Non-Linear Programming, High priority queue, low priority queue Non-pre-emptive priority.

1. Introduction

Queueing systems in which arriving customers who find all servers occupied may retry for service after a period of time are called Retrial queues ([1], [2], [3], [4]). Retrial queuing models are mainly motivated by applications to telephone switching systems, telecommunication networks and computer systems packet switch networks, shared bus local area networks operating under the Carrier-Sense Multiple Access Protocol and collision avoidance star local area networks ([5]).
Most retrial queues deal with one type of calls. But there are some practical models which deal with several types of calls ([6]). One example is a telephone switching system ([7]). In modern telephone exchanges, subscriber lines are usually connected to the so-called subscriber line modules. These modules serve both incoming and outgoing calls. An important difference between these two types of calls lies in the fact that in the case of blocking due to all channels busy in the module, outgoing calls can be queued, where as incoming calls get busy signal and must be retried in order to establish the connection. As soon as the channel is free, an outgoing call, if present, occupies the channel immediately. Therefore, incoming calls may not establish the connection as long as there are outgoing calls waiting. This fact implies that outgoing calls have non-pre-emptive priority over incoming calls.

The general single server retrial queues with two types of subscribers (Type I subscriber and Type II subscriber) arrive to the system. Service times of different types of subscribers are the same. If a subscriber arrives and finds the server free, he immediately occupies the server and leaves the system after service. When a Type I customer finds the server busy, he enters the priority queue and waits and is served as soon as the server is free. If a Type II customer finds the server busy on his arrival, he enters the retrial group in order to seek service again after a random amount of time. Type II customers can be served only when there are no Type I customers in the priority queue. The service discipline described above is said to be non-pre-emptive priority. In the pre-emptive priority case; when an arriving Type I customer finds the server busy with a Type II customer, he pre-empts the Type II customer in service and begins to be served, and after all Type I customers in the priority queue have been served the pre-empted Type II customer is served for remaining service time ([5]).

In the literature described above, the inter-arrival times of Type I and Type II customers, inter-retrial times and service times of customers are required of follow certain probability distributions with fixed parameters. However, in many real-world applications, the parameter distributions may only be characterized subjectively; rather than with complete probability distribution. In other words, these system parameters are both possibilistics and probabilistic. Thus, fuzzy retrial queues would be potentially much more useful and realistic than the commonly used crisp retrial queues ([8], [9], [10]).

Fuzzy queueing models have been described by such researches like Li and Lee ([9]), Negi and Lee [11], Kao et al. [12], Chen [13,14] are analyzed fuzzy queue using Zadeh's extension principle [15]. Kao et al [12] constructed the membership functions of the system characteristics for fuzzy queues and successfully applied them to four simple fuzzy queues models: M/F/1/∞, F/M/1/∞, F/F/1/∞ and FM/FM/1/∞. Recently, Chen [13,14] developed FM/FM/1/∞ and FM/FM(k)/1/∞ fuzzy systems using the same approach.

All previous research on fuzzy queuing models is focused on ordinary queues with one or two fuzzy variables. In this paper, we develop an approach that provides system characteristics for the retrial queues with priority customers having three fuzzy variables: fuzzified exponential arrival, retrial and service rates [8]. Through α-cuts and Zadeh's extension principle, we transform the fuzzy retrial queues with priority customers to a family of crisp retrial queues with priority customers. As α varies, the family of crisp retrial queues is described and solved using
parametric Non-Linear Programming (NLP). The NLP solutions completely and successfully yield the membership functions of the system characteristics.

In section 2, the system characteristics of standard and fuzzy retrial queuing models with priority customers is presented. In section 3, we develop a consolidated solution method for deriving the membership functions of these system characteristics. To demonstrate the validity of the proposed approach, a numerical illustration is described and solved. Discussion is provided in section 4 and conclusion are drawn from the discussion in section 5. For notational convenience, our model in this paper is hereafter denoted as

\[ \text{FM}_1, \text{FM}_2, \text{FM}/1-(\text{FR}) \] [16],

where \( \text{FM}_1 \) and \( \text{FM}_2 \) are fuzzified exponential arrival rate of Type I and Type II customers respectively, \( \text{FR} \) represents the fuzzified exponential retrial rate and I represents the single server.

2. Fuzzy queues with Priority customers

2.1. \( \text{M}_1, \text{M}_2/\text{M}/1 \) queues

We consider a single server retrial queuing system in which Type I and Type II customers arrive according to poisson process with rate \( \lambda_1 \) and \( \lambda_2 \) respectively. Type I customers can be identified as high priority customers and they are queued after blocking and then served in some discipline such as FCFS or random order. On the other hand, any low priority customers (those from Type II flow) who finds the server busy upon arrival leaves the system immediately, and enters the retrial group in order to seek service again after a random amount of time. The retrial times is assumed to be independent and exponentially distributed with parameta \( \theta \). Service times are independent and identically distributed and have the same exponential distribution \( \mu \).

Define: \( E[N_1] \), the expected number of customers in high priority queue and \( E[N_2] \), the expected number of customers in low priority queue.

\[
E[N_1] = \frac{\lambda_1 \lambda}{\mu^2 (1 - \rho_1)} \quad (1)
\]

\[
E[N_2] = \frac{\lambda_2 \lambda}{\mu^2 (1 - \rho_1)(1 - \rho_2)} + \frac{\rho_2}{\theta} \frac{\rho}{1 - \rho} \quad (2)
\]

where \( \lambda = \lambda_1 + \lambda_2, \rho = \rho_1 + \rho_2, \rho_1 = \frac{\lambda_1}{\mu} \) and \( \rho_2 = \frac{\lambda_2}{\mu} \). In steady state, it is necessary that we have \( \rho = \rho_1 + \rho_2 < 1 \).

From the results in G.I.Falin et al. [7], we can easily derive the system characteristics for high priority customers and low priority customer in terms of the system parameters.

2.2. \( \text{FM}_1, \text{FM}_2 [\text{FM}] 1 - (\text{FR}) \) queues

To extend the applicability of the standard queuing model with priority customer, we allow for fuzzy specification of system parameters. Suppose the arrival rate of high priority customers \( \lambda_1 \), the arrival rate of low priority customers \( \lambda_2 \), service rate \( \mu \) and retrial rate \( \theta \) are approximately
known and can be represented by the fuzzy sets, \( \tilde{A}_1, \tilde{A}_2, \tilde{\mu}_2 \) and \( \tilde{\theta} \) respectively. Let \( \eta_{\tilde{A}_1}(x_1), \eta_{\tilde{A}_2}(x_2), \eta_{\tilde{\theta}}(y) \) and \( \eta_{\tilde{\theta}}(\nu) \) denote the membership functions of \( \tilde{A}_1, \tilde{A}_2, \tilde{\mu}_2 \) and \( \tilde{\theta} \), respectively. We then have the following fuzzy sets:

\[
\tilde{A}_1 = \{ (x_1, \eta_{\tilde{A}_1}(x_1)) | x_1 \in X_1 \}, \quad (3a)
\]

\[
\tilde{A}_2 = \{ (x_2, \eta_{\tilde{A}_2}(x_2)) | x_2 \in X_2 \}, \quad (3b)
\]

\[
\tilde{\mu} = \{ (y, \eta_{\tilde{\mu}}(y)) | y \in Y \}, \quad (3c)
\]

\[
\tilde{\theta} = \{ (\nu, \eta_{\tilde{\theta}}(\nu)) | \nu \in \nu \}, \quad (3d)
\]

where \( X_1, X_2, Y \) and \( \nu \) are the crisp universal sets of the arrival rate of high priority, arrival rate of low priority, service rate and retrial rate respectively.

