CHAPTER 3
NUMERICAL SOLUTION OF HYBRID FUZZY DIFFERENTIAL EQUATIONS USING RK3 AND RKN3

3.1 INTRODUCTION

Today’s technological applications require the inevitable combination of multi-disciplinary methods. This is the spirit of hybrid dynamical systems, which have emerged as a framework with potential for improving the performance and capabilities of a wide variety of engineering systems. In the last few years, the numerical solution of hybrid fuzzy differential systems has been formed by several authors. Here Runge-Kutta method and Runge-Kutta Nystrom method of order three is discussed. The strongly generalized derivative is defined for a larger class of fuzzy valued function than the H-derivative, and fuzzy differential equations can have solutions which have a decreasing length of their support. So, this differentiability concept is used to solve hybrid fuzzy differential equations.

In this chapter, the numerical solution of hybrid fuzzy differential equations is obtained which is of the form

\[
\begin{align*}
(x(t) &= f(t, x(t), \lambda_k(x_k)), \quad t \in [t_k, t_{k+1}], \\
\lambda_k &= 0.
\end{align*}
\] (3.1)

Where \(0 \leq t_0 < t_1 < \ldots < t_k < \ldots \ldots \) \(t_k \rightarrow \infty\)

\(f \in C[R_+ \times E \times E, E], \quad \lambda_k \in C[E, E].\)
The solutions of equation (3.1) are piecewise differentiable in each interval for \( t \in [t_k, t_{k+1}] \) for a fixed \( x_k \in E \) and \( k = 0, 1, 2 \). Equation (3.1) can be replaced by an equivalent system when \( x(t) \) is considered as (i) differentiable fuzzy – valued function:

\[
\begin{align*}
\dot{x}(t) &= f(t, x, \lambda_k(x_k)) = F_k \left( t, x, \lambda_k(x_k) \right), \quad x(t_k) = x_k \\
\dot{\bar{x}}(t) &= \bar{f}(t, x, \lambda_k(x_k)) = G_k \left( t, \bar{x}, \lambda_k(x_k) \right), \quad \bar{x}(t_k) = \bar{x}_k
\end{align*}
\]

(3.2)

and also equation (3.1) is equivalent to the following system when \( x(t) \) is considered as (ii) differentiable fuzzy – valued function:

\[
\begin{align*}
\dot{\bar{x}}(t) &= \bar{f}(t, x, \lambda_k(x_k)) = F_k \left( t, x, \lambda_k(x_k) \right), \quad \bar{x}(t_k) = \bar{x}_k \\
\dot{x}(t) &= f(t, x, \lambda_k(x_k)) = G_k \left( t, \bar{x}, \lambda_k(x_k) \right), \quad x(t_k) = x_k
\end{align*}
\]

(3.3)

That is for each \( t \), the pair \([x(t; r), \bar{x}(t; r)]\) is a fuzzy number, where \( x(t; r), \bar{x}(t; r) \) are, respectively, the solutions of the parametric form given by:

\[
\begin{align*}
\dot{x}_k(t) &= F_k \left( t, x(t; r), \bar{x}(t; r), \lambda_k(x_k) \right), \quad x(t_k; r) = x_k(r) \\
\dot{\bar{x}}_k(t) &= G_k \left( t, x(t; r), \bar{x}(t; r), \lambda_k(x_k) \right), \quad \bar{x}(t_k; r) = \bar{x}_k(r)
\end{align*}
\]

(3.4)

in the sense of (i) differentiability and

\[
\begin{align*}
\dot{\bar{x}}_k(t) &= G_k \left( t, \bar{x}(t; r), \bar{x}(t; r), \lambda_k(x_k) \right), \quad \bar{x}(t_k; r) = \bar{x}_k(r) \\
\dot{x}_k(t) &= F_k \left( t, x(t; r), \bar{x}(t; r), \lambda_k(x_k) \right), \quad x(t_k; r) = x_k(r)
\end{align*}
\]

(3.5)

in the sense of (ii) differentiability and for each \( r \in [0, 1] \).
3.2 THIRD ORDER RUNGE KUTTA METHOD FOR FIVP

The Runge-Kutta method and Runge-Kutta Nystrom method of order three for the FIVP \( y' = f(t, y(t)), \ a \leq t \leq b \) and \( y(\alpha) = y_0 \) is as follows:

\[
y_{n+1}(r) - y_n(r) = \sum_{i=1}^{3} w_i k_i(t_n; y_n(r))
\]

\[
y_{n+1}(r) - \overline{y}_n(r) = \sum_{i=1}^{3} w_i \overline{k}_i(t_n; y_n(r)),
\]

Where \( w_i \)'s are constants and

\[
k_i(t_n; y_n(r)) = h f \left( t_n + c_i h, \ y(t_n) + \sum_{j=1}^{i-1} \alpha_{ij} k_j(t_n; y_n(r)) \right),
\]

\[
\overline{k}_i(t_n; y_n(r)) = h f \left( t_n + c_i h, \ \overline{y}(t_n) + \sum_{j=1}^{i-1} \alpha_{ij} \overline{k}_j(t_n; y_n(r)) \right)
\]

Let the exact and approximate solutions at \( t_n, \ 0 \leq n \leq N \) are denoted by

\[
Y(t_n; y_n(r)) = \begin{bmatrix} Y(t_n; y_n(r)) \ y(t_n; y_n(r)) \end{bmatrix}
\]

and \( y(t_n; y_n(r)) = \begin{bmatrix} y(t_n; y_n(r)) \ \overline{y}(t_n; y_n(r)) \end{bmatrix} \) respectively.

