Chapter - I

Introduction
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INTRODUCTION

1.1. GENERAL

The world of chaos and fractals is fascinating. The visual scrolls and patterns attract all of us but somehow most of us lack the mindset to understand the same. In fact, it has been argued that the answer to several economic depressions and stock market crashes lies in nonlinear dynamics and chaos theory. From the days of Sir Issac Newton and Pierre-Simon Laplace to that of Albert Einstein and Neil’s Bohr, there has always been a hunt for hidden parameter leading to unusual behavior of the physical world. It was not before 1905, that a mathematician named H. Poincare discovered the fascinating field of Chaos. In mathematics, chaos theory describes the behavior of certain dynamical systems i.e. systems whose state evolves with time that may exhibit dynamics that are highly sensitive to initial conditions. Chaos, along with Quantum Mechanics and the Theory of Relativity, has been hailed as one of the major discoveries of the 20th century. However, despite being such a fascinating field it remained almost dormant until E. Lorentz discovered the fact that weather is indeed a chaotic system. The lack of right mindset is attributed to the slow development of this field. It has been well pointed out by Prof. Peter Kennedy that we have a mindset of linearize, and then analyze which leads to ignoring several interesting dynamical behavior as noise. In fact, the name “Chaos Theory” in itself is a misnomer and is a manifestation of our inability to understand things beyond our developed mindset. The established mindset was so strong that during the early development of chaos theory, emphasis was paid on convincing the research community that chaos is
actually a phenomenon and not a mathematical fallacy. This is one of the reasons why Chua’s circuit (a circuit which celebrates itself as a paradigm of chaos) was rigorously proved to be chaotic both mathematically and experimentally to convince the community. With its presence in almost every field from weather to finance, from economics to hydraulics and now nanotechnology, Chaos theory is now witnessing a lot of increased enthusiasm in interdisciplinary sciences. However, there still lies a need to develop a mindset to understand patterns, fractals and other bizarre phenomenon.

The term chaos is ancient, has appeared even in religious texts such as Bhagvat Gita, and is often associated with a physical state or human behaviour without pattern and out of control. In normal terminology, it implies a breakdown of laws or traditions governing social events leading to riots, civil wars, etc. In scientific parlance, the term chaos is assigned to that class of motions in deterministic physical and mathematical systems whose time history has a high sensitive dependence on initial conditions [1–4]. This phenomenon of chaos was considered to be complex and was not given sufficient importance earlier because no straightforward analysis was possible, which could help one to appreciate its interesting aspects and explore its further consequences. Chaos is ubiquitous, and especially occurs when nonlinearity is present. The concept has permeated all fields of science and engineering [5–10].

Historically, the relaxation oscillator investigated by the Dutch engineer and physicist Balt-hazar van der Pol in his seminal paper, “Frequency Demultiplication” (Van der Pol and van der Mark) [11], may be considered as the earliest example of a nonlinear electronic circuit, exhibiting many of the characteristic features underlying bifurcations and chaos, though not fully understood at that time. Van der Pol noted that “often an irregular noise is heard in the telephone receiver (monitoring the signal
in some way) before the frequency jumps to the next level value”. We now know that this “noise” indeed corresponds to chaos.

Chaos refers to the idea that the system displays extreme sensitivity to initial conditions so that arbitrary small errors in measuring the initial state of the system grow exponentially in time and hence long-term predictability of the future state of the system is lost.

This behavior is often called the “butterfly effect” [2,12–14]. Dynamical chaos refers to the appearance of apparently random type motion in a deterministic dynamical system, that is, in a system with no random forcing. The discovery of this dynamical chaos in physical systems has led to the emergence of a new understanding of the laws of nature. Often chaos is unavoidable even for very simple physical systems, though unexpected order can arise within chaos.

Depending on external or initial conditions and the values of the control parameters, physical systems can show both regular and apparently random motions even in the absence of any external random forcing. This is a typical feature of dynamical chaos, or simply chaos. The onset of chaos does not even require a large number of degrees of freedom. For example, for some initial conditions a system consisting of two pendulum connected by spring generates motion which is quite complex.

Chaos can be thought of as deterministic randomness—“deterministic” because it arises from intrinsic causes and not from some extraneous noise or interference; and “randomness” referring to irregular, unpredictable behaviour. The appealing aspect of chaos is that it offers a way to understand complicated behaviour [15,16]. Over the past three decades or so physicists, biologists, mathematicians, and scientists from
many other disciplines have developed the science of nonlinear dynamics, including chaos, fractals and cellular automata in order to study complexity in nature. From the time the founders of chaos theory first began their studies; the discipline has grown into a science and is, as some researchers argue, the direction of the future. The contributions of Poincaré, Lorenz, Hénon, May, Feigenbaum and many others have led to our understanding of what chaos is and its importance in the natural world [12,13]. Though the study of predictable linear systems helps to understand many interesting phenomena, those systems are by far the exception, not the rule. Most natural systems are nonlinear and often chaotic.

