Chapter - VII

Hyperchaotic Behaviour of a Fourth-Order Autonomous Electric Circuit with a Diode Pair
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HYPERCHAOTIC BEHAVIOUR OF A FOURTH-ORDER AUTONOMOUS ELECTRIC CIRCUIT WITH A DIODE PAIR

7.1. INTRODUCTION

In original Chua's circuit, a nonlinear resistor is called Chua's diode is the unique nonlinear electric element. It plays an important role in the circuit. Due to the existence of this nonlinear element Chua's circuit exhibits a variety of nonlinear phenomena, such as chaos, bifurcation and so on [62,153,227–229]. The characteristic of Chua’s diode is described by a continuous piecewise-linear function with three segments and two nondifferential break points [152,154,180,181]. However, the characteristics of nonlinear devices in practical circuits are always smooth and the implementation of piecewise-linear function requires a large amount of circuitry compared with smooth cubic function. Therefore, it is significant to investigate Chua's circuit with a smooth cubic nonlinearity from practical viewpoint [184]. Hartley proposed to replace the piecewise-linear nonlinearity in Chua's circuit with a smooth cubic nonlinearity.

In the present report the behavior of a hyperchaotic fourth-order autonomous electric circuit has been studied. This circuit consists of two active elements, one linear negative conductance and one smooth cubic nonlinearity exhibiting a symmetrical piecewise-linear \( v-i \) characteristic. Two inductances \( (L_1, L_2) \), two capacitances \( (C_1, C_2) \) and the capacitance \( C_2 \) serve as the control parameter.

Hyperchaos is defined as a chaotic attractor with more than one positive Lyapunov exponents, i.e., its dynamics expand in more than one direction [62].
In other words, the dynamics expand not only small line segments, but also small area elements, there giving rise to a 'thick' chaotic attractor. Most hyperchaotic and bifurcation effects cited in the literature have been observed in electric circuits. They include the period-doubling route to chaos, the intermittency route to chaos, the quasiperiodicity route to chaos and of course the crisis [152,154,180,187,218,230]. This popularity is attributed to the advantages which electric circuits offer to experimental hyperchaos studies, such complicated hyperchaotic wave forms are expected to be utilized for realization of several hyperchaotic applications such a chaos communication system with robustness against various interferences including multi-user access [184,186,221,230–232]. The plan of the paper is as follows. In section 7.2, we present the details of realization of the proposed autonomous circuit. The results of the observations from the laboratory experimental simulation and the conformation through analytical calculation and numerical simulation on the dynamics of the circuit are presented in section 7.3. Finally, in section 7.4, we summarize and conclude the results and indicate further direction.

7.2. CIRCUIT DESCRIPTION AND EXPERIMENTAL RESULTS

The fourth-order hyperchaotic autonomous electric circuit we have studied is presented in Fig. 7.1. It consists of two active elements, one linear negative conductance \( G_1 \) and smooth cubic nonlinearity with an odd symmetric piecewise-linear \( v-i \) characteristic [184]. This fourth-order circuit is based on a third-order autonomous piecewise-linear circuit introduced by Chua and Lin, capable to realize every member of the Chua's circuit family [152]. Applying Kirchoff's laws, the set of four first-order coupled autonomous differential equations as given below:
\[ C_1 \frac{dV_1}{dt} = -i_N - i_{i_1} \]
\[ C_2 \frac{dV_2}{dt} = i_N - i_{i_2} \]
\[ L_1 \frac{di_{i_1}}{dt} = i_{i_1} - V_1 \]
\[ L_2 \frac{di_{i_2}}{dt} = V_2 \]

... (7.1)

While \( V_1 \) and \( V_2 \) are the voltages across the Capacitors \( C_1 \) and \( C_2 \), \( i_{i_1} \) and \( i_{i_2} \) denotes the currents through the inductances \( L_1 \) and \( L_2 \) respectively and the characteristics of linear negative conductance is mathematically represented by \( i_{G_1} = -G_1 V_1 \). Here the term \( i_N = f(V_1 - V_2) \) representing the characteristic of the smooth cubic nonlinearity can be expressed mathematically:

\[ f(V_1 - V_2) = a(V_1 - V_2) + b(V_1 - V_2)^3 \]

... (7.2)