**Theorem 3.2.1:**

For arbitrary fixed \( r, \ 0 \leq r \leq 1 \), the approximate solutions \( \overline{y}(t, r), \overline{y}(t, r) \) converge to the exact solutions \( Y(t, r), \overline{Y}(t, r) \) uniformly in \( t \) for \( Y, \overline{Y} \in C^3(a, b) \).

**Proof:** Similar to the proof of Ming Ma et.al (1999)
3.3 THE RUNGE–KUTTA METHOD & RUNGE-KUTTA NYSTROM METHOD OF ORDER THREE

In this section, for a hybrid fuzzy differential equation (3.1) a Runge-Kutta method and Runge-Kutta Nystrom method of order three is developed.

For a fixed r, to integrate the system of equation (3.4) and equation (3.5) in \([t_0, t_1], [t_1, t_2], \ldots, [t_k, t_{k+1}], \ldots\), each interval is replaced by a set of \(N_k + 1\) discrete equally spaced grid points at which the exact solution

\[
X(t, r) = (\overline{X}(t; r), \overline{\overline{X}}(t; r))
\]
is approximated by some

\[
(y_k(t; r), \overline{y}_k(t; r)).
\]

For the chosen grid points on \([t_k, t_{k+1}]\)

\[
t_{k,n} = t_k + nh_k = \frac{t_{k+1} - t_k}{N_k}, 0 \leq n \leq N_k,
\]

let \(Y_k(t; r), \overline{Y}_k(t; r)\) \(\equiv (\overline{X}(t; r), \overline{\overline{X}}(t; r))\).

\[
(Y_k(t; r), \overline{Y}_k(t; r)) \quad \text{and} \quad (y_k(t; r), \overline{y}_k(t; r))
\]
is denoted respectively by \(Y_{k,n}(r), \overline{Y}_{k,n}(r)\) and \(y_{k,n}(r), \overline{y}_{k,n}(r)\). The \(N_k\) s are allowed to vary over the \([t_k, t_{k+1}]\) s so that the \(h_k\) s may be comparable.

The Runge-Kutta method of order three (RK3) for equation (3.1) is developed as

\[
y_{k,n+1}(r) - y_{k,n}(r) = \sum_{i=1}^{3} w_i k_i (t_{k,n}, y_{k,n}(r))
\]
\[
\overline{y}_{k,n+1}(r) - \overline{y}_{k,n}(r) = \sum_{i=1}^{3} w_i \overline{k_i}(t_{k,n}; y_{k,n}(r)),
\]

Where \( w_1, w_2, w_3 \) are constants and

\[
\overline{k_1}(t_{k,n}; y_{k,n}(r))
\]

\[
= \min \left\{ h_k f \left( t_{k,n}, u, \lambda_k(u_k) \right) \right\},
\]

\[
\langle u \in \left[ y_{k,n}(r), \overline{y}_{k,n}(r) \right], u_k \in \left[ y_{k,0}(r), \overline{y}_{k,0}(r) \right] \rangle,
\]

\[
\overline{k_1}(t_{k,n}; y_{k,n}(r))
\]

\[
= \max \left\{ h_k f \left( t_{k,n}, u, \lambda_k(u_k) \right) \right\},
\]

\[
\langle u \in \left[ y_{k,n}(r), \overline{y}_{k,n}(r) \right], u_k \in \left[ y_{k,0}(r), \overline{y}_{k,0}(r) \right] \rangle,
\]

\[
\overline{k_2}(t_{k,n}; y_{k,n}(r))
\]

\[
= \min \left\{ h_k f \left( t_{k,n} + \frac{1}{2} h_k, u, \lambda_k(u_k) \right) \right\},
\]

\[
\langle u \in \left[ \overline{z}_{k,1}(t_{k,n}, y_{k,n}(r)), \overline{z}_{k,1}(t_{k,n}, y_{k,n}(r)) \right], u_k \in \left[ y_{k,0}(r), \overline{y}_{k,0}(r) \right] \rangle,
\]

\[
\overline{k_2}(t_{k,n}; y_{k,n}(r))
\]

\[
= \max \left\{ h_k f \left( t_{k,n} + \frac{1}{2} h_k, u, \lambda_k(u_k) \right) \right\},
\]

