Chapter - II

Linear and Nonlinear Circuits
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LINEAR AND NONLINEAR CIRCUITS

2.1. INTRODUCTION

In the earlier chapter we have seen that nonlinear electronic circuits can be used to model chaotic dynamical systems. These nonlinear circuits are essentially interconnections of various linear circuit elements like linear resistors, capacitors, inductors, etc. and at least one nonlinear element.

In this chapter, we introduce various linear and nonlinear elements, discuss the construction of nonlinear circuits using these circuit elements and explain how bifurcations and chaos set in a few simple nonlinear electronic circuits.

2.2. LINEAR AND NONLINEAR CIRCUIT ELEMENTS

In this subsection, we shall briefly discuss the characteristics of various circuit elements, namely resistors, capacitors and inductors [2,14,93]. They can be linear or nonlinear depending upon their characteristic curves, namely the \(v - i\) (voltage-current), \(v - q\) (voltage-charge) and \(i - \phi\) (current-magnetic flux) curves.

(a) Resistors

A two-terminal resistor (Fig. 2.1(a)) whose current \(i(t)\) and voltage \(v(t)\) falls on some fixed (characteristic) curve in the \((v,i)\) plane, represented by the equation

\[ f_R(v, i) = 0 \]

at any time \(t\), is called a time-invariant resistor. If the measured \((v,i)\) characteristic curve is a straight line passing through the origin as shown in Fig. 2.1(b), then the resistor is said to be linear one and it satisfies the Ohm’s law

\[ v(t) = Ri(t) \]

... (2.1)
Fig. 2.1: (a) A two terminal linear resistor \((R)\), (b) Its \((v−i)\) characteristics curve, (c) Schematic symbol of nonlinear resistor \((N_R)\) and (d–f) the \((v−i)\) characteristic curves for three practical nonlinear resistors.

If a resistor is characterized by a \((v−i)\) curve other than a straight line through the origin, it is called a nonlinear resistor. The schematic symbol of nonlinear resistor, \(N_R\) is shown in Fig. 2.1(c). In this case, the resistor can no longer be described by a single number and hence the entire \((v−i)\) curve must be given. This may be specified either graphically by a curve or analytically by a mathematical relationship. For example, the characteristic curves of certain typical nonlinear resistors are given in Figs. 2.1(d) - 2.1(f).

**Examples**

*Tunnel diode, Glow tube and Neon bulb.*
(b) Capacitors

A two-terminal circuit element (Fig. 2.2(a)) whose charge \( q(t) \) and voltage \( v(t) \) falls on some fixed (characteristic) curve in the \((v-q)\) plane, represented by the equation \( f_c(q, v) = 0 \), at any time \( t \), is called a time-invariant capacitor. If the measured \((q-v)\) characteristic curve is a straight line passing through the origin as shown in Fig. 2.2(b), then the capacitor is said to be linear one and it satisfies the current-voltage relation.

\[
i = C \frac{dv}{dt} \quad \ldots \quad (2.2)
\]

![Fig. 2.2: (a) A two terminal linear capacitor (C), (b) Its (q − v) characteristics curve, (c) Schematic symbol of nonlinear capacitor (Nc) and (d-f) the (q − v) characteristic curves for three practical nonlinear capacitors.](image)

In all other cases, the capacitor is a nonlinear one. The schematic symbol of nonlinear capacitor, \( N_c \) is shown in Fig. 2.2(c). The characteristic curves of certain typical nonlinear capacitors are also given in Figs. 2.2(d) – 2.2(f).
Example

Varactor diode

(c) Inductors

A two-terminal circuit element (Fig. 2.3(a)) whose flux $\phi(t)$ and current $i(t)$ falls on some fixed (characteristic) curve in the $(\phi - i)$ plane, represented by the equation $f_L(\phi, i) = 0$ at any time $t$, is called a time-invariant inductor. If the measured $(\phi - i)$ characteristic curve is a straight line passing through the origin as shown in Fig. 2.3(b), then the inductor is said to be linear one and it satisfies the voltage-current relation.

$$v = L \frac{di}{dt}$$

Fig. 2.3: (a) A two terminal linear inductor ($L$), (b) Its $(i - \phi)$ characteristics curve, (c) Schematic symbol of nonlinear inductor ($N_L$) and (d-f) the $(i - \phi)$ characteristic curves for three practical nonlinear inductors.
In all other cases, the inductor is nonlinear one. The schematic symbol of nonlinear inductor, \( N_L \) is shown in Fig. 2.3(c). The characteristic curves of certain nonlinear inductors are given in Figs. 2.3(d) - 2.3(f).

Example

*Josephson junction*

2.3. REALIZATION OF NONLINEAR RESISTORS

Of all the possible nonlinear circuit elements, nonlinear resistors are easy to build and model. In particular, negative resistance nonlinear elements are of considerable importance. There are many ways to synthesize a negative resistance element. In this thesis we make use of two different realizations, which are slight variations of each other. One of the methods is a piecewise linear approximation using Op-amp and passive resistances. This functional realization was first proposed by Chua and co-workers [143] and later on realized by Kennedy [144]. A second means of synthesizing a negative resistance element is by connecting three positive linear resistances to a *voltage controlled voltage source* (VCVS) to form a negative resistance converter [93, 144]. This arrangement can also be implemented by means of Op-amps.

2.3.1(a) Chua’s Diode

*Chua’s diode* [144] is a simple nonlinear resistor, \( N_R \), with a three segment piecewise-linear (*voltage* \( v \)–*current* \( i \)) characteristic curve in the region of interest as shown in Fig. 2.4. It is a simple voltage controlled two-terminal element. It is widely used by circuit theorists [145] and electronic engineers [141]. The most
interesting part of the typical \((v_R-i_R)\) characteristic curve of the Chua’s diode consists of the odd-symmetric three segment piecewise-linear form as shown in Fig. 2.4. Its most important qualitative features are its piecewise nature and the negative resistance behaviour as the slope is negative between the two break points.