Let \( f(x_1,x_2,y,\nu) \) denote the system characteristic of interest. Since \( \tilde{A}_1, \tilde{A}_2, \tilde{\mu} \) and \( \tilde{\theta} \) are fuzzy numbers, \( f(\tilde{A}_1,\tilde{A}_2,\tilde{\mu},\tilde{\theta}) \) is also a fuzzy numbers. Following Zadeh’s extension principle ([10], [15]), the membership function of the system characteristic \( f(\tilde{A}_1,\tilde{A}_2,\tilde{\mu},\tilde{\theta}) \) is defined as

\[
\eta_{f(\tilde{A}_1,\tilde{A}_2,\tilde{\mu},\tilde{\theta})}(z) = \text{Sup } \Omega \left\{ \eta_{\tilde{A}_1}(x_1), \eta_{\tilde{A}_2}(x_2), \eta_{\tilde{\mu}}(y), \eta_{\tilde{\theta}}(\nu) \mid z = f(x_1,x_2,y,\nu) \right\}, \quad (4)
\]

where the supremum is taken over the set

\[
\Omega = \left\{ x_1 \in X_1, x_2 \in X_2, y \in Y, \nu \in \nu \left| 0 < \frac{x_1 + x_2}{y} < 1 \right. \right\}.
\]

Assume the system characteristic of interest is the expected number of customers in queue. It follows from (1) that the expected number of customers in high priority queue is

\[
f(x_1,x_2,y) = \frac{x_1(x_1 + x_2)}{y(y - x_1)} \quad (5)
\]

The membership function for the expected number of customer in high priority queue is

\[
\eta_{\tilde{g}(x)}(z) = \text{Sup } \min_{\Omega} \left\{ \eta_{\tilde{A}_1}(x_1), \eta_{\tilde{A}_2}(x_2), \eta_{\tilde{\mu}}(y) \mid z = \frac{x_1(x_1 + x_2)}{y(y - x_1)} \right\} \quad (6)
\]

Like wise, from (2) the expected number of customers in low priority queue is

\[
f(x_1,x_2,y,\nu) = \frac{x_2(x_1 + x_2)}{(y - x_1 - x_2)} \left\{ \frac{1}{y} + \frac{1}{\nu} \right\} \quad (7)
\]

The membership function for the expected number of customers in the low priority queue is

\[
\eta_{\tilde{g}(x)}(z) = \text{Sup } \min_{\Omega} \left\{ \eta_{\tilde{A}_1}(x_1), \eta_{\tilde{A}_2}(x_2), \eta_{\tilde{\mu}}(y), \eta_{\tilde{\theta}}(\nu) \mid z = \frac{x_2(x_1 + x_2)}{(y - x_1 - x_2) \left\{ \frac{1}{y} + \frac{1}{\nu} \right\}} \right\} \quad (8)
\]

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Unfortunately, these membership functions are not expressed in the usual forms, making it very difficult to imagine their shapes. In this paper we approach the representation problem using a mathematical programming technique. Parametric NLPs are developed to find the \( \alpha \)-cuts of \( \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\mu}, \tilde{\Theta} \) based on the extension principle.

3. The Solution Procedure

One approach to construct the membership function \( \eta_{\tilde{E}[N_1]} \) of \( \tilde{E}[N_1] \) (for high priority) is on the basis of deriving the \( \alpha \) - cuts of \( \tilde{E}[N_1] \). Denote the \( \alpha \)-cuts of \( \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\mu}, \tilde{\Theta} \) as crisp interval as follows.

\[
\lambda_1(\alpha) = \left[ x_{1a}^L, x_{1a}^U \right] = \left[ \min_{x \in X_1} \left\{ x_1 \mid \eta_{\tilde{\lambda}}(x_1) \geq \alpha \right\}, \max_{x \in X_1} \left\{ x_1 \mid \eta_{\tilde{\lambda}}(x_1) \geq \alpha \right\} \right] \tag{9a}
\]

\[
\lambda_2(\alpha) = \left[ x_{2a}^L, x_{2a}^U \right] = \left[ \min_{x \in X_2} \left\{ x_2 \mid \eta_{\tilde{\lambda}}(x_2) \geq \alpha \right\}, \max_{x \in X_2} \left\{ x_2 \mid \eta_{\tilde{\lambda}}(x_2) \geq \alpha \right\} \right] \tag{9b}
\]

\[
\mu(\alpha) = \left[ y_a^L, y_a^U \right] = \left[ \min_{y \in Y} \left\{ y \mid \eta_{\tilde{\mu}}(y) \geq \alpha \right\}, \max_{y \in Y} \left\{ y \mid \eta_{\tilde{\mu}}(y) \geq \alpha \right\} \right] \tag{9c}
\]

\[
\theta(\alpha) = \left[ \nu_a^L, \nu_a^U \right] = \left[ \min_{\nu \in \nu} \left\{ \nu \mid \eta_{\tilde{\theta}}(\nu) \geq \alpha \right\}, \max_{\nu \in \nu} \left\{ \nu \mid \eta_{\tilde{\theta}}(\nu) \geq \alpha \right\} \right] \tag{9d}
\]

The constant arrival, service and retrial rates and shown as intervals when the membership functions are no less than a given possibility level for \( \alpha \). As a result, the bounds of these intervals can be described as functions of \( \alpha \) and can be obtained as: \( x_{1a}^L = \min \eta^{-1}_{\tilde{\lambda}}(\alpha) \), \( x_{1a}^U = \max \eta^{-1}_{\tilde{\lambda}}(\alpha) \), \( x_{2a}^L = \min \eta^{-1}_{\tilde{\lambda}}(\alpha) \), \( x_{2a}^U = \max \eta^{-1}_{\tilde{\lambda}}(\alpha) \), \( y_a^L = \min \eta^{-1}_{\tilde{\mu}}(\alpha) \), \( y_a^U = \max \eta^{-1}_{\tilde{\mu}}(\alpha) \), \( \nu_a^L = \min \eta^{-1}_{\tilde{\theta}}(\alpha) \) and \( \nu_a^U = \max \eta^{-1}_{\tilde{\theta}}(\alpha) \). Therefore, we can use the \( \alpha \)-cuts of \( \tilde{E}[N_1] \) and \( \tilde{E}[N_2] \) to construct its membership function since the membership function defined in (6) and (8) is parameterized by \( \alpha \).