In the last few decades or so a large number of theoretical investigations [17–21], numerical simulations [22–29] and experimental works [28–31] have been carried out on various dynamical systems in an effort to understand the different features associated with the occurrence of chaotic behaviour. Analytical methods have been developed to study the qualitative changes in the structure of the orbits and the mechanisms responsible for the onset of chaos. Particular mention can be made of the Melnikov method to predict the occurrence of horseshoe chaos [5,10,28], approximate methods like harmonic balance and multiple scales [32,33], center manifold reduction and normal form analysis [34] to investigate bifurcations and onset of chaos.

As there are limitations in employing analytical methods, one has to often take recourse to numerical simulations in exploring the system dynamics. In this direction, a number of subtle and versatile characteristic features have been identified. These include Lyapunov exponents [30,35–39], power spectrum analysis [35,40], dimensions [41,42] and auto-correlation functions [42], basins of attraction [43,44], invariant measures [45], and so on. These quantities have been used to distinguish the
chaotic attractors from regular motions. In addition, using numerical simulations, a number of routes such as period-doubling [2,15,37], quasiperiodic [46–49] and intermittency routes [49,50], through which transition from regular to chaotic motion occurs in different dynamical systems, have been identified in the literature.

Besides, a rich variety of phenomena such as period-adding sequence and Farey sequence [15,37,45], co-existence of multiple attractors with fractal basin boundaries [51,52], occurrence of Smale-horseshoe [5,10], devil’s staircase [53,54], strange attractors [10], strange nonchaotic attractors [55–59], border-collision bifurcation [60], and hyperchaotic attractors [61, 62] have been reported in relation to chaotic dynamics in various nonlinear systems.

Typical of the nonlinear systems studied in the literature are Duffing oscillator, Duffing-van der Pol oscillator, periodically driven LCR circuit with varactor diode [63], PN junction diode [64], piecewise linear capacitor [65] and noise affected periodically driven PN junction [66], Lorenz system [34], driven Toda oscillator [67], Rossler system [68], etc. Interestingly, all the effects and phenomena identified in the computer simulations have been realized experimentally in simple electronic circuits [2,69], in mechanical systems [70,71], lasers [72,73], chemical reactions [76–78], and so on. In spite of a large body of knowledge which has accumulated, there are many more old and new nonlinear systems to be investigated properly in order to realize newer features in chaotic dynamics.

Understanding chaos has long been the main focus of research in the field of nonlinear dynamics. There are many practical reasons for investigating chaos. First, “Chaotic” (i.e. in practice messy, irregular, or disordered) system carrying little useful information content is unlikely to be desirable. Second, chaos can lead a system to
harmful or even catastrophic situations. In these troublesome cases, chaos should be reduced as much as possible or even totally eliminated. Ironically, recent research has shown that chaos can actually be useful under certain circumstance, and there is growing interest in utilizing it in some special situations. For example, since chaotic signals have a continuous spectrum in the frequency domain, they are of significance in developing wideband communication system.

In general, dynamical systems can be classified into two main categories, namely, dissipative and conservative systems. The typical motions of these systems will therefore have different characteristic properties:

1.1.1. Dissipative systems

The time evolution of these systems leads to contraction of volume/area in phase-space and consequently trajectories approach asymptotically either a chaotic or a non-chaotic attractor. These attractors are bounded regions of phase-space towards which the system’s trajectory represented as a curve converges in the course of long time evolution [79]. Bifurcation of periodic attractors can occur leading to more complicated and chaotic structures as a control parameter is varied. The unique character of this chaotic attractor is its sensitivity to initial conditions as pointed above.

1.1.2. Conservative or Hamiltonian systems

Here chaotic orbits tend to visit all parts of a subspace of the phase-space uniformly and thus the phase-space volume is conserved. In general, the dynamics of a non-integrable conservative system is typically neither entirely regular nor entirely irregular, but the phase-space consists of a complicated mixture of regular and
irregular components. In the regular regions, the motion is quasiperiodic [6, 80–83] and the orbits lie on tori while in the irregular regions, the motion appears to be chaotic but they are not attractive in nature.