Fig. 7.1: Circuit realization of the fourth-order hyperchaotic autonomous electric circuit.
For our present experimental study we have chosen the following typical values of the circuit in Fig. 7.1. Were $L_1 = 85 \text{ mH}$, $L_2 = 33 \text{ mH}$, $C_1 = 10 \text{ nF}$, $C_2 = 33 \text{ nF}$ and the characteristics of linear negative conductance $G_1 = -0.5 \text{ mS}$. Here the variable capacitor ‘$C_2$’ is assumed to be the control parameter. By increasing the value of ‘$C_2$’ from 5 $\text{nF}$ to 50 $\text{nF}$, the circuit behavior of Fig. 7.1 is found to transit from a period-doubling route to chaos and then to hyperchaotic attractor through period-doubling bifurcation behavior followed by period-doubling windows and boundary crisis [230–232], etc. The hyperchaotic attractors of fourth-order autonomous circuit with the smooth cubic nonlinearity projected onto different planes are shown in Fig. 7.2. Experimental time series were registered using a simulation storage oscilloscope for discrete values of $C_1$ and $C_2$ are shown if Fig. 7.3.
Fig. 7.2: Experimental results of the projections of hyperchaotic attractor onto different planes.
The distribution of power in a signal $x(t)$ is the most commonly quantified by means of the power density spectrum or simply power spectrum. It is the magnitude-square of the Fourier transforms of the signal $x(t)$. It can detect the presence of hyperchaos when the spectrum is broad-banded. The power spectrum corresponding to the voltages $V_1(t)$ and $V_2(t)$ waveforms across the capacitors $C_1$ and $C_2$ for the hyperchaotic regimes are shown in Fig. 7.4 which resembles broad-band spectrum noise.
7.3. NUMERICAL CONFIRMATION

The hyperchaotic dynamics of circuit as shown in Fig. 7.1 is studied by numerical integration of the normalized differential equations [232]. For a convenient numerical analysis of the experimental system given by Eq. (7.1), we rescale the parameters as $V_1=Vx_1$, $V_2=Vx_2$, $i_{L_1} = \sqrt{\frac{C_1}{L_1}}Vx_3$, $i_{L_2} = \sqrt{\frac{C_1}{L_1}}Vx_4$, $a_1 = a\sqrt{\frac{L_1}{C_1}}$, $a_2 = b\sqrt{\frac{L_1}{C_1}}$, $\nu = \frac{C_1}{C_2}$, $\gamma = \frac{1}{G_1}\sqrt{\frac{C_1}{L_1}}$, $\beta = \frac{L_1}{L_2}$, and then redefine $\tau$ as $t$. Eq. (7.1) and Eq. (7.2) reduce to the following set of normalized equations of the fourth-order hyperchaotic autonomous electric circuit as given below:

$$
\begin{align*}
\dot{x}_1 &= -\left(\alpha_1 (x_1 - x_2) + \alpha_2 (x_1 - x_2)^3 + x_1\right) \\
\dot{x}_2 &= \nu\left(\alpha_1 (x_1 - x_2) + \alpha_2 (x_1 - x_2)^3 - x_4\right) \\
\dot{x}_3 &= \gamma x_3 - x_1 \\
\dot{x}_4 &= \beta x_2
\end{align*}$$

... (7.3)

The dynamics of Eq. (7.3) now depends upon the parameters $\alpha_1$, $\alpha_2$, $\nu$, $\gamma$ and $\beta$.

The experimental results have been verified by numerical simulation of the normalized Eq. (7.3) using the standard fourth-order Runge-Kutta method for a specific choice of system parameters employed in the experimental simulation results. That is, in the actual experimental set up the capacitor ‘$C_2$’ is varied from $C_2 = 5 \text{ nF}$ upward to $50 \text{ nF}$. Therefore in the numerical simulation, we study the corresponding Eq. (7.3) for in the range $C_2 = 5 \text{ nF}$ to $50 \text{ nF}$. From our numerical investigations, we find that for the value of ‘$C_2$’ above $5 \text{ nF}$ periodic limit cycle motions is obtained. When the value of ‘$C_2$’ is increased to higher than $50 \text{ nF}$ particularly in the range
$C_2 = (5 \text{ } nF \text{ to } 50 \text{ } nF)$ the system displays a period-doubling route to chaos and then to hyperchaos through boundary condition [180]. These numerical results of the hyperchaotic attractor of fourth-order autonomous circuit with the smooth cubic nonlinearity projected onto different planes are shown in Fig. 7.5. Fig. 7.6 shows the numerical chaotic time series was registered using a discrete value of ‘$C_2$’ serving as the control parameter. It is gratifying to note that the numerical results agree qualitatively very well with that of the experimental simulation results.
Fig. 7.5: Numerical results of the projections of hyperchaotic attractor onto different planes.
Fig. 7.6: Numerical results of the hyperchaotic time series.

7.3.1. One parameter Bifurcation diagram and Lyapunov exponents

The main features of the fourth-order hyperchaotic autonomous electric circuit can be summarized in the one parameter bifurcation diagram drawn in the $(C_2 - x_2)$ plane (Fig. 7.7(a)). Note that $x_2$ is the rescaled variable in Eq. (7.3), $x_2 = V_2/V$. This bifurcation diagram clearly indicates that in the region $C_2 = (5 nF, 50 nF)$ the system undergoes period-doubling bifurcation sequence to chaos, shows periodic windows through hyperchaotic region are observed in Fig. 7.1.