Using Zadeh’s extension principle, \( \eta_{\tilde{E}[N_1]}(z) \) is the minimum of \( \eta_{\tilde{\lambda}}(x_1), \eta_{\tilde{\lambda}}(x_2), \eta_{\tilde{\mu}}(y) \) and \( \eta_{\tilde{\theta}}(\nu) \). To derive the membership function \( \eta_{\tilde{E}[N_1]}(z) \), we need at least one of the following cases to hold such that

\[
z = \frac{x_2(x_1 + x_2)}{(y - x_1 - x_2)} \left\{ \frac{1}{y - x_1} + \frac{1}{\nu} \right\} \text{ satisfies } \eta_{\tilde{E}[N_1]}(z) = \alpha:
\]

Case (i): \( \eta_{\tilde{\lambda}}(x_1) = \alpha, \eta_{\tilde{\lambda}}(x_2) \geq \alpha, \eta_{\tilde{\mu}}(y) \geq \alpha, \eta_{\tilde{\theta}}(\nu) \geq \alpha \),

Case (ii): \( \eta_{\tilde{\lambda}}(x_1) \geq \alpha, \eta_{\tilde{\lambda}}(x_2) = \alpha, \eta_{\tilde{\mu}}(y) \geq \alpha, \eta_{\tilde{\theta}}(\nu) \geq \alpha \),

Case (iii): \( \eta_{\tilde{\lambda}}(x_1) \geq \alpha, \eta_{\tilde{\lambda}}(x_2) \geq \alpha, \eta_{\tilde{\mu}}(y) = \alpha, \eta_{\tilde{\theta}}(\nu) \geq \alpha \),

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Case (iv) : \( \eta_\hat{\lambda}\left(x_1\right) \geq \alpha, \eta_\hat{\lambda}\left(x_2\right) \geq \alpha, \eta_\hat{\mu}\left(y\right) \geq \alpha, \eta_\hat{\nu}\left(v\right) = \alpha \).

This can be accomplished using parametric NLP techniques. The NLP to find the lower and upper bounds of the \( \alpha \)-cuts of \( \eta_{E[N_2]} \) for case (i) are

\[
\left(E[N_2]\right)_\alpha^L = \min_{\Omega} \left( \frac{x_2(x_1 + x_2)}{(y - x_1 - x_2)} \left\{ \frac{1}{y - x_1} + \frac{1}{v} \right\} \right), \tag{10a}
\]

\[
\left(E[N_2]\right)_\alpha^U = \max_{\Omega} \left( \frac{x_2(x_1 + x_2)}{(y - x_1 - x_2)} \left\{ \frac{1}{y - x_1} + \frac{1}{v} \right\} \right), \tag{10b}
\]

for case (ii) are \( \left(E[N_2]\right)_\alpha^{L_2} = \min_{\Omega} \left( \frac{x_2(x_1 + x_2)}{(y - x_1 - x_2)} \left\{ \frac{1}{y - x_1} + \frac{1}{v} \right\} \right), \tag{10c} \)

\[
\left(E[N_2]\right)_\alpha^{U_2} = \max_{\Omega} \left( \frac{x_2(x_1 + x_2)}{(y - x_1 - x_2)} \left\{ \frac{1}{y - x_1} + \frac{1}{v} \right\} \right), \tag{10d}
\]

for case (iii) are \( \left(E[N_2]\right)_\alpha^{L_3} = \min_{\Omega} \left( \frac{x_2(x_1 + x_2)}{(y - x_1 - x_2)} \left\{ \frac{1}{y - x_1} + \frac{1}{v} \right\} \right), \tag{10e} \)

\[
\left(E[N_2]\right)_\alpha^{U_3} = \max_{\Omega} \left( \frac{x_2(x_1 + x_2)}{(y - x_1 - x_2)} \left\{ \frac{1}{y - x_1} + \frac{1}{v} \right\} \right), \tag{10f}
\]

for case (iv) are \( \left(E[N_2]\right)_\alpha^{L_4} = \min_{\Omega} \left( \frac{x_2(x_1 + x_2)}{(y - x_1 - x_2)} \left\{ \frac{1}{y - x_1} + \frac{1}{v} \right\} \right), \tag{10g} \)

\[
\left(E[N_2]\right)_\alpha^{U_4} = \max_{\Omega} \left( \frac{x_2(x_1 + x_2)}{(y - x_1 - x_2)} \left\{ \frac{1}{y - x_1} + \frac{1}{v} \right\} \right). \tag{10h}
\]

From the definitions of \( \lambda_1(\alpha), \lambda_2(\alpha), \mu(\alpha) \) and \( \theta(\alpha) \) in (9), \( x_1 \in \lambda_1(\alpha), x_2 \in \lambda_2(\alpha), y \in \mu(\alpha) \) and \( v \in \theta(\alpha) \) can be replaced by \( x_1 \in [x_{1a}^L, x_{1a}^U], x_2 \in [x_{2a}^L, x_{2a}^U], y \in [y_a^L, y_a^U] \) and \( v \in [v_a^L, v_a^U] \), respectively. The \( \alpha \)-cuts form a nested structure with respect to \( \alpha \) ([17], [18]); i.e., given \( 0 < \alpha_2 < \alpha_1 \leq 1 \), we have \( [x_{1a}^L, x_{1a}^U] \subseteq [x_{1a_2a}^L, x_{1a_2a}^U], [x_{2a}^L, x_{2a}^U] \subseteq [x_{2a_2}^L, x_{2a_2}^U], [y_a^L, y_a^U] \subseteq [y_a_2^L, y_a_2^U] \) and \( [v_a^L, v_a^U] \subseteq [v_a_2^L, v_a_2^U] \).

Therefore, (10a), (10c), (10e) and (10g) have the smallest element and (10b), (10d), (10f) and (10h) have the same largest element. To find the membership function \( \eta_{E[N_2]} \), it suffices to find the left and right shape functions of \( \eta_{E[N_2]} \), which is equivalent to finding the lower bound
\((E[N_2])^L_\alpha\) and upper bound \((E[N_2])^U_\alpha\) of the \(\alpha\)-cuts of \(\tilde{E}[N_2]\), which based on (8), can be rewritten as

\[
(E[N_2])^L_\alpha = \min_{\Omega} \left( \frac{x_2(x_1 + x_2)}{y - x_1 - x_2} \left( \frac{1}{y - x_1} + \frac{1}{\nu} \right) \right)
\]

s.t. \(x^L_{1\alpha} \leq x_1 \leq x^U_{1\alpha}, x^L_{2\alpha} \leq x_2 \leq x^U_{2\alpha}, y^L_\alpha \leq y \leq y^U_\alpha\) and \(v^L_\alpha \leq \nu \leq v^U_\alpha\) \hspace{1cm} (11a)

\[
(E[N_2])^U_\alpha = \max_{\Omega} \left( \frac{x_2(x_1 + x_2)}{y - x_1 - x_2} \left( \frac{1}{y - x_1} + \frac{1}{\nu} \right) \right)
\]

s.t. \(x^L_{1\alpha} \leq x_1 \leq x^U_{1\alpha}, x^L_{2\alpha} \leq x_2 \leq x^U_{2\alpha}, y^L_\alpha \leq y \leq y^U_\alpha\) and \(v^L_\alpha \leq \nu \leq v^U_\alpha\) \hspace{1cm} (11b)