1.2. HISTORICAL COMMENTS

Since chaotic behaviour can only occur in nonlinear systems, detailed investigations of the problem had to wait for arrival of powerful computers, although Poincaré was already fully aware of the existence of chaos [12,13]. It is only now that people can attempt to apply the theory of chaos to the solution of various problems in plasma physics and in the design of microwave devices.

The operation of all microwave tubes, whether amplifiers or oscillators, is based on an interaction process between electrons, usually in the form of a beam or plasma and an electromagnetic field, the kinetic or potential energy of the electrons being gradually converted into the RF energy of the field. In most cases the interaction process is highly nonlinear and the number of phase-space variables is usually sufficient for chaos to occur. It is therefore desirable to investigate such interaction processes more closely using the concept of chaos.

One of the early applications of the theory of chaos to the problem of plasma e/m field interaction dates back to the late 1970s. It was then shown that a surprisingly short electron relaxation time could be explained in terms of a transition to chaos. This effect proved to be of considerable practical interest in connection with electron cyclotron heating (ECH), which seems to be essential to the successful operation of plasma fusion devices. The investigations were further extended by taking into account relativistic kinematics which would be required in the case of
interaction with high-velocity particles. The possibility of chaos in gyrotrons has also been investigated, a somewhat unexpected conclusion having been reached that gyrotrons are singularly stable and resistant to the onset of chaos.

In the field of electronic circuits, the earliest experimental observations of chaos were in externally driven nonlinear oscillators, including the sinusoidally excited neon-bulb relaxation-oscillator studied by van der Pol and van del Mark [11], the forced negative-resistance oscillator of Ueda and the driven series-tuned RL-diode circuit. More recently chaos has been observed and studied in a variety of unforced autonomous electronic circuits such as Chua’s oscillator [65,69], hysteresis oscillator, phase-lock loop and a number of important discrete-time systems, including switched capacitor circuits, dc-dc converters, digital filters and sigma-delta modulators. It can be seen that the research into chaos has advanced tremendously even the well-known classical Colpitts oscillator has a chaotic performance according to recent investigations. These studies suggest that more time should be devoted to the study of chaos in conventional oscillator structures, which include single-transistor LC oscillators and simple RC oscillators. At the same time, as electronic oscillators lie at the heart of modern computing and communication systems, it is well worth while to investigate the dynamical behaviour of such classical oscillators. This would include the control of chaos when they are working in their usual oscillatory state or, alternatively, deliberate generation of a chaotic signal.

1.3. STUDY OF CHAOS USING ELECTRONIC CIRCUITS

In the study of chaotic phenomenon, often the available analytical techniques are not enough to understand all the bewildering variety of bifurcations and chaotic
orbits. It is often brute force numerical analysis which brings out the full picture; however, here again one requires much computer power and enormous time to scan the entire parameter space, particularly if more than one control parameters are involved. In this connection, studies of nonlinear systems through appropriate electronic circuits, using operational amplifiers (Op-amps) and multipliers, are often helpful in a dramatic way for a quick scanning of the parameter space and also to avoid long transient periods to reach the steady state as in the numerical studies.

The last decade has seen significant development in research on the dynamics of nonlinear electrical and electronic circuits and in the employment of these in the study of chaos. This is mainly due to two reasons.

Firstly, electrical and electronic circuits constitute a group of real physical systems in which observations and measurements are relatively easy to make. Various types of behaviour like bifurcations and chaos can be observed using general purpose laboratory tools such as oscilloscope and spectrum analyzer. More sophisticated specialized tools for tracing solution curves in three dimensions and taking Poincaré sections can be used to pursue experimental analysis even further. Also, general purpose circuits simulation programs such as Simulation Program for Integrated Circuit Emphasis (PSPICE) [84] provide an “experimental comfort” that enable one to study and confirm the existence of strange unexpected behaviour in almost every type of electronic circuits-oscillators, filters, instrumentation circuits, switched capacitor circuits, digital circuits, etc.

Secondly, electronic circuits offer an unprecedented opportunity for researchers. With the development of IC technology we can build cheap laboratory experimental setups that reflect properties of almost any proposed model and make measurements
in real time in a real physical system for a wide range of system parameters. This makes possible experiments that are not available in any other domain of research such as medicine, chemistry, biology, economics, etc. A good example of such a universal electronic circuit is the Chua’s circuit or the Murali-Lakshmanan-Chua (MLC) circuit which can mimic a large variety of bifurcations and chaotic phenomena when just one of the circuit elements is tuned; these circuits provide a useful paradigm for understanding different types of dynamical behaviours [85].

