CHAPTER 6

NUMERICAL APPROACH TO SOLVE HYBRID
INTUITIONISTIC FUZZY DIFFERENTIAL EQUATION

6.1 INTRODUCTION

Intuitionistic fuzzy sets (IFSs) have been proposed by Atanassov as a generalization mathematical framework of the traditional fuzzy sets originated from an early work of Zadeh. The main advantage of the IFSs is their property to cope with the hesitancy that may exist due to information impression. This is achieved by incorporating a second function, along with the membership function of the conventional fuzzy sets, called non-membership function. In this way, apart from the degree of the belongingness, the IFSs also combine the notation of the non-belongingness in order to better describe the real status of the information.

Since the first introduction of the IFSs and the consequent study on the fundamentals of the IFSs, a lot of attention has been paid on developing distance or similarity measures between the IFSs, as a way to apply them on several problems of the engineering life. In this chapter, hybrid intuitionistic fuzzy differential equation is framed and solved numerically.

6.2 BASIC DEFINITIONS

Definition 6.2.1: A binary operation \(*\): \([0,1] \times [0,1] \rightarrow [0,1]\) is continuous t-norm if \(*\) satisfies the following conditions:
(i) \( * \) is commutative and associative

(ii) \( * \) is continuous

(iii) \( a \ast 1 = a \) for all \( a \in [0,1], \)

(iv) \( a \ast b \leq c \ast d \) whenever \( a \leq c, b \leq d \) and \( a, b, c, d \in [0,1] \).

**Definition 6.2.2:** A binary operation \( \diamond : [0,1] \times [0,1] \to [0,1] \) is continuous t-conorm if \( \diamond \) satisfies the following conditions:

(i) \( \diamond \) is commutative and associative

(ii) \( \diamond \) is continuous

(iii) \( a \diamond 0 = a \) \( \forall a \in [0,1], \)

(iv) \( a \diamond b \leq c \diamond d \) whenever \( a \leq c, b \leq d \) and \( a, b, c, d \in [0,1] \).

**Definition 6.2.3:** Let \( * \) be the continuous t-norm, \( \diamond \) be a continuous t-conorm and \( V \) be a linear space over the field \( F (= R or C) \). An intuitionistic fuzzy norm on \( V \) is an object of the form

\[
A = \left\{ ((x, t), \mu(x, t), \nu(x, t)) : (x, t) \in V \times R^+ \right\}, \quad \text{where} \quad \mu \& \nu \quad \text{are fuzzy sets on} \ V \times R^+, \mu \text{denotes the degree of membership and} \nu \text{denotes the degree of non membership} \quad (x, t) \in V \times R^+ \text{ satisfying the following conditions:

(i) } \mu(x, t) + \nu(x, t) \leq 1 \quad \forall \quad (x, t) \in V \times R^+; \]

(ii) \( \mu(x, t) > 0; \)

(iii) \( \mu(x, t) = 1 \) if and only if \( x = \theta, \theta \) is a null vector;

(iv) \( \mu(cx, t) = \mu\left(x, \frac{t}{|c|}\right) \quad \forall \ c \in F \text{ and } c \neq 0; \)
\( \mu(x, s) \ast \mu(y, t) \leq \mu(x + y, s + t); \)

(vi) \( \mu(x, \cdot) \) is a non decreasing function of \( R^+ \) and \( \lim_{t \to \infty} \mu(x, t) = 1; \)

(vii) \( \nu(x, t) < 1; \)

(viii) \( \nu(x, t) = 0 \) if and only if \( x = \theta; \)

(ix) \( \nu(cx, t) = \nu\left(x, \frac{t}{|c|}\right) \quad \forall \ c \in F \) and \( c \neq 0; \)

(x) \( \nu(x, s) \circ \nu(y, t) \geq \nu(x + y, s + t); \)

(xi) \( \nu(x, \cdot) \) is a non increasing function of \( R^+ \) and \( \lim_{t \to \infty} \nu(x, t) = 0. \)

**Definition 6.2.4:** If \( A \) is an intuitionistic fuzzy norm on a linear space \( V \) then \( (V, A) \) is called an intuitionistic fuzzy normed linear space.