At least one \(x_1, x_2, y\) or \(\nu\) must hit the boundaries of their \(\alpha\)-cuts to satisfy \(\eta_{\tilde{E}[N_2]}(z) = \alpha\). This model is a set of mathematical programs with boundary constraints and lends itself to the systematic study of how the optimal solutions change with \(x^L_{1\alpha}, x^U_{1\alpha}, x^L_{2\alpha}, x^U_{2\alpha}, y^L_\alpha, y^U_\alpha, v^L_\alpha\) and \(v^U_\alpha\) as \(\alpha\) varies over \((0,1]\). The model is a special case of parametric NLPs[19].

The crisp interval \([E[N_2]]^L_\alpha, (E[N_2])^U_\alpha\] obtained from (11) represents the \(\alpha\)-cuts of \(\tilde{E}[N_2]\).

Again, by applying the results of Zimmermann [17] and Kaufmann [18] and convexity properties to \(\tilde{E}[N_2]\), we have \((E[N_2])^L_\alpha \geq (E[N_2])^L_{\alpha_1}\) and \((E[N_2])^U_\alpha \geq (E[N_2])^U_{\alpha_1}\), where \(0 < \alpha_2 < \alpha_1 \leq 1\).

In other words, \((E[N_2])^L_\alpha\) increases and \((E[N_2])^U_\alpha\) decreases as \(\alpha\) increases. Consequently, the member function \(\eta_{\tilde{E}[N_2]}(z)\) can be found from (11).

If both \((E[N_2])^L_\alpha\) and \((E[N_2])^U_\alpha\) in (11) invertible with respect to \(\alpha\), then a left shape function

\[
L(z) = \left[ (E[N_2])^L_\alpha \right]^{-1}
\]

a right shape function \(R(z) = \left[ (E[N_2])^U_\alpha \right]^{-1}\) can be derived, from which the membership function \(\eta_{\tilde{E}[N_2]}(z)\) is constructed.

\[
\eta_{\tilde{E}[N_2]}(z) = \begin{cases} 
L(z), & (E[N_2])^L_{\alpha=0} \leq z \leq (E[N_2])^L_{\alpha=1} \\
R(z), & (E[N_2])^U_{\alpha=1} \leq z \leq (E[N_2])^U_{\alpha=0}
\end{cases}
\]

(12)

In most cases, the values of \((E[N_2])^L_\alpha\) and \((E[N_2])^U_\alpha\) cannot be solved analytically. Consequently, a closed-form membership function for \(\tilde{E}[N_2]\) cannot be obtained. However, the numerical solutions for \((E[N_2])^L_\alpha\) and \((E[N_2])^U_\alpha\) at different possibility levels can be collected to approximate the shapes of \(L(z)\) and \(R(z)\). That is, the set of intervals \(\{(E[N_2])^L_\alpha, (E[N_2])^U_\alpha \mid \alpha \in [0,1]\}\) shows the shape of \(\eta_{\tilde{E}[N_2]}\), although the exact function is not known explicitly.
Note that the membership function of the expected number of customers in high priority queue can be derived in a similar manner.

Since the performance measures are described by membership function, the values conserve completely all of fuzziness of arrival rate, service rate and retrial rate. However the practitioners would prefer only crisp value for one system characteristics rather than a fuzzy set. In order to overcome this problem, we defuzzify the fuzzy values of system characteristics by Yager’s ranking index method [20]. Since the Yager’s ranking index method possesses the property of area compensation, we adopt this method for transforming the fuzzy values of system characteristics. The recommended suitable values of system characteristics are calculated by

$$\Omega(\tilde{\lambda}) = \frac{1}{2} \int_{0}^{1} (\tilde{\lambda}_a^L + \tilde{\lambda}_a^U) \, d\alpha,$$

where \( \tilde{\lambda} \) is a convex fuzzy number and \((\tilde{\lambda}_a^L, \tilde{\lambda}_a^U)\) is the \( \alpha \)-cuts of \( \tilde{\lambda} \).

Note that this method is a robust ranking technique that possesses the properties of compensation, linearity and additivity.

4. Numerical Example

Assume that a LAN has two kinds of users (nonpersistent user and persistent user) connected by a single bus (or channel). Communication between users is realized by the bus. Persistent users are controlled by the central system, so, that as soon as the channel is idle, the central system lets one persistent user occupy the channel in order, to send a message if there is any persistent user with a packet. Nonpersistent users with a packet try retransmission independently after a random amount of time. The server corresponds to the channel. The persistent users in the central system are regarded as high priority calls and the nonpersistent users in the non central system are regarded as low priority calls.

4.1. The fuzzy expected number customers in low priority queue \( \tilde{E}[N_2] \)

Suppose the arrival, service and retrial rates are triangular fuzzy numbers represented by \( \tilde{\lambda}_1 = [5,6,7] \), \( \tilde{\lambda}_1 = [3,4,5] \), \( \tilde{\mu} = [14,15,16] \) and \( \tilde{\theta} = [4,7,10] \).

First it is easy to find that \( [x_{1a}^L, x_{1a}^U] = [5+\alpha,7-\alpha] \), \( [x_{2a}^L, x_{2a}^U] = [3+\alpha,5-\alpha] \), \( [y_a^L, y_a^U] = [14+\alpha,16-\alpha] \). and \( [v_a^L, v_a^U] = [4+3\alpha,10-3\alpha] \). Next, it is obvious that when \( x_1 = x_{1a}^U \), \( x_2 = x_{2a}^U \), \( y = y_a^L \) and \( v = v_a^L \), the number of customers attains its maximum value and when \( x_1 = x_{1a}^L \), \( x_2 = x_{2a}^L \), \( y = y_a^U \) and \( v = v_a^U \), the number of customers attains its minimum value.