The Lyapunov exponent’s $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ were obtained using the Wolf algorithm. For periodic orbits $\lambda_1 = 0$, $\lambda_2$, $\lambda_3$, $\lambda_4 < 0$, for quasi-periodic orbits $\lambda_1 = \lambda_2 = 0$, $\lambda_3$, $\lambda_4 < 0$, while for chaotic attractor $\lambda_1 > 0$, $\lambda_2 = 0$, $\lambda_3$, $\lambda_4 < 0$ and for hyperchaotic attractor $\lambda_1 > \lambda_2 > 0$, $\lambda_3 = 0$, $\lambda_4 < 0$. The Lyapunov spectrum in the $(C_2 - \lambda_1, \lambda_2)$ plane, that is the first two maximal Lyapunov exponents versus fixed range of the control parameter as $C_2$ is increased, is shown in Fig. 7.7(b). This correlates to the bifurcation diagram, Fig. 7.7(a). In the range $(50 nF > C_2 > 5 nF)$ the system exhibits periodic windows with no positive Lyapunov exponent. When $C_2$ is increased further in the range $(50 nF > C_2 > 5 nF)$, the system becomes chaotic with a single positive Lyapunov exponent ($\lambda_1$).
Fig. 7.7(a): For the normalized Eq. (7.3): One parameter bifurcation diagram in the \((C_2 - x_2)\) plane at fixed range of control parameter of \(C_2 = (5 \text{ nF}, 50 \text{ nF})\).

Fig. 7.7(b): For the normalized Eq. (7.3): Two largest Lyapunov exponents versus \(C_2\) for two trajectories in the \((C_2 - \lambda_1, \lambda_2)\) plane.

The chaotic nature is also characterized by a single positive Lyapunov exponent \((\lambda_1)\). It is quite fascinating to look at the window region in the range \((50 \text{ nF} > C_2 > 5 \text{ nF})\), which corresponds to an entirely different dynamical behavior. It has been observed that for \(C_2 > 5 \text{ nF}\) the attractors of the system are in any one of the smooth regions of the piecewise segments. Correspondingly, the attractors exhibit
one of the generic types of bifurcations, namely period-doubling, saddle-node, or hopf-bifurcations. A section of the bifurcation diagram and the Lyapunov spectrum for the range $50 \, nF > C_2 > 5 \, nF$ are shown in Figs. 7.7(a) and 7.7(b), respectively, for clarity. The Lyapunov spectrum in the $(C_2 - \lambda_1, \lambda_2)$ plane, that is the first two maximal Lyapunov exponents versus fixed range of the control parameter as $C_2$ is increased, the hyper chaotic Lyapunov exponents shown in Fig. 7.7(b) for $C_2 = 47.739999 \, nF$ the Lyapunov exponents are $\lambda_1 = 0.02700$, $\lambda_2 = 0.00032$, $\lambda_3 = -0.58956$ and $\lambda_4 = -8.72141$.

The numerical computational results of system (Eq. (7.3)) are shown in Table 3.

**Table - 3**

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<th>$C_2$</th>
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<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
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</tr>
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### 7.4. MULTI - SIMULATION RESULTS

In recent years, circuit simulators such as multi-simulation have been used for the simulation of the chaotic circuits. Multi-simulation of the hyperchaotic behaviour of a fourth order autonomous electric circuit with a diode pair (Fig.7.1) were carried-out using $C_1 = 10 \, nF$, $C_2 = 33 \, nF$, $L_1 = 85 \, mH$, $L_2 = 33 \, mH$, $R_L = 6 \, \Omega$, and $R_N = -2,000 \, \Omega$. For the implementation of the non-linear element $G_N$, a negative impedance ($-R_N$) is connected in parallel with a diode pair (e.g. $1N4148$, $B_p = 0.65 \, V$, $R_D = 25 \, \Omega$ at $5 \, mA$). Typical phase portraits of the system corresponding to different regimes are shown in Fig. 7.8. There is good agreement between the numerical simulation (Fig. 7.5) and the multi-simulation results.
Fig. 7.8: Simulation results of the projections of hyperchaotic attractor onto different planes.
7.5. CONCLUSIONS

We have presented a fourth-order hyperchaotic autonomous electric circuit which has symmetrical piecewise-linear elements. We can confirm hyperchaotic attractor on computer simulation or circuit experiments. The attractive feature of this circuit is the presence of hyperchaotic attractor over a range of parameter values, which might be useful for applications in controlling of hyperchaos, synchronization and in secure communication system.