\[
\langle u \in \left[ \overline{z}_{k,1}(t_{k,n}, y_{k,n}(r)), \overline{z}_{k,1}(t_{k,n}, y_{k,n}(r)) \right], u_k \in \left[ y_{k,0}(r), \overline{y}_{k,0}(r) \right] \rangle,
\]
\[ k_2(t_{k,n}; y_{k,n}(r)) \]
\[ = \min \{ h_k f (t_{k,n} + h_k, u, \lambda_k(u_k)) \}
\[ \setminus u \in [z_{k2}(t_{k,n}, y_{k,n}(r)), \bar{z}_{k2}(t_{k,n}, y_{k,n}(r))), u_k \in [y_{k,0}(r), \bar{y}_{k,0}(r)] \} \]

\[ \bar{k}_3(t_{k,n}; y_{k,n}(r)) \]
\[ = \max \{ h_k f (t_{k,n} + h_k, u, \lambda_k(u_k)) \}
\[ \setminus u \in [z_{k2}(t_{k,n}, y_{k,n}(r)), \bar{z}_{k2}(t_{k,n}, y_{k,n}(r))), u_k \in [y_{k,0}(r), \bar{y}_{k,0}(r)] \} \]

and

\[ z_{k1}(t_{k,n}, y_{k,n}(r)) = \frac{1}{2} k_1(t_{k,n}, y_{k,n}(r)) \]

\[ z_{k1}(t_{k,n}, y_{k,n}(r)) = \frac{1}{2} \bar{k}_1(t_{k,n}, y_{k,n}(r)) \]

\[ z_{k2}(t_{k,n}, y_{k,n}(r)) = \frac{1}{2} k_2(t_{k,n}, y_{k,n}(r)) + 2 \bar{k}_2(t_{k,n}, y_{k,n}(r)) \]

\[ z_{k2}(t_{k,n}, y_{k,n}(r)) = \frac{1}{2} \bar{k}_2(t_{k,n}, y_{k,n}(r)) + 2 \bar{k}_2(t_{k,n}, y_{k,n}(r)) \]

Let us define

\[ F_k[t_{k,n}, y_{k,n}(r), \bar{y}_{k,n}(r)] \]
\[ = k_1(t_{k,n}, y_{k,n}(r)) + 4k_2(t_{k,n}, y_{k,n}(r)) + k_3(t_{k,n}, y_{k,n}(r)) \]

\[ G_k[t_{k,n}, y_{k,n}(r), \bar{y}_{k,n}(r)] \]
\[ = \bar{k}_1(t_{k,n}, y_{k,n}(r)) + 4\bar{k}_2(t_{k,n}, y_{k,n}(r)) + \bar{k}_3(t_{k,n}, y_{k,n}(r)) \]
Then the exact solution at \( t_{k,n+1} \) for the case of (i) – differentiability is given by

\[
\begin{align*}
\begin{cases}
Y_{k,n+1}(r) &\approx \frac{1}{6} F_k[t_{k,n}, y_{k,n}(r), \overline{y_{k,n}}(r)], \\
\overline{Y}_{k,n+1}(r) &\approx \frac{1}{6} G_k[t_{k,n}, y_{k,n}(r), \overline{y_{k,n}}(r)].
\end{cases}
\end{align*}
\]  \( (3.6) \)

and for the case of (ii) – differentiability is given by

\[
\begin{align*}
\begin{cases}
Y_{k,n+1}(r) &\approx \frac{1}{6} G_k[t_{k,n}, y_{k,n}(r), \overline{y_{k,n}}(r)], \\
\overline{Y}_{k,n+1}(r) &\approx \frac{1}{6} F_k[t_{k,n}, y_{k,n}(r), \overline{y_{k,n}}(r)].
\end{cases}
\end{align*}
\]  \( (3.7) \)

The approximated solution based on equations (3.6) and (3.7) is given by

\[
\begin{align*}
\begin{cases}
y_{k,n+1}(r) &\approx y_{k,n}(r) + \frac{1}{6} F_k[t_{k,n}, y_{k,n}(r), \overline{y_{k,n}}(r)], \\
\overline{y}_{k,n+1}(r) &\approx \overline{y}_{k,n}(r) + \frac{1}{6} G_k[t_{k,n}, y_{k,n}(r), \overline{y_{k,n}}(r)].
\end{cases}
\end{align*}
\]  \( (3.8) \)

and

\[
\begin{align*}
\begin{cases}
y_{k,n+1}(r) &\approx y_{k,n}(r) + \frac{1}{6} G_k[t_{k,n}, y_{k,n}(r), \overline{y_{k,n}}(r)], \\
\overline{y}_{k,n+1}(r) &\approx \overline{y}_{k,n}(r) + \frac{1}{6} F_k[t_{k,n}, y_{k,n}(r), \overline{y_{k,n}}(r)].
\end{cases}
\end{align*}
\]  \( (3.9) \)