Fig. 2.4: Example of a piecewise-linear characteristics \((v_R-i_R)\) curve for Chua’s diode.

In the figure, \(G_a = m_1\) and \(G_b = m_0\) are the inner and outer slopes respectively and the breakpoints are located at \(v_R = -B_p\) and \(v_R = B_p\) in a set of consistent units.

The mathematical representation of the characteristic curve [93, 143] is given by

\[
\begin{align*}
    i_R &= g(v_R) = \\
    &\begin{cases} \\
    G_a v_R + (G_a - G_b) Bp & v_R < -Bp \\
    G_a v_R & -Bp \leq v_R \leq Bp \\
    G_b v_R - (G_a - G_b) Bp & v_R > Bp \\
    \end{cases} \quad \text{(2.4)}
\end{align*}
\]

\[
\begin{align*}
    g(v_R) &= G_b v_R + 0.5 (G_a - G_b) \left( |v_R + Bp| - |v_R - Bp| \right) \quad \text{(2.5)}
\end{align*}
\]

where \(Bp > 0\) (break-point voltage). For small voltages, the nonlinear resistance is negative.

2.3.1(b) A Simple Practical Implementation of Chua’s Diode

The Chua’s diode \((N_R)\) can be implemented discretely using two operational amplifiers (Op-amps) and six linear resistors [144] or fabricated as an IC chip [146].
Fig. 2.5(a) shows a practical implementation using discrete components. The Op-amps used are analog devices AD712 (dual BiFET) Op-amps and two 9 volts batteries. The circuit functions as a nonlinear resistor, $N_R$ with point characteristics as shown in Fig. 2.5(b). Typical values of the resistances.

\[ R_1 - R_6 \text{ are } 220 \, \Omega, 220 \, \Omega, 2.2 \, k\Omega, 22 \, k\Omega, 22 \, k\Omega \text{ and } 3.3 \, k\Omega \text{ respectively.} \]

The two 9 V batteries used to power the Op-amps ($A_1$ and $A_2$) give the voltages $V_+ = 9V$ and $V_- = -9V$. The slopes of the characteristic curve are given by

\[
G_a = m_1 = -R_2 / (R_6 R_3) - R_5 / (R_4 R_5) = -0.758 \, mA / V \quad \text{and} \quad G_b = m_0 = -R_2 / (R_6 R_3) + (1/R_4) = -0.406 \, mA / V .
\]

The breakpoints are determined by the saturation voltages $E_{sat}$ of the Op-amps. They are calculated as $B_p = E_{Sat} R_2 / (R_2 + R_1) = 7.61 \, V$ and $B_p = E_{Sat} R_6 / (R_5 + R_6) = 1.08 \, V$. The $(v_R - i_R)$ characteristic curve (Fig. 2.5(b)) of the Op-amp based Chua’s diode differs from the desired piecewise linear characteristic shown in Fig. 2.4 in that, Section 2.3.1(a) the latter has only three segments while the former has five segments, the outer two of which have positive slopes $G_c = m_2 = (1/R_2) = (1/220) \, mA / V$. Every physical resistor is eventually passive,
meaning simply that for suitable voltage across its terminals, the instantaneous power

\[ P(t) = (v(t)i(t)) \]

comsumed by a real resistor is positive. For large enough \(|v|\) or \(|i|\), therefore, the \((v_R - i_R)\) characteristics must lie only in the first and third quadrants of the \((v - i)\) plane. Hence, the \((v_R - i_R)\) characteristic of a real Chua’s diode must include at least two outer segments with positive slopes in the first and third quadrants (Fig. 2.5). From a practical point of view, as long as the voltages and currents in a given nonlinear circuit with Chua’s diode are restricted to the negative resistance region of the characteristic, the above mentioned outer segments will not affect the circuit’s dynamical behaviour. This is what we will assume in the present study.

### 2.3.2. VCVS Based Negative Resistance Element

A controlled source is a resistive two-port four terminal element consisting of two branches: a primary branch which is either an open circuit or a short circuit and a secondary branch which is either a voltage source or a current source. The voltage or current waveform in the secondary branch is controlled by (or dependent upon) the voltage or current of the primary branch. Therefore there exists four types of controlled sources (i) *current-controlled voltage source* (CCVS), (ii) *voltage controlled current source* (VCCS), (iii) *current controlled current source* (CCCS) and (iv) *voltage controlled voltage source* (VCVS). The two-port four terminal networks of VCVS is shown in Fig. 2.6. The VCVS is governed by the equations

\[ i_1 = 0 \]

\[ v_2 = \mu v_1 \]

where \(\mu =\) voltage transfer ratio. In matrix form these equations are written as
\[
\begin{bmatrix}
i_1 \\
v_2
\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix} \begin{bmatrix}
i_2 \\
v_1
\end{bmatrix}
\] ... (2.6c)

The characteristic properties of a VCVS are (i) no current flows into or out of the input terminals \((i_i = 0)\) and (ii) the voltage which appears across the output terminals is a function of the potential difference \((v_i)\) between the input terminals that is \(v_2 = f(v_i)\). This is shown graphically in Fig. 2.6(b). An operational amplifier (Op-amp) processes these properties. Hence it can be conveniently used a VCVS. By connecting passive resistance to this Op-amp as VCVS, we can configure a negative impedance converter (NIC). The configuration of the NIC, its schematic symbol and its characteristics \((v-i)\) relationship are shown in Figs. 2.6(c) and 2.6(d) respectively.