For the intuitionistic fuzzy normed linear space \( (V, A) \), it is assumed that \( \mu, \nu, \ast, \circ \) satisfy the following axioms:

(i) \( a \circ a = a, \ a \ast a = a \) for all \( a \in [0,1]. \)

(ii) \( \mu(x, t) > 0, \) for all \( t > 0 \) \( \Rightarrow x = \theta. \)

(iii) \( \nu(x, t) < 1, \) for all \( t > 0 \) \( \Rightarrow x = \theta. \)

(iv) For \( x \neq \theta, \mu(x, \cdot) \) is a continuous function of \( R \) and strictly increasing on the subset \( \{t: 0 < \mu(x, t) < 1\} \) of \( R. \)

(v) For \( x \neq \theta, \nu(x, \cdot) \) is a continuous function of \( R \) and strictly decreasing on the subset \( \{t: 0 < \nu(x, t) < 1\} \) of \( R. \)

**Definition 6.2.5:** A sequence \( \{x_n\}_n \) in an intuitionistic fuzzy normed linear space \( (V, A) \) is said to converge to \( x \in V \) if for given \( r > 0, t > \)
0, 0 < r < 1, there exist an integer n₀ ∈ N such that μ(xₙₙ - x, t) > 1 - r
and ν(xₙₙ - x, t) < r for all ≥ n₀.

**Definition 6.2.6:** Let (U, A) and (V, B) be two intuitionistic fuzzy normed linear space over the same field F. A mapping f from (U, A) to (V, B) is said to be intuitionistic fuzzy continuous at x₀ ∈ U, if for any given ε > 0, α ∈ (0, 1), ∃δ(α, ε). 0, β = β(α, ε) ∈ (0, 1)such that for all x ∈ U,

\[ μ_U(x - x₀, δ) > 1 - β \Rightarrow μ_V(f(x) - f(x₀), ε) > 1 - α \]

\[ ν_U(x - x₀, δ) < β \Rightarrow ν_V(f(x) - f(x₀), ε) < α. \]

**Definition 6.2.7:** Let (R, μ₁, ν₁,*ι, φ) and (R, μ₂, ν₂,*ι, φ) be two intuitionistic fuzzy normed linear space over the same field R. A mapping f from (R, μ₁, ν₁,*ι, φ) to (R, μ₂, ν₂,*ι, φ) is said to be intuitionistic fuzzy differentiable at x₀ ∈ R, if for any given ε > 0, α ∈ (0, 1), ∃ δ = δ(α, ε) > 0, β = β(α, ε) ∈ (0, 1) such that for all x(≠ x₀) ∈ R,

\[ μ_1(x - x₀, δ) > 1 - β \Rightarrow μ_2\left(\frac{f(x) - f(x₀)}{x - x₀} - f'(x₀), ε\right) > 1 - α \]

\[ ν_1(x - x₀, δ) < β \Rightarrow ν_2\left(\frac{f(x) - f(x₀)}{x - x₀} - f'(x₀), ε\right) < α. \]

Intuitionistic fuzzy derivative of f at x₀ is denoted by f'(x₀).

**6.3 HYBRID INTUITIONISTIC FUZZY CAUCHY PROBLEM**

Consider a first order hybrid intuitionistic fuzzy differential equation
\[ x'(t) = f(t, x(t), \lambda_k(x_k)) \]

Where \( x \) is an intuitionistic fuzzy function of the crisp variable \( t \), \( f(t, x(t), \lambda_k(x_k)) \) is an intuitionistic fuzzy function of the crisp variable \( t \) and the intuitionistic fuzzy variable \( x \) where \( f \in C[R^+ \times \mathbb{E} \times \mathbb{E}, \mathbb{E}], \lambda_k \in C[\mathbb{E}, \mathbb{E}] \) and \( x' \) is the hybrid intuitionistic fuzzy derivative.

If an initial value \( x(t_k) = x_k \) \{intuitionistic fuzzy number\}, then an hybrid intuitionistic fuzzy Cauchy problem of first order

\[
\begin{align*}
\begin{cases}
x'(t) = f(t, x(t), \lambda_k(x_k)), & t \in [t_k, t_{k+1}], \\
x(t_k) = x_k
\end{cases}
\end{align*}
\]

is obtained,

\[ 0 \leq t_0 < t_1 < \cdots < t_k < \cdots, \quad t_k \to \infty, \]

\[ f \in C[R^+ \times \mathbb{E} \times \mathbb{E}, \mathbb{E}], \lambda_k \in C[\mathbb{E}, \mathbb{E}]. \]

To be specific the system will be as follows

\[
\begin{align*}
x_0'(t) &= f(t, x_0(t), \lambda_0(x_0)), & x(t_0) = x_0, & t \in [t_0, t_1], \\
x_1'(t) &= f(t, x_1(t), \lambda_1(x_1)), & x(t_1) = x_1, & t \in [t_1, t_2], \\
&\vdots & & \\
x_k'(t) &= f(t, x_k(t), \lambda_k(x_k)), & x(t_k) = x_k, & t \in [t_k, t_{k+1}],
\end{align*}
\]