The Runge-Kutta Nystrom method of order three (RKN3) for equation (3.1) is defined as

\[
\begin{align*}
y_{k,n+1}(r) - y_{k,n}(r) &= \sum_{i=1}^{3} \omega_i k_i(t_{k,n}; y_{k,n}(r)), \\
\overline{y}_{k,n+1}(r) - \overline{y}_{k,n}(r) &= \sum_{i=1}^{3} \omega_i \overline{k}_i(t_{k,n}; y_{k,n}(r)),
\end{align*}
\]
Where $w_1, w_2, w_3$ are constants and

$$ k_1(t_{k,n}; y_{k,n}(r)) $$

$$ = \min \left\{ h_k f \left( t_{k,n}, u, \lambda_k(u_k) \right) \right\} $$

$$ \forall u \in [y_{k,n}(r), \overline{y}_{k,n}(r)], u_k \in [y_{k,0}(r), \overline{y}_{k,0}(r)] \right\}, $$

$$ k_2(t_{k,n}; y_{k,n}(r)) $$

$$ = \min \left\{ h_k f \left( t_{k,n} + \frac{2}{3} h_k, u, \lambda_k(u_k) \right) \right\} $$

$$ \forall u \in [z_k(t_{k,n}, y_{k,n}(r)), \overline{z}_k(t_{k,n}, y_{k,n}(r)), u_k \in [y_{k,0}(r), \overline{y}_{k,0}(r)] \right\}, $$

$$ k_3(t_{k,n}; y_{k,n}(r)) $$

$$ = \min \left\{ h_k f \left( t_{k,n} + \frac{2}{3} h_k, u, \lambda_k(u_k) \right) \right\} $$

$$ \forall u \in [z_k(t_{k,n}, y_{k,n}(r)), \overline{z}_k(t_{k,n}, y_{k,n}(r)), u_k \in [y_{k,0}(r), \overline{y}_{k,0}(r)] \right\}, $$
$$\overline{k_3}(t_{k,n}; y_{k,n}(r))$$

$$= \max \left\{ h_k f \left( t_{k,n} + \frac{2}{3} h_k, u, \lambda_k(u_k) \right) \right\}$$

$$\cup u \in [\overline{z_{k2}}(t_{k,n}, y_{k,n}(r)), \overline{z_{k2}}(t_{k,n}, y_{k,n}(r)), u_k \in [\overline{y_{k,6}}(r), \overline{y_{k,0}}(r)] \right\}$$

and

$$\overline{z_{k1}}(t_{k,n}, y_{k,n}(r)) = \overline{y_{k,n}}(r) + \frac{2}{3} \overline{k_1}(t_{k,n}, y_{k,n}(r))$$

$$\overline{z_{k1}}(t_{k,n}, y_{k,n}(r)) = \overline{y_{k,n}}(r) + \frac{2}{3} \overline{k_1}(t_{k,n}, y_{k,n}(r))$$

$$\overline{z_{k2}}(t_{k,n}, y_{k,n}(r)) = \overline{y_{k,n}}(r) + \frac{2}{3} \overline{k_2}(t_{k,n}, y_{k,n}(r))$$

$$\overline{z_{k2}}(t_{k,n}, y_{k,n}(r)) = \overline{y_{k,n}}(r) + \frac{2}{3} \overline{k_2}(t_{k,n}, y_{k,n}(r))$$

Let us define

$$S_k[t_{k,n}, \overline{y_{k,n}}(r)]$$

$$= 2 \overline{k_1}(t_{k,n}, y_{k,n}(r)) + 3 \overline{k_2}(t_{k,n}, y_{k,n}(r)) + 3 \overline{k_3}(t_{k,n}, y_{k,n}(r))$$

$$T_k[t_{k,n}, \overline{y_{k,n}}(r)]$$

$$= 2 \overline{k_1}(t_{k,n}, y_{k,n}(r)) + 3 \overline{k_2}(t_{k,n}, y_{k,n}(r)) + 3 \overline{k_3}(t_{k,n}, y_{k,n}(r))$$

Then the exact solution at $$t_{k,n+1}$$ for the case of (i) differentiability is given by
\[
\begin{align*}
\left\{ Y_{k,n+1}(r) & \approx Y_{k,n}(r) + \frac{1}{8} S_k \left[ t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r) \right], \\
\left\{ Y_{k,n+1}(r) & \approx \overline{y}_{k,n}(r) + \frac{1}{8} T_k \left[ t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r) \right].
\end{align*}
\] (3.10)

and for the case of (ii) – differentiability is given by

\[
\begin{align*}
\left\{ Y_{k,n+1}(r) & \approx y_{k,n}(r) + \frac{1}{8} S_k \left[ t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r) \right], \\
\left\{ \overline{y}_{k,n+1}(r) & \approx \overline{y}_{k,n}(r) + \frac{1}{8} T_k \left[ t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r) \right].
\end{align*}
\] (3.11)