For this Op-amp circuit shown in Fig. 2.6(c),

**Fig. 2.6:** Voltage Controlled Voltage Source (VCVS): \(i_d = 0\) and \(v_o = f(v_d)\).
(a) block diagram, (b) voltage transfer characteristic of linear VCVS with gain A, (c) circuit configuration of Op-amp based negative converter using VCVS and (d) its \((v-i)\) characteristics.
\[ v_2 = \left( 1 + \frac{R_2}{R_1 R_3} \right) v_1 \]  

... (2.6d)

By applying Kirchoff’s current law and voltage law, it can be shown that for large values of voltage transfer ratio (\( \mu \)), the current that is flowing into the circuit is

\[ i = -\left( \frac{R_2}{R_1 R_3} \right) v_1 \]  

... (2.6e)

Hence obviously the impedance as seen from the point at which the voltage is applied is negative and is given by

\[ Z_n = -\left( \frac{R_1 R_3}{R_2} \right) \]  

... (2.6f)

By choosing \( R_1 = R_2 \),

\[ Z_n = -R_3 \]  

... (2.6g)

or the conductance

\[ g_n = -\left( \frac{1}{R_3} \right) \]  

... (2.6h)

This NIC is an attenuating element. Hence it should be always used in conjunction with an active device such as a PN junction diode to maintain stability of the circuit. This parallel combination of a NIC and PN junction diode is used as a nonlinear resistive element in chapter 4 and chapter 6.

2.4. DIODES AND CHARACTERIZATION

2.4.1. PN Junction Diode and Characterization

The effect described in the previous tutorial is achieved without any external voltage being applied to the actual PN junction resulting in the junction being in a state of equilibrium. However, if we were to make electrical connections at the ends
of both the $N$–type and the $P$–type materials and then connect them to a battery source, an additional energy source now exists to overcome the barrier resulting in free charges being able to cross the depletion region from one side to the other. The behaviour of the PN junction with regards to the potential barrier width produces an asymmetrical conducting two terminal device, better known as the **Junction Diode**. The symbol of PN junction diode is shown in Fig. 2.7(a).

![](image)

**Fig. 2.7(a): PN Junction Basic Symbol of Diode.**

A diode is one of the simplest semiconductor devices, which has the characteristic of passing current in one direction only. However, unlike a resistor, a diode does not behave linearly with respect to the applied voltage as the diode has an exponential $I - V$ relationship and therefore we can not described its operation by simply using an equation such as Ohm’s law.

If a suitable positive voltage (**forward bias**) is applied between the two ends of the PN junction, it can supply free electrons and holes with the extra energy they require to cross the junction as the width of the depletion layer around the PN junction is decreased. By applying a negative voltage (**reverse bias**) results in the free charges being pulled away from the junction resulting in the depletion layer width being increased. This has the effect of increasing or decreasing the effective resistance of the junction itself allowing or blocking current flow through the diode.
The current $I = \left( I_0 e^{\frac{V_2}{AT}} \right)$; then the depletion layer widens with an increase in the application of a reverse voltage and narrows with an increase in the application of a forward voltage. This is due to the differences in the electrical properties on the two sides of the PN junction resulting in physical changes taking place. One of the results produces rectification as seen in the PN junction diodes static $I - V$ (current-voltage) characteristics. Rectification is shown by an asymmetrical current flow when the polarity of bias voltage is altered as shown below. But before we can use the PN junction as a practical device or as a rectifying device we need to firstly bias the junction, i.e. connect a voltage potential across it. On the voltage axis above, “Reverse Bias” refers to an external voltage potential which increases the potential barrier. An external voltage which decreases the potential barrier is said to act in the “Forward Bias” direction. The PN junction diode is a simple nonlinear element, with a three segment smooth piecewise $v - i$ characteristics curve in the region of interest as shown in Fig. 2.7(b). There are two operating regions and three possible “biasing” conditions for the standard Junction Diode and these are:

![Fig. 2.7(b): Junction Diode Symbol and Static V – I Characteristics.](image)
Zero Bias: No external voltage potential is applied to the PN-junction.

Reverse Bias: The voltage potential is connected negative, (−ve) to the $P$-type material and positive, (+ve) to the $N$-type material across the diode which has the effect of increasing the PN-junction width.

Forward Bias: The voltage potential is connected positive, (+ve) to the $P$-type material and negative, (−ve) to the $N$-type material across the diode which has the effect of decreasing the PN-junction width.

2.4.2. Zener Diode and Characterization

The symbol of Zener diode is shown in Fig. 2.8(a). The Zener Diode or “Breakdown Diode” as they are sometimes called, are basically the same as the standard PN junction diode but are specially designed to have a low pre-determined Reverse Breakdown Voltage that takes advantage of this high reverse voltage. The zener diode is the simplest types of voltage regulator and the point at which a zener diode breaks down or conducts is called the “Zener Voltage” ($V_z$).

![Zener Diode Symbol](image)

**Fig. 2.8(a): Basic Symbol of Zener Diode.**

The Zener diode is like a general-purpose signal diode consisting of a silicon PN junction. When biased in the forward direction it behaves just like a normal signal diode passing the rated current, but as soon as a reverse voltage applied across the zener diode exceeds the rated voltage of the device, the diodes breakdown voltage $V_B$.
is reached at which point a process called Avalanche Breakdown occurs in the semiconductor depletion layer and a current starts to flow through the diode to limit this increase in voltage.

The current now flowing through the zener diode increases dramatically to the maximum circuit value (which is usually limited by a series resistor) and once achieved this reverse saturation current remains fairly constant over a wide range of applied voltages. This breakdown voltage point, $V_B$ is called the “zener voltage” for zener diodes and can range from less than one volt to hundreds of volts.

The point at which the zener voltage triggers the current to flow through the diode can be very accurately controlled (to less than 1% tolerance) in the doping stage of the diodes semiconductor construction giving the diode a specific zener breakdown voltage, ($V_z$) for example, 4.3 V or 7.5 V. This zener breakdown voltage is almost a vertical straight line characteristics curve in the region of interest as shown in Fig. 2.8(b).