According to (11), the \( \alpha \)-cuts of \( \tilde{E}[N_2] \) are

\[
\begin{align*}
E[N_2]_a^L &= \frac{-10\alpha^3 - 28\alpha^2 + 174\alpha + 504}{-18\alpha^3 + 207\alpha^2 - 754\alpha + 880} \\
E[N_2]_a^U &= \frac{10\alpha^3 - 88\alpha^2 + 58\alpha + 660}{18\alpha^3 + 99\alpha^2 + 142\alpha + 56}
\end{align*}
\]
With the help of MATLAB®7.0, the inverse functions of $E[N_2]^L$ and $E[N_2]^U$ exist, yielding the membership function.

$$\eta_{E[N_2]}(z) = \begin{cases} 
L(z), & 63 \leq z \leq \frac{128}{63} \\
R(z), & \frac{128}{63} \leq z \leq \frac{165}{14}
\end{cases}$$

where $L(z) = 1 / 6 \cdot (-5 + 9z) (-39208 + 6300522z - 486000z^2 + 76545z^3 - 30 (-1318707z^4 - 143137722z^3 - 1126373907z^2 - 9225427848z - 26142912)^{\frac{1}{2}} + 54 (-1318707z^4 - 143137722z^3 - 1126373907z^2 - 9225427848z - 26142912)^{\frac{1}{2}} + 1 / 6(6004 + 24816z + 2133z^2) / (-5 + 9z) / (-39208 + 6300522z - 486000z^2 + 76545z^3 - 30 (-1318707z^4 - 143137722z^3 - 1126373907z^2 - 9225427848z - 26142912)^{\frac{1}{2}} + 54 (-1318707z^4 - 143137722z^3 - 1126373907z^2 - 9225427848z - 26142912)^{\frac{1}{2}} + 1 / 6(28 + 207z) / (-5 + 9z)$.

$$R(z) = 1 / 6 \cdot (-5 + 9z)(-39208 - 6300522z + 486000z^2 - 76545z^3 - 30 (-1318707z^4 - 143137722z^3 - 1126373907z^2 - 9225427848z - 26142912)^{\frac{1}{2}} + 54(-1318707z^4 - 143137722z^3 - 1126373907z^2 - 9225427848z - 26142912)^{\frac{1}{2}} + 1 / 6(6004 + 24816z + 2133z^2) / (-5 + 9z) / (-39208 - 6300522z + 486000z^2 - 76545z^3 - 30 (-1318707z^4 - 143137722z^3 - 1126373907z^2 - 9225427848z - 26142912)^{\frac{1}{2}} + 54 (-1318707z^4 - 143137722z^3 - 1126373907z^2 - 9225427848z - 26142912)^{\frac{1}{2}} + 1 / 6(28 + 207z) / (-5 + 9z)$, as shown in Fig. 1. The overall shape turns out as expected. The membership functions $L(z)$ and $R(z)$ have complex values with their imaginary parts approaching zero when $\frac{63}{110} \leq z \leq \frac{128}{63}$ for $L(z)$ and $\frac{128}{63} \leq z \leq \frac{165}{14}$ for $R(z)$. Hence, the imaginary parts of these two functions have no influence on the computational results and can be disregarded.

Applying the $V_d$ ranking index method stated (13), the suitable queue length of low priority is given by

$$O'(E[N_2]) = \frac{1}{2} \left[ -10a^2 - 28a^2 + 174a + 504 + 10a^3 - 88a^2 + 58a + 660 \right] da = 3.1241$$

Next, we perform $\alpha$-cuts of arrival, service and retrial rates and fuzzy expected number of customers in low priority queue at eleven district $\alpha$ values 0.0, 0.1, 0.2, .........., 1.0. Crisp intervals for fuzzy expected number of customers in low priority queue at different possibilistic $\alpha$ levels are presented in Table. 1.
Table 1
\( \alpha \)-cuts of arrival, retrial and service rates and fuzzy queue length of low priority

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( x_1^{L} )</th>
<th>( x_1^{U} )</th>
<th>( x_2^{L} )</th>
<th>( x_2^{U} )</th>
<th>( y_1^{L} )</th>
<th>( y_1^{U} )</th>
<th>( y_2^{L} )</th>
<th>( y_2^{U} )</th>
<th>( \nu_1^{L} )</th>
<th>( \nu_1^{U} )</th>
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From the table we find that for fuzzy expected queue length of low priority \( \tilde{E}[N_2] \), the value of \( \tilde{E}[N_2] \) at \( \alpha=1 \) is 2.03, indicating that it is definitely possible that the number of low priority customers in the queue is 2.03. Moreover the range of \( \tilde{E}[N_2] \) at \( \alpha=0 \) is [0.6,11.8], indicating that the number of customers in the queue will never exceed 11.8 or fall below 0.6.

![Fig.1 The membership function for fuzzy queue length for low priority customer](image)

4.2. The fuzzy expected number of customer in high priority queue \( \tilde{E}[N_1] \)

By the same argument and Eq.(5), the \( \alpha \) - cuts of \( \tilde{E}[N_1] \) are

\[
\tilde{E}[N_1]_a^L = \frac{2\alpha^2+18\alpha+40}{2\alpha^2-43\alpha+176} \quad \tilde{E}[N_1]_a^U = \frac{2\alpha^2-26\alpha+84}{2\alpha^2-35\alpha+98}
\]

The membership function is
On Fuzzy Retrial Queue with Priority Subscribers

\[ \eta_{E[N_1]}(z) = \begin{cases} 
\frac{(18+4z)-\sqrt{(4+376z+441z^2)}}{2(-2+2z)} & ; \frac{5}{22} \leq z \leq \frac{4}{9} \\
\frac{(-26-35z)+\sqrt{(4+3276z+441z^2)}}{2(-2+2z)} & ; \frac{4}{9} \leq z \leq \frac{6}{7} 
\end{cases} \]

as shown in fig. 2. Applying the Yager ranking index method stated (13), the suitable queue length of high priority is given by,

\[ O\tilde{E}[N_1] = \int_{0}^{1} \frac{1}{2} \left[ \frac{2\alpha^2 + 18\alpha + 40}{2\alpha^2 - 43\alpha + 176} + \frac{2\alpha^2 - 26\alpha + 84}{2\alpha^2 + 35\alpha + 98} \right] d\alpha = 0.4730 \]

Table 2 reports \( \alpha \)- cuts of arrival, service and retrial rates and the fuzzy expected number of customers in high priority queue for the same eleven selected \( \alpha \)- values. From Table 2, we again insight into the possible expected number of customers in high priority queue.

Fig.2. The membership function for fuzzy expected queue length of high priority customers

<table>
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5. Conclusion

This paper applies the concepts of α-cuts and Zadeh’s extension principles to retrial queuing, model with priority customers and constructs membership functions of the queue length of high priority and low priority customer using paired NLP models. Following the proposal approach, α-cuts of the membership functions are found and their interval limits inverted to attain closed form expressions for the system characteristics. The approach proposed in this paper provides practical information for system manager and practitioners.

References

DSW ALGORITHMIC APPROACH TO FUZZY RETRIAL QUEUES WITH PRIORITY DISCIPLINE

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ABSTRACT

Retrial queueing models with priority discipline have a wide range of application in practical situations and are widely used in communication systems, computer, etc., In this paper, the priority disciplined retrial queueing models (preemptive priority, non preemptive priority) in which the arrival rate, service rate and retrial rate are fuzzy numbers. Approximate method of Extension namely DSW (Dong, Shah and Wong) algorithm is used to define membership functions of the system characteristics of priority retrial queuing system. DSW algorithm is based on the $\alpha$-cut representation of fuzzy sets in a standard interval analysis. The system characteristics obtained for the priority retrial queueing models are fuzzy subsets containing the whole initial information. The priority retrial queues with uncertain data have more use and have broader range of applications. To demonstrate, the validity of the proposed procedure, a numerical example is illustrated.