In recent times a variety of nonlinear electronic circuits consisting of either real nonlinear physical devices such a nonlinear diodes, capacitors, inductors and resistors [86–88] or devices constructed with ingenious piecewise-linear circuit elements [87,89] have been utilized as veritable black boxes to explore different properties of chaotic dynamics. These circuits are unique in being easy to build, easy to analyze and easy to model. Due to their rich repertoire of dynamical phenomena, nonlinear electronic circuits provide a convenient framework to understand the various mechanisms underlying the onset of chaos and its possible technological applications. Thus it is clear that the study of nonlinear circuits is an important and active area of study in nonlinear dynamics.

1.4. CIRCUITS AS DYNAMICAL SYSTEMS

A circuit is said to be dynamic if it contains energy storing elements such as inductors and/or capacitors. A dynamic circuit containing resistive elements (resistors and voltage and current sources) and n-energy storage elements (capacitors and/or inductors) can be described by the initial value problem of a system of ordinary differential equations, by applying Kirchoff’s laws for current and voltage to the various branches of the circuit [87,89–91],
\[ \dot{X}(t) = F(X(t),t), \quad X(0) = X_0 \] ... (1.1)

where \( X(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n \) is the state vector and 
\[ F(X(t),t) = \left[ F_1(x(t),t), F_2(x(t),t), \ldots, F_n(x(t),t) \right]^T \] is an \( n \)-dimensional map. We shall assume that \( F \) is sufficiently well behaved for there to exist a unique solution \( X(t) \) of Eq. (1.1) such that \( X(0) = X_0 \) for given \( X_0 \in \mathbb{R}^n \); moreover we assume that \( X(t) \) is nonsingular at least for \( t \) less than some finite value depending on \( X_0 \) as well as \( F \). A solution \( X(t) \) is often called a trajectory and \( F(X(t),t) \) is called a vector field because it defines the direction and speed of a trajectory at every point in the state-space (known as the phase-space) and at every instant of time. If the vector field depends explicitly on the time variable \( t \), that is time variable \( t \) appears explicitly whenever the circuit contains time-dependent voltage and/or current sources or time-varying circuit elements, then the system is said to be non-autonomous. If the vector field depends only on the state variables and is independent of time \( t \), then the system is said to be autonomous and may be represented as 
\[ \dot{X}(t) = F(X(t)), \quad X(0) = X_0 \] ... (1.2)
or simply
\[ \dot{X}(t) = F(X), \quad X(0) = X_0 \] ... (1.3)

When looking at the time evolution of the state variables given by Eqs. (1.2) or (1.3), one usually distinguishes between the transient behaviour, which disappears after certain time, and the permanent features which persist in time. The later is referred to as the asymptotic behaviour of the circuit, since permanent features of the time evolution have to be extracted in the limit as time goes to infinity. Instead of studying
the waveforms, that is, the state variables as functions of time, one often concentrates on the orbits in state-space, namely, the phase-space. In particular, one can consider the two dimensional projections for any two state variables. For each initial state the time evolution of the system is represented by a single curve in phase-portrait, the so-called orbit. The time dependence is not visible anymore on the orbit, but the direction of increasing time can be indicated by arrows. The phase-portrait is valuable because it reveals important dynamical behaviours of the associated circuit in a single picture [89,92].

1.4.1. Nonlinear circuits: Asymptotic behaviour

A circuit is said to be nonlinear if the circuit contains at least one nonlinear circuit element like a nonlinear resistor, nonlinear capacitor and/or a nonlinear inductor. Another basic inventory in nonlinear circuit analysis is the use of piecewise-linear circuit elements designed ingeniously for specific needs whose characteristic curves are piecewise-linear. These elements include piecewise-linear resistors, capacitors and inductors [87,93].

In the literature a large number of nonlinear circuits have been widely discussed, especially in the study of chaotic phenomena. These nonlinear circuits include both driven nonlinear LCR circuits and undriven nonlinear LCR networks. There are at least three groups of studies here. In the first group of circuits, a number of authors have reported circuits with typical nonlinear elements such as nonlinear capacitors (varactor diode, junction diode) [66,94], nonlinear inductors (saturable core inductor, Josephson junctions, ferroresonant power systems) [95,96] and nonlinear resistors (tunnel diode, thyristor, dead-zone conductor, serially connected Zener
diodes, neon bulb, etc.) [97,98]. In the second set of circuits, circuits with ingeniously devised piecewise-linear circuit elements like piecewise-linear resistors, capacitors and inductors have been discussed [90,99–101]. These studies comprise of both low-order and high-order circuits. In the third group, nonlinear circuits including phase-locked loops (PLL) [102,103], digital filters [104], flip-flops [105,106], adaptive filters [107], power converters and power supplies [108], cellular neural networks [104], RC-ladder phase-shift networks [109], DC-DC converter [110], etc., have been extensively discussed. Among the many nonlinear elements, nonlinear resistors play an important role from circuit theoretic point of view over nonlinear capacitors and nonlinear inductors. This is because conceptually nonlinear resistors are fairly easy to devise, easy to model and easy to implement.