![Fig. 2.8(b): Zener Diode V – I Characteristics.](image_url)
The **Zener Diode** is used in its “reverse bias” or reverse breakdown mode, i.e. the diodes anode connects to the negative supply. From the $I - V$ characteristics curve above, we can see that the zener diode has a region in its reverse bias characteristics of almost a constant negative voltage regardless of the value of the current flowing through the diode and remains nearly constant even with large changes in current as long as the zener diodes current remains between the breakdown current $I_{Z(min)}$ and the maximum current rating $I_{Z(max)}$.

This ability to control itself can be used to great effect to regulate or stabilize a voltage source against supply or load variations. The fact that the voltage across the diode in the breakdown region is almost constant turns out to be an important application of the zener diode as a voltage regulator. The function of a regulator is to provide a constant output voltage to a load connected in parallel with it in spite of the ripples in the supply voltage or the variation in the load current and the zener diode will continue to regulate the voltage until the diodes current falls below the minimum $I_{Z(min)}$ value in the reverse breakdown region.

### 2.4.3. Tunnel Diode and Characterization

A tunnel diode or Esaki diode is a type of semiconductor diode which is capable of very fast operation, well into the microwave frequency region, by using the quantum mechanical effect called tunneling diode as shown in Fig. 2.9(a). It was invented in August 1957 by Leo Esaki when he was with Tokyo Tsushin Kogyo, now known as. In 1973 he received the Nobel Prize in Physics, jointly with Brian Josephson, for discovering the electron tunneling effect used in these diodes. Robert Noyce independently came up with the idea of a tunnel diode while working for William Shockley, but was discouraged from pursuing.
These diodes have a heavily doped PN junction only some 10 nm wide. The heavy doping results in a broken band gap, where conduction band electron states on the N-side are more or less aligned with valence band hole states on the P-side.

Tunnel diodes were first manufactured by Sony in 1957 followed by General Electric and other companies from about 1960, and are still made in low volume today. Tunnel diodes are usually made from germanium, but can also be made in gallium arsenide and silicon materials. They are used in frequency converters and detectors. They have negative differential resistance in part of their operating range and therefore are also used as oscillators, amplifiers, and in switching circuits using hysterics.

![Fig. 2.9(a): Basic Symbol of Tunnel Diode.](image)

**Forward bias operation** - Under normal forward bias operation, as voltage begins to increase, electrons at first tunnel through the very narrow PN junction barrier because filled electron states in the conduction band on the N-side become aligned with empty valence band hole states on the P-side of the PN junction. As voltage increases further these states become more misaligned and the current drops—this is called negative resistance because current decreases with increasing voltage. As voltage increases yet further, the diode begins to operate as a normal diode, where electrons travel by conduction across the PN junction, and no longer by tunneling through the PN junction barrier. Thus the most important operating region for a tunnel diode is the negative resistance region.
**Reverse bias operation** - When used in the reverse direction they are called back diodes and can act as fast rectifiers with zero offset voltage and extreme linearity for power signals (they have an accurate square law characteristic in the reverse direction). Under reverse bias filled states on the $P$-side become increasingly aligned with empty states on the $N$-side and electrons now tunnel through the PN junction barrier in reverse direction.

**Technical comparisons** - $V – I$ curve similar to a tunnel diode characteristic curve as shown in Fig. 2.9(b). It has negative resistance in the shaded voltage region, between $v_1$ and $v_2$.

In a conventional semiconductor diode, conduction takes place while the PN junction is forward biased and blocks current flow when the junction is reverse biased. This occurs up to a point known as the “reverse breakdown voltage” when conduction begins (often accompanied by destruction of the device). In the tunnel diode, the dopant concentration in the $P$ and $N$ layers are increased to the point where the reverse breakdown voltage becomes zero and the diode conducts in the reverse direction. However, when forward-biased, an odd effect occurs called quantum mechanical tunnelling” which gives rise to a region where an increase in forward voltage is accompanied by a decrease in forward current. This negative resistance region can be exploited in a solid state version of the normally uses a tetrode thermion valve (or tube).

![V-I Characteristics of Tunnel Diode](image)

**Fig. 2.9(b): Forward Bias $V – I$ Characteristics of Tunnel Diode.**
The tunnel diode showed great promise as an oscillator and high-frequency threshold (trigger) device since it would operate at frequencies far greater than the tetrode would, well into the microwave bands. Applications for tunnel diodes included local oscillators for UHF television tuners, trigger circuits in oscilloscopes, high speed counter circuits, and very fast-rise time pulse generator circuits. The tunnel diode can also be used as low-noise microwave amplifier. However, since its discovery, more conventional semiconductor devices have surpassed its performance using conventional oscillator techniques. For many purposes, a three-terminal device, such as a field-effect transistor, is more flexible than a device with only two terminals. Practical tunnel diodes operate at a few mill amperes and a few tenths of a volt, making them low-power devices. The Gunn diode has similar high frequency capability and can handle more power. Tunnel diodes are also relatively resistant to nuclear radiation, as compared to other diodes. This makes them well suited to higher radiation environments, such as those found in space applications.

2.4.4. Varactor Diode and Characterization

The basic symbol of Varactor diode is shown in Fig. 2.10(a). Varactor diodes, also known as varicap diodes, are a simple electronic component. A type of simple semiconductor diode commonly used in electronics such as parametric amplifiers, filters, oscillators and frequency synthesizers, varactor diodes have a variable capacitance, which is a function of the voltage impressed on its terminals. In electronics, varactor diodes are mostly utilized as voltage-controlled capacitors.