The approximated solution based on equations (3.10) and (3.11) is given by

\[
\begin{align*}
\left\{ y_{k,n+1}(r) & = y_{k,n}(r) + \frac{1}{8} S_k \left[ t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r) \right], \\
\left\{ \overline{y}_{k,n+1}(r) & = \overline{y}_{k,n}(r) + \frac{1}{8} T_k \left[ t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r) \right].
\end{align*}
\] (3.12)

and

\[
\begin{align*}
\left\{ y_{k,n+1}(r) & = y_{k,n}(r) + \frac{1}{8} T_k \left[ t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r) \right], \\
\left\{ \overline{y}_{k,n+1}(r) & = \overline{y}_{k,n}(r) + \frac{1}{8} S_k \left[ t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r) \right].
\end{align*}
\] (3.13)

### 3.4 CONVERGENCE AND STABILITY

The following result proves that convergence is point wise in \( r \) for a fixed \( k \).

**Lemma 3.4.1:**

Suppose \( i \in \mathbb{Z}^+, \varepsilon_i > 0, r \in [0,1] \) and \( h_i < 1 \) are fixed. Let \( \{Z_{i,n}(r)\}_{n=0}^{N_i} \) be the Runge-Kutta approximation with \( N = N_i \) to the fuzzy IVP:
\begin{align*}
(x(t) &= f(t, x(t), \lambda_i(x_i)), \ t \in [t_i, t_{i+1}], \\
(x(t_i) &= x_i
\end{align*}

If \( \{y_{i,n}(r)\}_{n=0}^{N_i} \) denotes the result (3.8), or (3.9) and (3.12), or (3.13) from some \( y_{i,0}(r) \), then there exists a \( \delta_i > 0 \) such that

\[
|z_{i,0}(r) - y_{i,0}(r)| < \delta_i, \quad |\overline{z}_{i,0}(r) - \overline{y}_{i,0}(r)| < \delta_i
\]

Implies \( |z_{i,N_i}(r) - y_{i,N_i}(r)| < \varepsilon_i, \quad |\overline{z}_{i,N_i}(r) - \overline{y}_{i,N_i}(r)| < \varepsilon_i \)

**Proof:** Similar to the proof of Pederson & Sambandham (2007)

**Theorem 3.4.1:**

Consider the systems (3.2), and (3.8) or system (3.3) and (3.9) also consider (3.2), and (3.12) or (3.3), and (3.13). For a fixed \( k \in \mathbb{Z}^+ \) and \( r \in [0,1] \),

\[
\lim_{h_0, \ldots, h_k \to 0} y_{k,N_k}(r) = \bar{x}(t_{k+1}; r),
\]

\[
\lim_{h_0, \ldots, h_k \to 0} \overline{y}_{k,N_k}(r) = \overline{x}(t_{k+1}; r).
\]

**Proof:** Similar to the proof of Pederson & Sambandham (2007)

**Theorem 3.4.2:**

Consider the hybrid fuzzy differential equations (3.1). For \( k = 0,1,2, \ldots \) and for each \( f_k: [t_k, t_{k+1}] \times \mathbb{E} \to \mathbb{E} \) is such that

(i) \[
f_k(t, x, r) =
[F_k \left(t, \bar{x}(t; r), \overline{x}(t; r), \lambda_k(x_k)\right), G_k \left(t, \bar{x}(t; r), \overline{x}(t; r), \lambda_k(x_k)\right)]
\]
(ii) \( F_k \left( t, x(t; r), \bar{x}(t; r), \lambda_k(x_k) \right) \) and \( G_k \left( t, \bar{x}(t; r), \bar{x}(t; r), \lambda_k(x_k) \right) \)
are equicontinuous (that is, for \( \varepsilon > 0 \) there is a \( \delta_k(\varepsilon) \) such that
\[ |F_k(t, x, y) - G_k(t, x, y)| < \varepsilon \quad \text{for all} \quad r \in [0,1], \]
whenever \( (t, x, y), (t_1, x_1, y_1) \in [t_k, t_{k+1}] \times \mathbb{R}^2 \) and \( \|(t, x, y) - (t_1, x_1, y_1)\) < \( \delta_k(\varepsilon) \)\] and uniformly bounded on any bounded set.