Keywords: Fuzzy set theory, Retrial Queueing models, Priority discipline, DSW algorithm.

1. INTRODUCTION

Queueing system with repeated attempts (trials) are characterized by the phenomenon that a customer finding all the servers busy upon arrival is obliged to leave the service area and repeat his request for service after some random time. Between trials, the blocked customer joins a pool of unsatisfied customers called “Orbit”. Retrial queues have been widely used to model many problems arising in telephone switching system, telecommunication networks, computer networks and computer systems etc., [1, 2, 3].
Retrial queueing models considered have the property that the customers are receiving service on a first come-first served basis. This is not only the manner of service and there are many alternatives such as last come-first served, selection in random order and selection by priority.

In priority schemes, customers with the highest priority begins to be served immediately and customers from the lowest priority cannot be queued, but after some random repeat an attempt to get service [1, 4]. There are two further refinements possible in priority situation, namely preemption and non-preemption.

In non-preemptive case, when a high priority customer finds the server busy, he enters the priority queue and waits and is served as soon as the server is free. If a low priority customer finds the server busy on his arrival, he enters the retrial group in order to seek service after a random amount of time. Low priority customers can be served only when there are no high priority customers in the priority queue. In the preemptive priority case; when an arriving high priority customer finds the server busy with a low priority customer, he pre-empts the low priority customer in service and begins to be served, after, all high priority customers in the priority queue have been served. The preempted low priority customer is served for remaining service time [5].

In practice, the retrial queueing model with priority discipline, the input data arrival rate, service rate and retrial rate are uncertainly know. Uncertainty is resolved by using fuzzy set theory. Hence the classical retrial queueing model with priority discipline will have more application if it is expanded using fuzzy models. Relatively few articles have been published on the topic of fuzzy queues. Based on Zadeh’s extension principle [6], researchers like Li and Lee [7], Negi and Lee [8], Kao et al [9], Chen[10,11] have analyzed fuzzy queues. Using the same principle, fuzzy retrial queues are studied by [12, 13, 14, 15].

2. RETRIAL QUEUES WITH PRIORITY DISCIPLINE

Consider a single server retrial queueing system with priority discipline, infinite calling population, in which the rate of arrival is \( \lambda \), rate of service is \( \mu \) and rate of retrial is \( \theta \). To establish the priority discipline in fuzzy retrial queueing model, we must compare the average total cost of inactivity for the three cases: no priority discipline, preemptive priority discipline, non-preemptive priority discipline which are denoted respectively by \( C \), \( C' \) and \( C'' \).

(a) No priority retrial queueing model:
Average total cost of inactivity when there is no priority discipline, \( C \).

\[
C = (C_1\lambda_1+C_2\lambda_2) W, \quad \text{where} \quad W = \frac{\lambda + \theta}{(\mu - \lambda)\theta}
\]

(b) Preemptive Priority retrial queueing model
Average total cost of inactivity when there is Preemption priority, \( C' \)

\[
C' = C_1L_{P1} + C_2L_{P2}, \quad \text{where} \quad L_{P1} = \frac{\rho_1^2 (1 + \rho_1 \cdot \rho)}{(1-\rho_1)^2} \quad \text{and}
\]
DSW ALGORITHMIC APPROACH TO FUZZY RETRIAL QUEUES WITH PRIORITY DISCIPLINE

\[ L_{p2} = \rho_2 + \frac{\rho_1}{1-\rho_1} \left( \frac{\lambda_2}{\theta} \right) + \frac{\rho_2}{(1-\rho_1)^2} \left( \frac{\rho_2 + \rho \rho_2}{1-\rho_1} \right) + \frac{3\rho_2}{2(1-\rho_1)^2} \]

(c) Non-Preemptive Priority retrial queueing model

Average total cost of inactivity when there is non preemptive priority, \( C'' \)

\[ C'' = C_1 L_1 + C_2 L_2, \] where \( L_1 = \frac{\rho \rho_1}{1-\rho_1} + \rho \) and \( L_2 = \frac{\rho \rho_2}{(1-\rho_1)^2} + \frac{\lambda_2}{\theta} \frac{1 - \rho}{\mu} + \frac{\rho_2}{\mu} ; \rho_1 = \frac{\lambda_1}{\mu} \) and \( \rho_2 = \frac{\lambda_2}{\mu} \)

Comparison of the three total costs shows which of priority discipline minimizes the average total cost function of inactivity.

3.1 Fuzzy Retrial Queues with Priority Discipline

Fuzzy retrial queues with priority discipline are described by fuzzy set theory. This paper develops fuzzy retrial queueing models with priority discipline in which the input source arrival rate, service rate and retrial rate are uncertain parameters. Approximate methods of extension is propagating fuzziness for continuous valued mapping determined the membership functions for the output variable.

DSW algorithm [16] is one of the approximate methods make use of intervals at various \( \alpha \)-cut levels in defining membership functions. It was the full \( \alpha \)-cut intervals in a standard interval analysis. The DSW algorithm greatly simplifies manipulation of the extension principle for continuous valued fuzzy variables, such as fuzzy numbers defined on the real line. It prevents abnormality in the output membership function due to application of the discrimination teaching on the fuzzy variable domain, and it can prevent the widening of the resulting functional value set due to multiple occurrence of variables in the functional expression by conventional interval analysis method.

3.2 Interval analysis Arithmetic

Let \( I_1 \) and \( I_2 \) be two interval numbers defined by ordered pairs of real numbers with lower and upper bounds.

\[ I_1 = [a, b], \ a \leq b; \ I_2 = [c, d], \ c \leq d. \]

Define a general arithmetic property with the symbol \(*\), where \(* = [+, -, \times, \div]\) symbolically the operation.

\[ I_1 * I_2 = [a, b] * [c, d] \]

represents another interval. The interval calculation depends on the magnitudes and signs of the elements \( a, b, c \) and \( d \).

\[ [a, b] + [c, d] = [a + c, b + d] \]
\[ [a, b] - [c, d] = [a - d, b - c] \]
\[ [a, b] \times [c, d] = [\min (ac, ad, bc, bd), \max (ac, ad, bc, bd)] \]
\[ [a, b] \div [c, d] = [a, b] \times \left[ \frac{1}{d}, \frac{1}{c} \right] \] Provided that \( 0 \notin [c, d] \)

\[ \alpha [a, b] = \begin{cases} [aa, ab] & \text{for } \alpha > 0 \\ [ab, aa] & \text{for } \alpha < 0 \end{cases} \]

where ac, ad, bc, bd are arithmetic products and \( \frac{1}{d}, \frac{1}{c} \) are quotients.