![Varactor Diode Symbol](Fig. 2.10(a): Basic symbol of Varactor Diode.)
The reverse breakdown voltage \( (V_B) \) and the reverse leakage current \( (I_R) \) are typically measures of the intrinsic quality of the semiconductor diode. Their effect on the frequency or phase tuning behavior is only indirect and of secondary importance. The \( I - V \) characteristics of a good-quality diode and a weak diode are depicted in the following Fig. 2.10(b).

![Fig. 2.10(b): Varactor Diode \( V - I \) Characteristics.](image)

The reverse breakdown voltage is normally measured at 10 \( \mu A \) of reverse leakage current. In a well constructed diode, the breakdown occurs when the electric field across the diode reaches the limit that causes an avalanche of conductors through the diode. The breakdown voltage, therefore, defines the operating limit for the reverse bias across the diode. A rule of thumb is to specify the reverse breakdown voltage a minimum of 5 \( V \) above the maximum operating reverse \( DC \) voltage. The breakdown voltage of the diode is determined by the density of dopants in the semiconductor. Higher dopant density translates into a lower breakdown voltage. An equally important factor determining the breakdown is the defect density (mostly an outcome of wafer fabrication processes). Hence, when a diode breakdown voltage is low, it could be either intentional in an effort to lower the resistance and increase the diode \( Q \), or unintentional - simply an outcome of poor wafer processing. Because of this latter factor, a low breakdown voltage is not necessarily an indicator of a high
diode $Q$. Therefore, it is not a good idea to specify an upper limit on breakdown voltage as a means of specifying high diode $Q$. It is better to specify directly a minimum acceptable limit on diode $Q$.

The reverse leakage current drawn by the diode is a direct measure of the diode quality as opposed to the reverse breakdown voltage. In a well-constructed varactor diode, depending on the geometry and the junction size, the leakage current can be less than a nano-ampere to a couple of hundreds of nano-amperes. A larger leakage current is usually the result of excessive defects in the semiconductor that present shortcut passage for movement of electrons and holes. As shown in Fig. 2.10(b), a good quality diode draws a very small leakage current up to the avalanche breakdown point. A soft diode, on the other hand, draws a greater and greater leakage current as the bias applied to the diode is increased. The reverse leakage current is typically specified at 80 percent of the rated breakdown voltage. The reverse leakage current is also the best indicator of the diode stability through a stress cycle such as burn-in. Of all the measurable parameters, a shift in leakage current at a given bias voltage is the most sensitive measure of the diode’s ability to withstand the burn-in stress. The reverse leakage current of a varactor diode increases rapidly with temperature as the motion of carriers is enhanced by the thermal energy. A rule of thumb is that a fifty fold increase in leakage current is obtained by an increase in temperature from 25 °C to 125 °C, or double the current for every 10 °C.

2.5. LINEAR RESONANT CIRCUITS

In studying the dynamics electronically, resonant oscillatory circuits play an important role. This is because they are easier to understand analytically and build using simple components like linear resistor, capacitor and inductor.
2.5.1. The Resonant Series LCR Circuit

The resonant series LCR circuit is a simple interesting second order linear circuit, which exhibits all the important characteristics of linear systems. It consists of a linear resistor, $R$, a linear capacitor, $C$, a linear inductor, $L$ and a time-dependent voltage source $f(t) = F_s \sin \omega t$ as shown in Fig. 2.11(a). Applying Kirchoff’s voltage law to the circuit of Fig. 2.11(a), we obtain

\[ L \frac{di_L}{dt} + Ri_L + v = F_s \sin \omega t \]  \hspace{1cm} \text{(2.7)}

with $i_L(0) = i_{o_0}$ and $v(0) = v_0$. Substituting $i_L = C \frac{dv}{dt}$ into Eq. (2.7) and simplifying, we obtain the second-order linear inhomogeneous ordinary differential equation with constant coefficients,

\[ \frac{d^2v}{dt^2} + \left( \frac{R}{L} \right) \frac{dv}{dt} + \left( \frac{1}{LC} \right)v = \left( \frac{F_s}{LC} \right) \sin \omega t \]  \hspace{1cm} \text{(2.8)}

In order to study the dynamics of this circuit of Fig. 2.11(a), one can consider the following three possibilities separately:

1. Lossless, zero-input response ($R = 0, F_s = 0, L, C \gg 0$)
2. Under-damped, zero-input response ($F_s = 0, R, L, C \gg 0$) and
3. Resonant oscillations ($F_s, R, L, C \gg 0$).

We will briefly outline the dynamical behaviour of each of the above cases.

**Fig. 2.11:** Linear LCR Circuits: (a) a LCR circuit driven by a time-dependent voltage source, (b) Undamped unforced LC circuit and (c) series LCR circuit.
(i) Losseless, zero-input response \((R = 0, F_s = 0)\)

This limiting case of circuit (2.11(a)) is shown in Fig. 2.11(b). By applying Kirchhoff’s laws to this circuit, the state equations can be written as

\[
L \frac{di_L}{dt} = -v, \quad C \frac{dv}{dt} = i_L
\]

... (2.9)

with \(i_L(0) = i_{L0}\) and \(v(0) = v_0\). Here \(C > 0\) and \(L > 0\). The above equation can be rewritten as

\[
\frac{d^2v}{dt^2} + \left(\frac{1}{LC}\right)v = 0, \quad v(0) = v_0, \quad \frac{dv}{dt}\bigg|_{t=0} = \frac{i_{L0}}{C}
\]

... (2.10)

In this case, the capacitor voltage \(v\) becomes

\[
v(t) = v_0 \cos\left(\sqrt{\frac{1}{LC}} t + \theta\right)
\]

... (2.11)

Where \(\theta\) is the phase difference determined by the initial value of the current \(i_{L0}\). Then the circuit of Fig. 2.11(b) is said to be undamped. The energy that was initially stored in the capacitor and inductor cannot be dissipated, but it simply oscillates back and forth between these two elements. In the absence of damping, this sinusoidal oscillation will continue indefinitely; the circuit is then called a harmonic oscillator. Typical voltage waveform for this circuit of Fig. 2.11(b) is shown in Fig. 2.12(a). The current-voltage phase portrait is shown in Fig. 2.12(b).