In all these cases, the dynamics changes radically when nonlinear circuit elements are added to the linear circuits discussed in the previous section. Apart from the “normal” behaviour, many qualitatively different time evolutions are possible [89,93]. The convergence of every voltage and current (steady state response) in such a nonlinear circuit may be anyone of the following forms, depending upon the parametric choice.

A. Regular behaviour

1. **DC steady state:** All trajectories in the state-space move towards a single point, namely, the fixed point. It means that after transients the state of the system does not alter with time. It just comes to rest.

2. **Periodic response:** All solutions converge to a periodic waveform having the natural frequency $\omega_0$ for the autonomous case. For the non-autonomous case the natural frequency $\omega$ is equal to the input signal frequency $\omega_s$; that is, $\omega = \omega_s$. 
3. **Subharmonic response:** All solutions converge to a periodic waveform whose frequency.

4. $\omega$ is a submultiple of the input signal frequency $\omega_s$; that is, $\omega = \frac{\omega_s}{n}$, where $n > 1$ (non-autonomous case). For the autonomous case the frequency of oscillation is $\omega = \frac{\omega_0}{n}$. Here $\omega_0$ is the fundamental frequency of the oscillator.

5. **Superharmonic response:** All solutions converge to a periodic waveform whose frequency $\omega$ is multiple of the input signal frequency $\omega_s$; that is, $\omega_0 = n\omega_s$, where $n > 1$ (non-autonomous case). For the autonomous case the frequency of oscillation is $\omega = n\omega_0$.

6. **Almost-periodic (quasiperiodic) response:** All solution waveforms are made up of periodic components whose fundamental frequencies are incommensurate and hence not periodic but quasiperiodic. Here the trajectory in phase space will describe torus attractors.

**B. Chaotic behaviour**

The asymptotic behaviour of certain nonlinear circuits and systems may be even more complicated. It may even fail to be approximately periodic. The orbits in state-space appear to occupy finite regions. Furthermore, close initial states lead to time evolutions that become more and more distant from each other in finite time, within an enclosed region of phase-space. This kind of asymptotic, irregular and exotic behaviour is the chaos, which we mentioned in the beginning, and the state-space trajectory is called a *strange chaotic attractor*. A strange attractor denotes a steady-state behaviour, which belongs to none of the traditional types of asymptotic
behaviour described earlier. We remark that the above identification of attractors in the phase-plane has to be further quantified using the standard characterizations such as Lyapunov exponents, power spectrum, fractal dimension and so on.

1.5. ROUTES TO CHAOS

Considering the time evolution of dissipative systems which include the standard nonlinear electronic circuits, one usually distinguishes between transient behaviour, which disappears after certain time, and permanent features which persist in time. The latter is often referred to as the asymptotic behaviour discussed earlier. As already noted, one encounters fixed point, periodic, quasiperiodic, strange nonchaotic and chaotic behaviours in the analysis of dissipative nonlinear systems. One essentially tries to vary one or more of the control parameters in these systems so that the parameter ranges for which regular and periodic behaviours occur, and the regimes for which chaotic behaviour occurs, can be identified. In many nonlinear dissipative dynamical systems, chaotic motion is found to set in mainly through one of the three predominant routes, which are all now very familiar in the chaos literature. In addition, there exist several other less prominent routes too [111–113].

1.5.1. Period-doubling route (Feigenbaum scenario)

Although Grossmann and Thomae [114], Coullet and Tresser [115] and of course May [15] had studied this route to chaos, Feigenbaum [116] was among the first to investigate this route to chaos and its scaling phenomenon, which is considered to be a major development in the field. He has also considered a simple difference equation which, for example, has been used to describe the time dependence of populations in biology and found that the population oscillated in time between stable
values (fixed points) whose number doubles at distinct values of an external parameter. This continues until the number of fixed points become infinite at a finite parameter value, where the variation in time of the population becomes irregular and chaotic. The results are not restricted to this special model alone but are in fact universal and hold good for a large variety of physical, chemical and biological systems ([25,29], and references therein). This discovery has triggered an explosion of theoretical and experimental activity in this field. It refers to a sequence of periodic motions in which the period doubles as some parameter in the system is varied. Beyond a critical accumulation parameter value chaotic motions occur. The ratio of successive differences between period-doubling bifurcation parameters approaches the universal constant $4.6692016...$ [116].