### 4.1 DSW Algorithm

Any continuous membership function can be represented by a continuous sweep of \( \alpha \)-cut in term from \( \alpha = 0 \) to \( \alpha = 1 \). Suppose we have single input mapping given by \( y = f(x) \) that is to be extended for fuzzy sets \( \tilde{B} = f(\tilde{A}) \) and we want to decompose \( \tilde{A} \) in to the series of \( \alpha \)-cut intervals say \( I_{\alpha} \). It uses the full \( \alpha \)-cut intervals in a standard interval analysis. The DSW algorithm [16] consists of the following steps:

(i) Select a \( \alpha \)-cut value where \( 0 \leq \alpha \leq 1 \).

(ii) Find the intervals in the input membership functions that correspond to this \( \alpha \).

(iii) Using standard binary interval operations, compute the interval for the output membership function for the selected \( \alpha \)-cut level.

(iv) Repeat steps (i) – (iii) for different values of \( \alpha \) to complete a \( \alpha \)-cut representation of the solution.

### 4.2 Solution Procedure

Decisions relating the priority discipline for a retrial queueing system are mainly based on a cost function.

\[ C = \sum_{i=1}^{n} C_i L_i \]

where \( C_i \) is the unit cost of inactivity for units in class \( i \), \( L_i \) is the average length in the system for unit of class \( i \). Let us consider a retrial queueing model with two unit classes arrive at \( \alpha_1 \) of arrivals belong to one of the classes, and \( \alpha_2 \) are in the other class. The average arrival rate at the system follows a Poisson process, is approximately known and is given by the triangular fuzzy number \( \tilde{\lambda} \), the service rate from a single server is the same for both unit classes, follows an exponential pattern and is distributed according to the triangular fuzzy number \( \tilde{\mu} \) and the retrial of the low priority customers follows an exponential pattern and is given by the triangular fuzzy number \( \tilde{\theta} \). The membership function of arrival rate, service rate and retrial rate, service rate and retrial rate is denoted as \( \eta_{\tilde{\lambda}}, \eta_{\tilde{\mu}} \) and \( \eta_{\tilde{\theta}} \) respectively and is given as follows:
The possible distribution of unit cost of inactivity for unit in the same class, is established by a triangular fuzzy number $\tilde{C}_A$, $\tilde{C}_B$ with membership function.

$$\eta_{\tilde{C}_A} = \begin{cases} \frac{C_A-a_4}{b_4-a_4}, & a_4 \leq C_A \leq b_4 \\ \frac{c_4-C_A}{c_4-b_4}, & b_4 \leq C_A \leq c_4 \\ 0, & \text{else where} \end{cases}$$

$$\eta_{\tilde{C}_B} = \begin{cases} \frac{C_B-a_5}{b_5-a_5}, & a_5 \leq C_B \leq b_5 \\ \frac{c_5-C_B}{c_5-b_5}, & b_5 \leq C_B \leq c_5 \\ 0, & \text{else where} \end{cases}$$
We choose three values of $\alpha$, viz, 0, 0.5 and 1. For instance when $\alpha = 0$, we obtain 5 intervals as follows:

$$\tilde{\lambda}_0 = [a_1, c_1]; \tilde{\mu}_0 = [a_2, c_2]; \tilde{\theta}_0 = [a_3, c_3]; \tilde{C}_{A,0} = [a_4, c_4]; \tilde{C}_{B,0} = [a_5, c_5].$$

Similarly when, $\alpha = 0.5$, 1, we obtain 10 intervals and it is denoted by $\tilde{\lambda}_{0.5}$, $\tilde{\mu}_{0.5}$, $\tilde{C}_{A,0.5}$, $\tilde{C}_{B,0.5}$, $\tilde{\lambda}_1$, $\tilde{\mu}_1$, $\tilde{\theta}_1$, $\tilde{C}_{A,1}$ and $\tilde{C}_{B,1}$.

The average total cost of inactivity in three situation (a) No priority discipline, (b) Preemptive priority discipline, (c) Non-preemptive priority discipline are calculated for different $\alpha$ level values. Interval arithmetic is used for computational efficiency.

(a) Average cost of inactivity when there is no priority discipline

$$\tilde{C}_0 = \left( \tilde{C}_{A,0} \tilde{\lambda}_{1,0} + \tilde{C}_{B,0} \tilde{\lambda}_{2,0} \right) \left( \frac{\tilde{\lambda}_0 + \tilde{\theta}_0}{(\tilde{\mu}_0 - \tilde{\lambda}_0) \tilde{\theta}_0} \right)$$

$$\tilde{C}_{0.5} = \left( \tilde{C}_{A,0.5} \tilde{\lambda}_{1,0.5} + \tilde{C}_{B,0.5} \tilde{\lambda}_{2,0.5} \right) \left( \frac{\tilde{\lambda}_{0.5} + \tilde{\theta}_{0.5}}{(\tilde{\mu}_{0.5} - \tilde{\lambda}_{0.5}) \tilde{\theta}_{0.5}} \right)$$

$$\tilde{C}_1 = \left( \tilde{C}_{A,1} \tilde{\lambda}_{1,1} + \tilde{C}_{B,1} \tilde{\lambda}_{2,1} \right) \left( \frac{\tilde{\lambda}_1 + \tilde{\theta}_1}{(\tilde{\mu}_1 - \tilde{\lambda}_1) \tilde{\theta}_1} \right)$$

(b) Average total cost of inactivity when there is preemptive discipline

$$\tilde{C}_{0'} = \tilde{C}_{A,0} \left( \frac{\tilde{\lambda}^2_{1,0}}{\tilde{\mu}_0} \left( 1 + \frac{\tilde{\lambda}_{2,0}}{\tilde{\mu}_0} \right) \left( \frac{\tilde{\lambda}_0}{\tilde{\mu}_0} \right) \right) + \tilde{C}_{B,0} \left( \frac{\tilde{\lambda}_{2,0}}{\tilde{\theta}_0} + \frac{\tilde{\lambda}_0}{\tilde{\mu}_0} \left( \frac{\tilde{\lambda}_0}{\tilde{\mu}_0} \right) \right) + \frac{\tilde{\lambda}_{2,0}}{\tilde{\mu}_0} \left( \frac{\tilde{\lambda}_{2,0}}{\tilde{\theta}_0} + \frac{\tilde{\lambda}_0}{\tilde{\mu}_0} \left( \frac{\tilde{\lambda}_0}{\tilde{\mu}_0} \right) \right)$$

$$+ \frac{3 \tilde{\lambda}_0}{\tilde{\mu}_0} \left( \frac{\tilde{\lambda}_{2,0}}{\tilde{\mu}_0} \right) + \frac{3 \tilde{\lambda}_0}{\tilde{\mu}_0} \left( \frac{\tilde{\lambda}_{2,0}}{\tilde{\mu}_0} \right)$$