![Fig. 2.12: (a) Time waveform of \(v(t)\) of circuit of Fig. 2.11(b) for \(R = 0\ \Omega\) and \(F_s = 0\) (undamped oscillations), (b) phase portrait in the \((v - i_L)\) plane for \(R = 0\ \Omega\).](image-url)
(ii) Underdamped, zero-input LCR response \((F_i = 0)\)

In the absence of the external periodic forcing \(F_i (F_i = 0)\), the circuit of Fig. 2.11(a) becomes an unforced LCR circuit of Fig. 2.11(c). By applying Kirchhoff’s laws to this circuit, the state equations can be given as

\[
L \frac{di}{dt} = -v - Ri \quad C \frac{dv}{dt} = i
\]

... (2.12)

With \(i(0) = i_o\) and \(v(0) = v_o\). It can obviously be expressed as

\[
\frac{d^2 v}{dt^2} + \left( \frac{R}{L} \right) \frac{dv}{dt} + \left( \frac{1}{LC} \right) v = 0
\]

... (2.13)

Fig. 2.13: (a) Time waveform of \(v(t)\) of circuit of Fig. 2.11(c) (under damped oscillations) and (b) Phase portrait in the \((v - i)\) plane for \(R = 360 \Omega\), \(L = 21 \text{ mH}\), and \(C = 10 \text{ nF}\).

Then the undamped solution of Eq. (2.15) becomes for \(0 \subset \left( \frac{R}{2L} \right) \subset \left( \frac{1}{\sqrt{LC}} \right) \)

\[
v(t) = v_o \exp\left(-\frac{R}{2L} t\right) \cos\left( \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} t - \theta \right)
\]

... (2.14)

where the phase constant \(\theta\) is determined from the initial value of \(i_o\).

If \(R\) is positive, the resistor is said to be dissipative. The energy initially stored in the capacitor and inductor is dissipated as heat in the resistor as the magnetic and
electric fields collapse, and \( v \) and \( i_L \) approach zero in the form of exponentially decaying signals as shown in Fig. 2.13(a). The corresponding phase portrait is given in Fig. 2.13(b).

(iii) **Forced LCR circuit** \((F_s > 0)\)

Finally, let us consider the dynamics of the full resonant circuit of Fig. 2.11(a) under the influence of the external sinusoidal forcing \( F_s > 0 \). The governing equation of motion of such a circuit is represented by Eq. (2.11). Then the solution of this equation is represented by

\[
v(t) = v_0 \exp\left(-\frac{R}{2L}t\right) \left( \cos \left( \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} t - \theta \right) \right) + F_s \sin(\omega_s t - \phi) \quad \text{... (2.16)}
\]

Where

\[
F_p = \frac{F_s/LC}{\left\{(1/LC)^2 - \omega_s^2\right\}^2 + (R\omega_s/L)^2}, \quad \gamma = \tan^{-1}\left[\frac{L\left(\omega_s^2 - (1/LC)^2\right)}{R\omega_s}\right] \quad \text{... (2.16)}
\]

Initial value \( i_{L_i} \). Here again the phase constant \( \theta \) is determined from the initial value \( i_{L_i} \). The asymptotic orbit of the circuit of Fig. 2.11(a) is a closed orbit (ellipse), corresponding to the sinusoidal steady state of the circuit. All the waveforms \( v \) and \( i_L \) converge to a periodic waveform having the same frequency \( \omega_s \), responding with the input signal of frequency \( \omega_s \); that is \( \omega_s = \omega_L \) for \( F > 0 \), as shown in Fig. 2.14(a) and Fig. 2.14(b). The actual experimental circuit result in the form of a phase portrait in the \((v-i_L)\) plane in a CRO is shown in Fig. 2.14(c), Fig. 2.14(d) and its mimics exactly the phase portrait of the damped and driven linear harmonic oscillator. This is the most regular asymptotic behavior which one can expect from a linear circuit with a sinusoidal source.
2.5.2. The Resonant Parallel LCR Circuit

Further let us consider the circuit of Fig. 2.15(a). It consists of a parallel arrangement of a linear resistor, a linear inductor, a linear capacitor and a time-dependent voltage source \( f(t) = F_p \sin \omega t \). Applying Kirchhoff’s voltage law to the circuit of Fig. 2.15(a), we obtain

\[
C \frac{dv}{dt} + \left( \frac{1}{R} \right) v + i_L = \left( \frac{F_p}{R} \right) \sin \omega t
\]

... (2.17)
With \( i_L(0) = i_{L_0} \) and \( v(0) = V_0 \). Substituting \( v = L \frac{di_L}{dt} \) into Eq. (2.17) and simplifying, we obtain the second-order linear inhomogeneous ordinary differential equation with constant coefficients:

\[
\frac{d^2i_L}{dt^2} + \left( \frac{1}{RC} \right) \frac{di_L}{dt} + \left( \frac{1}{LC} \right) i_L = \left( \frac{F_p}{LCR} \right) \sin \omega_p t \quad \text{... (2.18)}
\]

Now solving Eq. (2.18), one can easily write down the solution to the current \( i_L \) as

\[
i_L(t) = \frac{i_{L_0}}{\beta \sqrt{LC}} \exp \left( \frac{-1}{2RC} t \right) \cos (\beta t - \delta) + F_p \cos (\alpha t - \gamma) \quad \text{... (2.19)}
\]

Where

\[
\beta = \frac{1}{\sqrt{LC - \left( \frac{1}{2RC} \right)^2}}, \quad \delta = \tan^{-1} \left( \frac{1}{2RC \beta} \right)
\]