1.5.2. Quasiperiodic route (Ruelle-Takens-Newhouse scenario)

In this route when the control parameter of the system is changed, the initial stationary state becomes unstable and undergoes Hopf bifurcation. Further change of the control parameter makes the system to undergo one more Hopf bifurcation so that a doubly periodic orbit occurs. By Hopf bifurcation, here, we mean a bifurcation that transforms from an equilibrium state into a limit cycle. The precursor to chaotic motion is that the frequencies of these oscillations are in-commensurate with each other so that the motion is said to be quasiperiodic. This quasiperiodic motion then bifurcates to chaotic motion as the control parameter is further changed. However, there is a possibility for emergence of the chaotic attractor following the appearance of a two-frequency quasiperiodic motion [117]. In this route, first, a fixed point loses stability due to a supercritical Hopf bifurcation, resulting in a periodic attractor.
Subsequently, this periodic attractor experiences a supercritical secondary Hopf bifurcation, resulting in a two-period quasiperiodic attractor. As the control parameter in the system is further varied, a rich variety of bifurcations of the torus take place. The post bifurcation state can be one of the following: (1) a complex periodic behaviour (phase-locked or mixed-mode oscillations), (2) a nonstrange attractor whose corresponding orbit exhibits the feature of intermittent excursions, and (3) a chaotic behaviour. Following the identification of transition from quasiperiodicity to chaos, a number of investigations have been carried out to elucidate the associated local and global universal properties (see details in ref. [25]). In addition, a large number of studies on experimental realizations have also been made [50,118,119].

1.5.3. Intermittency route (Pomeau-Manneville scenario)

There is yet another important route to chaos called Pomeau-Manneville scenario [20,119] or intermittency route which proceeds through the appearance of intermittent dynamical behaviour. When a motion alternates at random between long regular or laminar phases and relatively short irregular burst, it is said that the motion is intermittent, or that there is an intermittency. In the intermittent route to chaos, initially the time series consists of regular laminar motion interrupted by irregular bursts. The laminar motions between two successive bursts have different durations which are distributed at random over the time intervals. As a control parameter is increased, the length of the laminar region decreases and the bursts become very frequent so that at a critical value of the parameter the laminar phases disappear altogether and the motion becomes fully chaotic. In other words, for a control parameter, say, $p$, less than $p_c$ the attractor is a periodic orbit. For $p$ slightly larger than
there are long laminar phases during which the orbit appears to be periodic and closely resembles the orbit for $p < p_c$ in addition to short chaotic bursts. As $p$ increases sufficiently above $p_c$, the motion becomes fully chaotic without laminar phase. This intermittent behaviour was first identified by Pomeau and Manneville in 1979 in the Lorenz equations [50,119]. Intermittency transition can occur through one of the three bifurcations, namely saddle-node, Hopf and subcritical period-doubling and the corresponding intermittency transitions are called type-I, type-II and type-III, respectively. The loss of stability of a periodic orbit via one of the aforementioned three generic bifurcations is not sufficient for intermittency to occur. The other necessary condition is the existence of a global relaminarization mechanism that repeatedly reinjects the trajectory in the neighborhood of the original periodic orbit. In recent years it has also been found that new kinds of intermittencies namely, on-off intermittency [120], type-X intermittency [121], type-Y intermittency [122], etc. exist in different systems.

Type-I intermittency has been observed experimentally in Rayleigh-Bénard convection, Belousov-Zhabotinsky reaction, electronic oscillators, Taylor-Couette flow, coherently pumped laser, analog simulation of Joseph junctions and ammonia oxidation on a platinum wire experiment. Similarly, type-II intermittency has been observed in an electronic oscillator experiment, in the oxidation of methanol on zeolithe-supported palladium, in certain hydrodynamic systems and in the wall region of a turbulent boundary layer. Experimental reports of type-III intermittency have appeared again in the Belousov-Zhabotinsky reaction, in the Rayleigh-B´enard convection, in a laser system and in a semiconductor system ([25, 29] references therein for all types of intermittencies).
These are the three prominent routes to chaos. Besides, there are many other not so common routes, such as period-adding sequences, Farey sequences, reverse period-doubling bifurcation, antimonotonocity, band merging, equal-periodic bifurcations and so on [34,118] in typical dynamical systems.