(iii) There exists \( l_k > 0 \) such that
\[ |F_k(t, x_2, y_2) - G_k(t, x_1, y_1)| < l_k \max\{|x_2 - x_1|, |y_2 - y_1|\} \]

Then equation (3.1) and the hybrid system of ODEs

\[
\begin{align*}
    x_k'(t; r) &= F_k \left( t, x(t; r), \bar{x}(t; r), \lambda_k(x_k) \right) \\
    \bar{x}_k'(t; r) &= G_k \left( t, \bar{x}(t; r), \bar{x}(t; r), \lambda_k(x_k) \right) \\
    x_k(t_k; r) &= x_{k-1}(t_k; r) \quad \text{if} \quad k > 0, \quad x_0(t_0; r) = x_0(r) \\
    \bar{x}_k(t_k; r) &= \bar{x}_{k-1}(t_k; r) \quad \text{if} \quad k > 0, \quad \bar{x}_0(t_0; r) = \bar{x}_0(r)
\end{align*}
\]

are equivalent when \( x(t) \) is (i) – differentiable and equation (3.1) and the
hybrid system of ODEs

\[
\begin{align*}
    \bar{x}_k'(t; r) &= F_k \left( t, x(t; r), \bar{x}(t; r), \lambda_k(x_k) \right) \\
    x_k'(t; r) &= G_k \left( t, x(t; r), \bar{x}(t; r), \lambda_k(x_k) \right) \\
    x_k(t_k; r) &= x_{k-1}(t_k; r) \quad \text{if} \quad k > 0, \quad x_0(t_0; r) = x_0(r) \\
    \bar{x}_k(t_k; r) &= \bar{x}_{k-1}(t_k; r) \quad \text{if} \quad k > 0, \quad \bar{x}_0(t_0; r) = \bar{x}_0(r)
\end{align*}
\]

are equivalent when \( x(t) \) is (ii) – differentiable.

**Proof:** It is completely similar to the proof of Pederson &
Sambandham (2009).
3.5 NUMERICAL ILLUSTRATION

To give a clear overview of our study and to illustrate the above discussed technique the following examples are considered. Manual Computation is very tedious hence Java version 1.5 is used to solve the examples. The obtained data are imported into MATLAB and represented graphically.

Example 3.5.1:

Consider the following hybrid fuzzy IVP,

$$\begin{align*}
\begin{cases}
x(t) = x(t) + m(t) \lambda_k(x(t_k)), & t \in [t_k, t_{k+1}], \\
x(0, r) = [0.75 + 0.25r, 1.125 - 0.125r], & 0 \leq r \leq 1.
\end{cases}
\end{align*}$$

(3.14)

Where $m(t) = \begin{cases} 
2(t \text{mod} 1), & \text{if } t \text{mod} 1 \leq 0.5 \\
2(1 - t \text{mod} 1), & \text{if } t \text{mod} 1 > 0.5,
\end{cases}$ and

$$\lambda_k(\mu) = \begin{cases} 
0, & \text{if } k = 0 \\
\mu, & \text{if } k \in [1,2, \ldots \ldots ]
\end{cases}$$

The HFDE (3.14) is equivalent to the following system of FDEs:

$$\begin{align*}
\begin{cases}
x_0'(t) = x_0(t), & t \in [0,1], \\
x_0(0; r) = [0.75 + 0.25r, 1.125 - 0.125r], & 0 \leq r \leq 1, \\
x_i'(t) = x_i(t) + m(t)x_i(t_i), & t \in [t_i, t_{i+1}], x_i(t_i) = x_{i-1}(t_i), i = 1,2,3 \ldots
\end{cases}
\end{align*}$$

In (3.14), $y(t) + m(t)\lambda_k(x(t_k))$ is a continuous function of $t, y,$ and $\lambda_k(x(t_k)).$ For each $k = 0,1,2 \ldots$ the fuzzy IVP

$$\begin{align*}
\begin{cases}
y(t) = y(t) + m(t)\lambda_k(y(t_k)), & t \in [t_k, t_{k+1}], \quad t_k = k \\
y(t_k) = y_{tk}
\end{cases},
\end{align*}$$

has a unique solution on $[t_k, t_{k+1}].$
The exact solution of (3.14) for $t \in [0,1]$ in the sense of (i) differentiability is

$$x(t; r) = [(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t]$$

For $t \in [1,1.5]$, the exact solution of (3.14) satisfies

$$x(t; r) = x(1, r)(3e^{t-1} - 2t)$$

and for $t \in [1.5,2]$, the exact solution of (3.14) satisfies

$$x(t; r) = x(1, r)[2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)]$$
in the sense of (i) differentiability.

Figure 3.1 Comparison of RK3 and RKN3 with exact solution

The solution $y(2, r)$ is compared at different $r$ values with exact, Euler, RK3 and RKN3 at N=10. It is evident from the figure that RK3 and RKN3 give better solution than the Euler method.
Figure 3.2 Comparison at 0-level and 1-level

Solid curve: real value; dotted curve: RK3; o: RKN3.

The comparison of the solution $y$ with different $t$ at the 0-level and 1-level are depicted in Figure 3.2. It is observed that the approximated solution better matches the exact solution at different $t$ values.