Fig. 2.15: (a) Linear forced parallel LCR circuit and (b) phase portrait in the \((v - i_L)\) plane of the experimental circuit of Fig. 2.15(a) for \( R = 1475 \ \Omega \), \( L = 445 \ mH \), \( C = 10 \ nF \) and frequency \( F_p = 1120 \ Hz \) after discarding the transients (asymptotic solution).
\[ F_p = \frac{F/\text{LCR}}{\left[ \left( \frac{1}{\text{LC}} - \omega^2 \right)^2 + \left( \frac{\omega}{\text{RC}} \right)^2 \right]^2}, \quad \text{and} \quad \gamma = \tan^{-1} \left[ \frac{\text{L} \left( \omega^2 - \left( \frac{1}{\text{LC}} \right)^2 \right)}{\text{R} \omega} \right] \] ... (2.20)

Obviously, the asymptotic orbit of the LCR circuit driven by a sinusoidal force is a closed orbit (ellipse) in the \((v-i_L)\) plane corresponding to the sinusoidal steady state of the circuit. All the waveforms \(v\) and \(i_L\) converge to a periodic waveform with a frequency that resonates with the input signal of frequency \(\omega\). The actual experimental result is shown as the phase portrait in the \((v-i_L)\) Plane on a CRO (Fig. 2.15(b)) and it mimics exactly the phase portrait of the damped and driven linear harmonic oscillator. This is the most regular asymptotic behavior which one can expect again from a linear circuit with a sinusoidal source.

### 2.5.3. Higher order linear circuit

Interestingly, further if one includes an additional linear passive capacitor, \(C_2\) in parallel with the inductor, \(L\) of Fig. 2.11(a), but without adding any signal generating source then we have a third-order autonomous linear circuit of Fig. 2.16(a) represented by the state equations

\[ C_1 \frac{dv_1}{dt} = \frac{1}{R} (v_2 - v_1) \] ... (2.21a)

\[ C_2 \frac{dv_2}{dt} = \frac{1}{R} (v_1 - v_2) + i_L \] ... (2.21b)

\[ L \frac{di_L}{dt} = -v_2 \] ... (2.21c)
Fig. 2.16: (a) A third-order linear circuit, (b) Time waveform of $v_1(t)$ and (c) phase portrait in the $(v_1 - v_2)$ plane for $R = 1730 \ \Omega$, $L = 21 \ mH$, $C_1 = 10 \ nF$ and $C_2 = 100 \ nF$ approaching the equilibrium position as the time progress.

Where $v_1, v_2$ and $i_L$ are the voltage across the capacitor, $C_1$ the voltage across the capacitor, $C_2$ and the current through the inductor, $L$ respectively, with appropriate initial values. All the circuit element elements are assumed to be positive. System (2.21) is equivalent to a third-order linear differential equation, which can again be solved explicitly. For a specific choice of parameters namely, $C_1 = 10 \ nF$, $C_2 = 100 \ nF$, $R = 1749 \ \Omega$ and $L = 18 \ mH$, the trajectories approach an equilibrium point and corresponding phase portrait as indicated in Fig. 2.16(b) and Fig. 2.16(c). Similarly for a different set of parameter choice, one can observe a periodic solution as well. One can check similar phenomena even for still higher order linear circuits. One can conclude that at the most one expects either a fixed point solution or a periodic steady state oscillation from a linear circuit for chosen nominal circuit parameter.
2.6. SIMPLE NONLINEAR CIRCUITS

2.6.1. Nonautonomous case

(a) Murali-Lakshmanan-Chua (MLC) circuit

Let us discuss the possibility of adding nonlinear circuit elements in the most convenient form to the linear circuits discussed above and study the circuit dynamics. The most natural extension to the linear circuit theory to the world of nonlinear circuits is through piecewise-linear circuit modeling. Therefore, we modify the serial/parallel LCR circuit by placing with it a piecewise-linear nonlinear resistor. By placing a piecewise-linear (nonlinear) resistor, namely the Chua’s diode ($N_R$) (discussed above) with a nonlinear characteristics shown in Fig. 2.5(b), in parallel to our familiar series LCR circuit of Fig. 2.11(a), we have a second order dissipative nonautonomous nonlinear circuit. The modified circuit shown in Fig. 2.17(a). This is one of the simplest and widely studied of all real nonlinear nonautonomous dynamical systems. It is called the Murali-Lakshmanan-Chua (MLC) circuit and was first reported by Muali, Lakshmanan and Chua [147].

The governing equations of motion are:

\[
C \frac{dv}{dt} = i_L - g(v) \quad \text{... (2.22a)}
\]

\[
L \frac{di_L}{dt} = -R_L i_L - v + f \sin(\Omega t) \quad \text{... (2.22b)}
\]
Fig. 2.17: (a) Circuit realization of simple non-autonomous (MLC) circuit, (b) experimental, (c) PSPICE and (d) numerical results of typical double band chaotic attractor in the \( (v - i_L) \) plane.

Here the form of the piecewise linear function \( g(v) \) is as given Eq. (2.4). For this remarkable circuit the presence of chaos has been reported experimentally, confirmed numerically and proven mathematically [148]. This circuit can be readily constructed at low cost using standard electronic components and its complete chaotic dynamics and rich variety of bifurcation [113]. A typical chaotic attractor of the circuit of Fig. 2.17(a) is shown in Figs. 2.17(b)-(d).

*(b) Variant of Murali-Lakshmanan-Chua Circuit*

Now let us include again the nonlinear resistor, namely, the Chua’s diode \( (N_R) \) discussed above to our familiar linear forced parallel LCR circuit of Fig. 2.15(a). The Modified circuit is shown in Fig. 2.18(a). This is also another simplest nonautonomous
circuit and has been first reported by Thamilmaran, Murali and Lakshmanan [149].