1.5.4. Synchronization of chaos and secure communication

Synchronization is a basic phenomenon in physics. In a classical sense, it means adjustment of frequencies of periodic oscillators due to a weak interaction. This effect is well studied and finds a lot of practical applications in electrical and mechanical engineering.

Interestingly, in recent times, the notion of synchronization has been extended to chaotic systems as well. In particular Pecora and Carroll proposed that a subsystem of a chaotic system could be synchronized with a separate chaotic system under certain conditions [123]. This idea of synchronization has been successfully applied to obtain chaos synchronization in many important nonlinear systems including coupled Lorenz systems [123–130], Rossler systems [123–128], the hysteretic circuits [123–128], Chua’s circuits [131], DVP oscillators [132,133], phase-locked loops [134–136] and so on. Very recently the concept of synchronization of chaos has been generalized. The classification into complete synchronization (CS), lag synchronization (LS), generalized synchronization (GS), and phase synchronization (PS) [137–140] has greatly enhanced our understanding of chaotic systems and circuits and led to innovative technological applications.

In recent years, much interest has focused particularly on chaos synchronization based secure communication and applications. Several chaos-based modulation techniques have been pro-posed in the literature-Chaos-Based Modulation Techniques,
Coherent Chaos-Based Communication Systems, Noncoherent Chaos-Based Communication Systems, Chaotic Pulse-Position Modulation, Spread-Spectrum Communications Using Chaos, Filtering of Chaotic Signals, Non-linear Circuits for Chaos-Based Communications and Chaos and Cryptography are some of them [141,142].

Considerable advances have been made both in theory and in experiments and the topic continues to be an active area of research. With the above developments in mind, a detailed study has been made of various nonlinear oscillator systems modeled by nonlinear electronic circuits in this thesis in order to understand their dynamics and chaotic behaviour.

1.6. PRESENT WORK

In particular, in this thesis we wish to investigate both experimentally and numerically, and also by PSPICE simulation and analytic study, the various features associated with the rich variety of bifurcations, chaos, hyperchaos, synchronization of chaos and its possible applications in spread spectrum secure communications in four important families of nonlinear electronic circuits namely

(a) Chaotic phenomena in a simple-3D autonomous circuit with a diode pair.
(b) Study the Dynamics of Period Doubling phenomena in a simple-3D chaotic oscillator with a diode pair.
(c) Simple-4D hyperchaotic canonical Van der Pol Duffing oscillator using current feedback op-amp.
(d) Experimental and numerical realization of simple-4D hyperchaotic circuit with “Cubic-Like” two ideal diodes.
(e) Hyperchaotic behaviour of a fourth-order autonomous electric circuit with a diode pair.
Experimental and numerical realization of higher order autonomous Van der Pol-Duffing oscillator.

Optical communication using chaos based signal transmission scheme.

The specific details are as follows.

1.6.1. Linear and nonlinear circuits

The realization of nonlinearities for the purpose of simulating chaotic systems in the laboratory can be done conveniently with the help of linear and nonlinear circuit elements. The practical implementation of these elements, the configuration of simple linear and possible nonlinear electronic circuits using them form the subject of discussion in chapter 2.

1.6.2. Chaotic phenomena in a simple-3D autonomous circuit with a diode pair

In chapter 3, we introduce the chaotic phenomena in a simple-3D chaotic dynamics of an autonomous oscillator circuit was studied by measuring its responsible in the form of phase-portrait, power spectrum and chaotic time series. The new realization combines attractive features of the current feedback op-amp operating in both voltage and current modes to construct the active linear negative conductance. The component count is reduced and the chaotic spectrum is extended to higher frequencies. In addition, a buffered and isolated voltage output directly representing a state variable is made available. The circuit consists of just three linear elements (two capacitors and one inductor), one linear negative conductance and two ideal diodes. The power spectra are presented to confirm the strong chaotic nature of the oscillations of the circuit. The performance of the circuit is investigated by means of experimental simulation and numerical confirmation of the appropriate differential
equations. The features of the obtained results are respected for various engineering systems such as chaos communication systems with robustness against various interferences.

1.6.3. Period doubling phenomena in a simple-3D chaotic oscillator with a diode pair

In chapter 4, we investigate the period phenomena of third-order chaotic oscillator circuit with a smooth cubic nonlinearity, different kinds of attractors, time waveforms and corresponding power spectra of systems are presented, respectively. The perturbation transforms an unpredictable chaotic behavior into a predictable chaotic or periodic motion via stabilization of unstable, aperiodic, or periodic orbits of the strange chaotic attractor. One advantage of the method is its robustness against noise. A theoretical analysis of the circuit equations is presented, along with experimental and numerical results.