Example 3.5.2

Consider the following hybrid fuzzy IVP,

$$
\begin{align*}
\{ & x(t) = x(t) + m(t)\lambda_k(x(t_k)), t \in [t_k, t_{k+1}], t_k = k, k = 0,1,2, \ldots, \\
& x(0, r) = [0.75 + 0.25r, 1.125 - 0.125r], 0 \leq r \leq 1.
\end{align*}
$$

(3.15)

Where $m(t) = |\sin(\pi t)|, k = 0,1,2, \ldots \ldots$ and

$$
\lambda_k(\mu) = \begin{cases} 
0, & \text{if } k = 0 \\
\mu, & \text{if } k \in \{1,2, \ldots \ldots \}
\end{cases}
$$

The HFDE (3.15) is equivalent to the following system of FDEs:
\[
\begin{align*}
  x_0'(t) &= x_0(t), t \in [0,1], \\
  x_0(0; r) &= [0.75 + 0.25r, 1.125 - 0.125r], 0 \leq r \leq 1, \\
  x_i'(t) &= x_i(t) + m(t)x_i(t), t \in [t_i, t_{i+1}], x_i(t_i) = x_{i-1}(t_i), i = 1,2,3 ... \\
\end{align*}
\]

In (3.15), \( y(t) + m(t)\lambda_k(x(t_k)) \) is a continuous function of \( t, y, \) and \( \lambda_k(x(t_k)) \). For each \( k = 0,1,2 \ldots \) the fuzzy IVP

\[
\begin{align*}
  y(t) &= y(t) + m(t)\lambda_k(y(t_k)), t \in [t_k, t_{k+1}], t_k = k \\
  y(t_k) &= y(t_k) \\
\end{align*}
\]

has a unique solution on \([t_k, t_{k+1}]\).

The exact solution of (3.15) for \( t \in [0,1] \) in the sense of (i) differentiability is

\[
x(t; r) = [(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t]
\]

For \( t \in [1,2] \), the exact solution of (3.15) satisfies

\[
x(t; r) = x(1, r) \frac{\pi \cos(\pi r) + \sin(\pi r)}{\pi^2 + 1} + e^t \frac{\pi}{e} x(1, r) \left(1 + \frac{\pi}{\pi^2 + 1}\right) \text{ in the sense of (i) differentiability.}
\]
The solution \( y(2, r) \) is compared at different \( r \) values with exact, Euler, RK3 and RKN3 at \( N=10 \). From the figure it is noted that RK3 and RKN3 give better solution than the Euler method.

![Figure 3.4 0-level and 1-level comparison](image)

Solid curve: real value; dotted curve: RK3; \( \cdot \): RKN3.

The comparison of the solution \( y \) with different \( t \) at the 0-level and 1-level are depicted in Figure 3.4. It is observed that the approximated solution better matches the exact solution at different \( t \) values.

**Example 3.5.3**

Consider the following hybrid fuzzy IVP,

\[
\begin{cases}
    x(t) = -x(t) + m(t)\lambda_k(x(t_k)), & t \in [t_k, t_{k+1}], \\
    x(0, r) = [0.75 + 0.25r, 1.125 - 0.125r], & 0 \leq r \leq 1.
\end{cases}
\]  

(3.16)

Where \( m(t) = \begin{cases} 
    2(t(\text{mod} 1)), & \text{if } t(\text{mod } 1) \leq 0.5 \\
    2(1 - t(\text{mod } 1)), & \text{if } t(\text{mod } 1) > 0.5,
\end{cases} \)

\[
\lambda_k(\mu) = \begin{cases} 
    0, & \text{if } k = 0 \\
    \mu, & \text{if } k \in \{1, 2, \ldots \}
\end{cases}
\]
In this example (ii) differentiability is applicable and hence HFDE in equation (3.16) is solved under (ii) differentiability.

The HFDE (3.16) is equivalent to the following system of FDEs:

\[
\begin{aligned}
& x'_0(t) = -x_0(t), \quad t \in [0, 1], \\
& x_0(0, r) = [0.75 + 0.25r, 1.125 - 0.125r], 0 \leq r \leq 1, \\
& x'_i(t) = -x_i(t) + m(t)x_i(t_i), \quad t \in [t_i, t_{i+1}], x_i(t_i) = x_{i-1}(t_i), i = 1, 2, 3 \ldots
\end{aligned}
\]

In (3.16), \(-y(t) + m(t)\lambda_k(x(t_k))\) is a continuous function of \(t, y, \) and \(\lambda_k(x(t_k))\). For each \(k = 0, 1, 2 \ldots\) the fuzzy IVP

\[
\begin{aligned}
& y (t) = -y(t) + m(t)\lambda_k(y(t_k)), t \in [t_k, t_{k+1}], \ t_k = k \\
& y(t_k) = y_{tk}
\end{aligned}
\]

has a unique solution on \([t_k, t_{k+1}]\).