The governing equations of motion of the circuit are:

\[
C \frac{dv}{dt} = \frac{1}{R} (f \sin(\Omega t) - v) - i_L - g(v) \quad \text{... (2.23a)}
\]

\[
L \frac{di_L}{dt} = v \quad \text{... (2.23b)}
\]

As in the earlier case the presence of chaos and rich variety of bifurcation and different routes to chaos have been reported experimentally, confirmed numerically and proven mathematically by us recently [113]. A typical chaotic attractor of the circuit of Fig. 2.18(a) is shown in Figs. 2.18(b) - 2.18(d).
(c) Forced Negative Conductance Series LCR circuit with Diode

By including a linear negative conductance $g_N$ and a PN junction diode in parallel to the familiar linear forced series LCR circuit instead of Chua's diode, $N_R$, we can construct the chaotic circuit Fig. 2.19(a). It is shown in Fig. 2.19(a). The governing equations of motion of the circuit are

$$L \frac{di_L}{dt} = -Ri_L - v + E_j \sin(\omega t) \quad \ldots \ (2.24a)$$

$$C \frac{dv}{dt} = i_L - i_j + g_N v \quad \ldots \ (2.24b)$$

![ Circuit realization of forced negative conductance LCR circuit, with experimental, PSPICE and numerical results of typical chaotic attractor in the $(\sin \omega t - v)$ plane.](image)

The presence of strong chaos and quasiperiodic behaviour in this circuit has been reported experimentally, confirmed numerically and proven mathematically by us recently [150]. A typical chaotic attractor of the circuit of Fig. 2.19(a) is shown in Figs. 2.19(b) - 2.19(d).
2.6.2. Autonomous Case

(a) Chua’s circuit

One can identify a very nonlinear autonomous circuit, by placing a piecewise-linear (nonlinear) resistor $N_R$ in parallel with the capacitor $C_1$ of the higher order linear circuit of Fig. 2.20(a). The resultant third-order autonomous circuit is nothing but the well known Chua’s autonomous circuit which is shown in Fig. 2.20(a). Here the nonlinear resistor $N_R$ is represented by its characteristics curve shown in Fig. 2.5(b). Chua’s circuit is one of the simplest and most widely studied of real nonlinear dynamical systems [143, 151] and is of autonomous type.

![Circuit Diagram](image)

Fig. 2.20: (a) Circuit realization of third order autonomous Chua’s circuit, (b) experimental, (c) PSPICE and (d) numerical results of typical double-scroll chaotic attractor in the $(v_I - v_2)$ plane.
The circuit equations of Fig. 2.20(a) are represented as

\[ C_1 \frac{dv_1}{dt} = \frac{1}{R} (v_2 - v_1) - g(v_1) \]  \hspace{1cm} \text{... (2.25a)}

\[ C_2 \frac{dv_2}{dt} = \frac{1}{R} (v_1 - v_2) + i_L \]  \hspace{1cm} \text{... (2.25b)}

\[ L \frac{di_L}{dt} = -v_2 \]  \hspace{1cm} \text{... (2.25c)}

Where \( g(v_1) \) is the mathematical representation of the characteristic curve of the nonlinear element \( N_R \) represented by Eq. (2.4) or Eq. (2.5), where \( v_n \) now corresponds to \( v_1 \). The presence of various bifurcations and chaos in this circuit have been reported experimentally and verified by numerical simulations and mathematical analysis [141]. A typical double scroll chaotic attractor of the circuit (Fig. 2.20(a)) is shown in Figs. 2.20(b) - 2.20(d).

(b) Canonical Chua’s circuit

Let us consider another third-order piecewise linear circuit capable of realizing every member of the Chua circuit family namely the canonical Chua circuit, introduced by Chua and Lin [152, 153]. This circuit is of considerable importance in the field of nonlinear electronic circuits. It is canonical in the sense that (i) it can exhibit all possible phenomena associated with any three-region symmetric piecewise linear continuous vector field, and (ii) it contains the minimum number (six) of circuit elements needed for such a circuit. Each number of this family consists of linear resistors, three linear dynamic elements (capacitors and/or inductors) with two active elements, one linear negative conductance and one nonlinear resistor with an odd-symmetric piecewise-linear \((v-i)\) characteristic. The original canonical Chua’s
The circuit shown in Fig. 2.17(a) is one of the most simple third-order autonomous electronic generators of chaotic signals.

The state equations of the circuits in Fig. 2.17(a) are given by

\[
C_1 \frac{dv_1}{dt} = i_L - g(v_1) \quad \text{(2.26a)}
\]

\[
C_2 \frac{dv_2}{dt} = g_\alpha v_2 - i_L \quad \text{(2.26b)}
\]

\[
L \frac{di_L}{dt} = v_2 - v_1 - R i_L \quad \text{(2.26c)}
\]

Where \( g(v_1) = G_b v_1 + 0.5(G_a - G_b) \left[ v_1 + B_\rho \right] - \left| v_1 - B_\rho \right| \)

The chaotic behavior of the circuit was studied numerically, confirmed by analytical and laboratory experiments \[153\]. A typical double scroll chaotic attractor
of the circuit (Fig. 2.21(a)) is shown in Figs. 2.21(b) - 2.21(c). In order to study the existence of hyperchaos, we have recently modified the canonical Chua’s circuit. The presences of chaos and hyperchaos have been reported experimentally and confirmed numerically [154].

2.7. CONCLUSIONS

In this chapter we have presented some introductory aspects of the simple two-terminal linear and nonlinear elements, their realization and some related circuits. An important experimental realization of nonlinear resistances, namely, Chua’s diode and linear negative conductance has been introduced. The associated circuits exhibit periodic oscillations, different types of bifurcations and chaotic behavior. More details will be discussed in the following chapters.
Chapter - III

Chaotic Phenomena in a Simple-3D Autonomous Circuit with a Diode Pair