$$\tilde{C}_{0.5'} = \tilde{C}_{A,0.5} \left( \frac{\tilde{\lambda}^2_{1,0.5}}{\tilde{\mu}_{0.5}} \left( 1 + \frac{\tilde{\lambda}_{2,0.5}}{\tilde{\mu}_{0.5}} \right) \left( \frac{\tilde{\lambda}_{0.5}}{\tilde{\mu}_{0.5}} \right) \right) + \tilde{C}_{B,0.5} \left( \frac{\tilde{\lambda}_{2,0.5}}{\tilde{\theta}_{0.5}} + \frac{\tilde{\lambda}_{0.5}}{\tilde{\mu}_{0.5}} \left( \frac{\tilde{\lambda}_{0.5}}{\tilde{\mu}_{0.5}} \right) \right) + \frac{3 \tilde{\lambda}_{0.5}}{\tilde{\mu}_{0.5}} \left( \frac{\tilde{\lambda}_{2,0.5}}{\tilde{\mu}_{0.5}} \right) + \frac{3 \tilde{\lambda}_{0.5}}{\tilde{\mu}_{0.5}} \left( \frac{\tilde{\lambda}_{2,0.5}}{\tilde{\mu}_{0.5}} \right)$$
DSW ALGORITHMIC APPROACH TO FUZZY RETRIAL QUEUES WITH PRIORITY DISCIPLINE

\[
\hat{C}_1 = \hat{C}_{A,1} \left( \frac{\tilde{\lambda}_{2,0.5}}{\mu_{0.5}} \right)^2 + \hat{C}_{B,1} \left( 1 - \frac{\tilde{\lambda}_{1,1,0.5}}{\mu_{0.5}} \right)^2 \left( \frac{\tilde{\lambda}_{2,1,0.5}}{\mu_{0.5}} \right) + \hat{C}_{B,1} \left( 1 - \frac{\tilde{\lambda}_{1,1,0.5}}{\mu_{0.5}} \right)^2 \left( \frac{\tilde{\lambda}_{2,2,1,0.5}}{\mu_{0.5}} \right) + \hat{C}_{B,1} \left( 1 - \frac{\tilde{\lambda}_{1,1,0.5}}{\mu_{0.5}} \right)^2 \left( \frac{\tilde{\lambda}_{2,3,2,1,0.5}}{\mu_{0.5}} \right)
\]

(c) Average total cost of inactivity when there is a non-preemptive priority discipline

\[
\hat{C}_0^* = \hat{C}_{A,0} \left( \frac{\tilde{\lambda}_{2,0}}{\mu_{0.5}} \right) + \hat{C}_{B,0} \left( 1 - \frac{\tilde{\lambda}_{1,0}}{\mu_{0.5}} \right) \left( \frac{\tilde{\lambda}_{2,0}}{\mu_{0.5}} \right) + \hat{C}_{B,0} \left( 1 - \frac{\tilde{\lambda}_{1,0}}{\mu_{0.5}} \right) \left( \frac{\tilde{\lambda}_{2,1}}{\mu_{0.5}} \right) + \hat{C}_{B,0} \left( 1 - \frac{\tilde{\lambda}_{1,0}}{\mu_{0.5}} \right) \left( \frac{\tilde{\lambda}_{2,2}}{\mu_{0.5}} \right) + \hat{C}_{B,0} \left( 1 - \frac{\tilde{\lambda}_{1,0}}{\mu_{0.5}} \right) \left( \frac{\tilde{\lambda}_{2,3}}{\mu_{0.5}} \right)
\]

\[
\hat{C}_{0.5}^* = \hat{C}_{A,0.5} \left( 1 - \frac{\tilde{\lambda}_{1,0.5}}{\mu_{0.5}} \right) + \hat{C}_{B,0.5} \left( 1 - \frac{\tilde{\lambda}_{1,0.5}}{\mu_{0.5}} \right) \left( \frac{\tilde{\lambda}_{2,0.5}}{\mu_{0.5}} \right) + \hat{C}_{B,0.5} \left( 1 - \frac{\tilde{\lambda}_{1,0.5}}{\mu_{0.5}} \right) \left( \frac{\tilde{\lambda}_{2,1.0.5}}{\mu_{0.5}} \right) + \hat{C}_{B,0.5} \left( 1 - \frac{\tilde{\lambda}_{1,0.5}}{\mu_{0.5}} \right) \left( \frac{\tilde{\lambda}_{2,2.0.5}}{\mu_{0.5}} \right) + \hat{C}_{B,0.5} \left( 1 - \frac{\tilde{\lambda}_{1,0.5}}{\mu_{0.5}} \right) \left( \frac{\tilde{\lambda}_{2,3.2.1.0.5}}{\mu_{0.5}} \right) + \hat{C}_{B,0.5} \left( 1 - \frac{\tilde{\lambda}_{1,0.5}}{\mu_{0.5}} \right) \left( \frac{\tilde{\lambda}_{2,3.3.2.1.0.5}}{\mu_{0.5}} \right)
\]

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Comparison of the three total costs shows which of the priority discipline is preferable.

5. NUMERICAL EXAMPLE

Consider a telephone switching system in which calls arrive in two class with utilization of 15% and 85% calls arrive at this system in accordance with a Poisson process, the service times and retrial times follow an exponential distribution. The arrival rate, service rate and retrial rate are triangular fuzzy numbers given by \( \tilde{\lambda} = [26, 30, 32] \), \( \tilde{\mu} = [38, 40, 42] \) and \( \tilde{\theta} = [22, 24, 26] \) per minute, respectively. The possibility distribution of unit cost of inactivity for units of the two classes are triangular fuzzy number \( \tilde{C}_A = [15, 17, 19] \) and \( \tilde{C}_B = [2, 3, 5] \) respectively. The system manager wants to evaluate the total cost of inactivity when there is no priority discipline, preemptive priority discipline, non-preemptive priority discipline in the retrial queue.

No Priority discipline: \( \tilde{C}_0 = [11.84, 99.83], \tilde{C}_{0.5} = [19.88, 57.54], \tilde{C}_1 = [40.16, 40.16] \).

Preemptive Priority discipline: \( \tilde{C}_{0.5} = [9.65, 92.11], \tilde{C}_{0.5} = [11.93, 36.67], \tilde{C}_1 = [30.29, 30.29] \).

Non-Preemptive Priority discipline: \( \tilde{C}_0 = [8.02, 63.14], \tilde{C}_{0.5} = [17.04, 25.75], \tilde{C}_1 = [27.27, 27.27] \).
6. CONCLUSION

Comparison of the three total costs shows which of the priority disciplines minimizes the average total cost function of inactivity. Even though they are overlapping fuzzy numbers, so minimum average total cost of inactivity is achieved with non-preemptive discipline. The conclusion can therefore be made that the optimum selection of a priority discipline selection of a priority discipline for the fuzzy retrial queueing model that we studied entails establishing a non preemptive priority discipline, in which class A units will be assigned a higher priority.

REFERENCES