The exact solution of (3.16) for \(t \in [0, 1]\) in the sense of (ii) differentiability we get

\[
x(t; r) = [(0.75 + 0.25r)e^{-t}, (1.125 - 0.125r)e^{-t}]
\]

For \(t \in [1, 1.5]\), the exact solution of (3.16) satisfies

\[
x(t; r) = x(1; r)(3e^{1-t} - 2t)
\]

and for \(t \in [1.5, 2]\), the exact solution of (3.16) satisfies

\[
x(t; r) = x(1; r)[2t - 2 + e^{1.5-t}\left(3 + 4\right)] \text{ in the sense of (ii) differentiability.}
\]
Table 3.1 Results of lower end points of RK3 and RKN3 with exact solution

<table>
<thead>
<tr>
<th>r</th>
<th>$Y_{k,n+1}$</th>
<th>RK3 $Y_{k,n+1}$</th>
<th>RKN3 $Y_{k,n+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.186933036</td>
<td>0.186925992</td>
<td>0.186925993</td>
</tr>
<tr>
<td>0.1</td>
<td>0.193164138</td>
<td>0.193156858</td>
<td>0.193156859</td>
</tr>
<tr>
<td>0.2</td>
<td>0.199395239</td>
<td>0.199387724</td>
<td>0.199387725</td>
</tr>
<tr>
<td>0.3</td>
<td>0.20562634</td>
<td>0.205618592</td>
<td>0.205618592</td>
</tr>
<tr>
<td>0.4</td>
<td>0.211857441</td>
<td>0.211849458</td>
<td>0.211849458</td>
</tr>
<tr>
<td>0.5</td>
<td>0.218088542</td>
<td>0.218080325</td>
<td>0.218080325</td>
</tr>
<tr>
<td>0.6</td>
<td>0.224319644</td>
<td>0.224311191</td>
<td>0.224311191</td>
</tr>
<tr>
<td>0.7</td>
<td>0.230550745</td>
<td>0.230542658</td>
<td>0.230542658</td>
</tr>
<tr>
<td>0.8</td>
<td>0.236781846</td>
<td>0.236772923</td>
<td>0.236772924</td>
</tr>
<tr>
<td>0.9</td>
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<td>0.243003789</td>
<td>0.243003790</td>
</tr>
<tr>
<td>1.0</td>
<td>0.249244049</td>
<td>0.249234657</td>
<td>0.249234657</td>
</tr>
</tbody>
</table>

Table 3.2 Results of upper end points of RK3 and RKN3 with exact solution

<table>
<thead>
<tr>
<th>r</th>
<th>$Y_{k,n+1}$</th>
<th>RK3 $Y_{k,n+1}$</th>
<th>RKN3 $Y_{k,n+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.280399555</td>
<td>0.280388989</td>
<td>0.280388990</td>
</tr>
<tr>
<td>0.1</td>
<td>0.277284004</td>
<td>0.277273556</td>
<td>0.277273556</td>
</tr>
<tr>
<td>0.2</td>
<td>0.274168453</td>
<td>0.274158123</td>
<td>0.274158124</td>
</tr>
<tr>
<td>0.3</td>
<td>0.271052903</td>
<td>0.271042689</td>
<td>0.271042689</td>
</tr>
<tr>
<td>0.4</td>
<td>0.267937352</td>
<td>0.267927256</td>
<td>0.267927256</td>
</tr>
<tr>
<td>0.5</td>
<td>0.264821802</td>
<td>0.264811823</td>
<td>0.264811823</td>
</tr>
<tr>
<td>0.6</td>
<td>0.261706251</td>
<td>0.261696390</td>
<td>0.261696390</td>
</tr>
<tr>
<td>0.7</td>
<td>0.25859070</td>
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<td>0.258580957</td>
</tr>
<tr>
<td>0.8</td>
<td>0.25547515</td>
<td>0.255465523</td>
<td>0.255465524</td>
</tr>
<tr>
<td>0.9</td>
<td>0.252359599</td>
<td>0.252350090</td>
<td>0.252350090</td>
</tr>
<tr>
<td>1.0</td>
<td>0.249244049</td>
<td>0.249234657</td>
<td>0.249234657</td>
</tr>
</tbody>
</table>
The results of exact solution with approximated solution RK3 and RKN3 for N=10 is tabulated in Table 3.1 and Table 3.2.

![Graph showing comparative results of RK3 and RKN3 with exact solution]

Figure 3.5 Comparative results of RK3 and RKN3 with exact solution

The solution \( y(z, r) \) is compared at different \( r \) values with exact, Euler, RK3 and RKN3 at N=10. From the figure it is noted that RK3 and RKN3 give better solution than the Euler method under (ii) differentiability.

3.6 CONCLUSION

In this chapter the Runge-Kutta method of order three and Runge-Kutta Nystrom method of order three is applied for finding the numerical solution of hybrid fuzzy differential equations. The strongly generalized derivative is defined for a larger class of fuzzy valued function than the H-derivative, and fuzzy differential equations can have solutions which have a decreasing length of their support. So, this differentiability concept is used to solve HFDEs. So far Runge Kutta methods are not used for finding solution under (ii) differentiability. An attempt is made to solve HFDE under (ii) differentiability. Several numerical examples are illustrated to prove the theory. It is found that the approximated result obtained by RK3 and RKN3 gives better solution than the Euler method.