1.6.4. Simple-4D hyperchaotic canonical Van der Pol Duffing oscillator using current feedback op-amp

In chapter 5, we introduce the some interesting phenomena of fourth-order hyperchaotic Canonical Van der Pol Duffing oscillator circuit with a smooth cubic nonlinearity, different kinds of attractors, time waveforms and corresponding Lyapunov exponent spectra of systems are presented, respectively. The perturbation transforms an unpredictable hyperchaotic behavior into a predictable hyperchaotic or periodic motion via stabilization of unstable, aperiodic, or periodic orbits of the strange hyperchaotic attractor. One advantage of the method is its robustness against noise. A theoretical analysis of the circuit equations is presented, along with experimental simulation and numerical results.
1.6.5. Experimental and numerical realization of simple-4D hyperchaotic circuit with “Cubic-Like” two ideal diodes

In chapter 6, we introduce the simple-4D hyperchaotic dynamics of an autonomous oscillator circuit was studied by measuring its responsible in the form of phase-portrait, power spectrum and hyperchaotic time series. The circuit consists of just four linear elements (two capacitors and two inductors), one linear negative conductance and two ideal diodes. The power spectrums are presented to confirm the strong hyperchaotic nature of the oscillations of the circuit. The performance of the circuit is investigated by means of experimental and numerical confirmation of the appropriate differential equations. The features of the obtained results are respected for various engineering system such as chaos based secure communication systems with robustness against various interferences.

1.6.6. Hyperchaotic behaviour of a fourth-order autonomous electric circuit with a diode Pair

In chapter 7, we introduce the some interesting phenomena of fourth-order hyperchaotic autonomous electric circuit with a smooth cubic nonlinearity, different kinds of attractors, time waveforms and corresponding power spectra of systems are presented, respectively. The perturbation transforms an unpredictable hyperchaotic behavior into a predictable hyperchaotic or periodic motion via stabilization of unstable, aperiodic, or periodic orbits of the strange hyperchaotic attractor. One advantage of the method is its robustness against noise. A theoretical analysis of the circuit equations is presented, along with experimental simulation and numerical results.
1.6.7. Experimental and numerical realization of higher order autonomous Van der Pol - Duffing oscillator

In chapter 8, the application of the hyperchaotic dynamics will enhance the engineering system such as chaos based security of communication in information technology. Interest in such systems has been increasing. Some interesting phenomena of higher order autonomous Van der Pol-Duffing oscillator based on fifth order hyperchaotic circuit to improve secure communication have been studied in the present paper. This circuit, which is capable of realizing the behaviour of every member of the autonomous Van der Pol-Duffing family, consists of just six linear elements (resistor, inductors and capacitors) and a smooth cubic non-linearity. In this circuit, we can confirm period doubling routes to chaos and then to hyperchaos through boundary condition, when the system parameter varies. The hyperchaotic dynamics, characterized by broad band power spectrum, is presented to confirm the hyperchaotic nature of the oscillations of the circuit. This has been investigated extensively using laboratory experiments and numerical integration of the appropriate differential equations.

1.6.8. Optical communication using chaos based signal transmission scheme

In chapter 9, Optical communication system using a chaos based signal transmission scheme has been proposed to transmit digital information signal by using the conventional synchronization of chaos and digital transmission approaches. In this scheme either a chaotic or hyper chaotic system is used to generate a chaotic signal. This signal along with the information digital signal is used to generate the transmitted signal. The transmitted signal is then masked by one of the chaotic signal of the transmitter and is transmitted through the channel to the receiver as well as
used to drive the transmitter chaotic system using the concept of self-modulation. At
the receiver end, suitable subtractor circuit is constructed for unmasking and the
reception rule is used to recover the information signal. The effect of typical
perturbing factors like channel noise and parameter mismatch are also included and
their corresponding performance analysis has been done. By considering appropriate
circuit configuration, the results of experiment are also presented.

1.6.9. Limitations of our study

The distinctive patterns of different dynamic phenomena can be visualized on
the oscilloscope screen as:

(i) Time waveform

(ii) Phase portrait and

(iii) Fast Fourier transform

Though the hardware facilities in our laboratory are sufficient, the non-
availability of certain electronic components imposed restrictions on the range of
experiments that could be performed. Hence we have limited our experiments to just
the first four items mentioned above. Also it was not possible for us at present to
perform experiments to study the effect of noise parameter mismatch, generalized and
phase synchronization aspects. We hope to develop our laboratory suitably in the near
future to perform these crucial studies experimentally.