With respect to the solution of equation (6.1), the following function is determined:
\[
    x(t) = \begin{cases} 
        x_0(t), & t \in [t_0, t_1] \\
        x_1(t), & t \in [t_1, t_2] \\
        \vdots \\
        x_k(t), & t \in [t_k, t_{k+1}] \\
        \vdots
    \end{cases}
\]

It is noted that the solutions of equation (6.1) are piecewise differentiable in each interval for

\[ t \in [t_k, t_{k+1}] \] for a fixed \( t_k \in \mathbb{E} \) and \( k = 0,1,2 \ldots \)

As each intuitionistic fuzzy number is represented by a pair of fuzzy numbers, hence equation (6.1) can be replaced by an equivalent system as follows:

\[
    \underline{x^+} = \underline{f^+(t, x)} = \min\{f(t, u) \mid u \in [\underline{x^+}, \overline{x^+}]\} = \underline{F_k(t, \underline{x^+}, \overline{x^+})}, \quad \underline{x^+(t_k)} = \underline{x^+}_k
\]

\[
    \overline{x^+} = \overline{f^+(t, x)} = \max\{f(t, u) \mid u \in [\underline{x^+}, \overline{x^+}]\} = \overline{G_k(t, \underline{x^+}, \overline{x^+})}, \quad \overline{x^+(t_k)} = \overline{x^+}_k
\]

\[
    \underline{x^-} = \underline{f^-(t, x)} = \min\{f(t, v) \mid v \in [\underline{x^-}, \overline{x^-}]\} = \underline{S_k(t, \underline{x^-}, \overline{x^-})}, \quad \underline{x^-(t_k)} = \underline{x^-}_k
\]

\[
    \overline{x^-} = \overline{f^-(t, x)} = \max\{f(t, v) \mid v \in [\underline{x^-}, \overline{x^-}]\} = \overline{T_k(t, \underline{x^-}, \overline{x^-})}, \quad \overline{x^-(t_k)} = \overline{x^-}_k
\]

The system of equations given in equations (6.2) and (6.3) will have unique solution \([\underline{x^+}, \overline{x^+}] \in B = \overline{C}[0,1] \times \overline{C}[0,1]\) and the system of equations given in (6.4) and (6.5) will have unique solution \([\underline{x^-}, \overline{x^-}] \in B = \overline{C}[0,1] \times \overline{C}[0,1]\)

Therefore the system given from (6.2) to (6.5) possesses unique solution
\[ x = \left[ x^+, x^+ \right], \left[ x^-, x^- \right] \in B \times B \] which is an intuitionistic fuzzy function.

(i.e) for each \( t, x(t; r) = \left[ x^+(t; r), x^+(t; r) \right], \left[ x^-(t; r), x^-(t; r) \right] \), \( r \in [0,1] \) is an intuitionistic fuzzy number.

The parametric form of the system of equations (6.2) to (6.5) is given by

\[
\begin{align*}
\bar{x}^+(t; r) &= \bar{f}^+(t, x(t; r)) = \bar{F}_k \left( t, x^+(t; r), x^+(t; r) \right), \quad x^+(t_k; r) = x^+(r) \quad (6.6) \\
\bar{x}^-(t; r) &= \bar{f}^+(t, x(t; r)) = \bar{G}_k \left( t, x^+(t; r), x^+(t; r) \right), \quad x^+(t_k; r) = x^+(r) \quad (6.7) \\
x^-(t; r) &= \bar{f}^-(t, x(t; r)) = \bar{S}_k \left( t, x^-(t; r), x^-(t; r) \right), \quad x^-(t_k; r) = x^-(r) \quad (6.8) \\
\bar{x}^-(t; r) &= \bar{f}^-(t, x(t; r)) = \bar{T}_k \left( t, x^-(t; r), x^-(t; r) \right), \quad x^-(t_k; r) = x^-(r) \quad (6.9)
\end{align*}
\]

for \( r \in [0,1] \)

A solution to the system of equations (6.6) to (6.9) must solve equations (6.2) to (6.5).

For every prefixed \( r \), each equation from (6.6) to (6.9) represents an ordinary Cauchy problem for which any converging classical numerical procedure can be applied. In the next section Euler method is proposed and a complete error analysis guarantees the method’s convergence to the unique solution to Equation (6.1).

6.4 THE EULER METHOD

To integrate the system given in (6.6) to (6.9) in \( [t_0, t_1], [t_1, t_2], ..., [t_k, t_{k+1}], ..., \) replace each interval by a set of \( N_k + 1 \)
discrete equally spaced grid points at which the exact solution \( \{x^+(t; r), x^-(t; r), \bar{x}^+(t; r), \bar{x}^-(t; r)\} \) is approximated by some
\[ \{\left(y^+_k(t; r), \bar{y}^+_k(t; r)\right), \left(y^-_k(t; r), \bar{y}^-_k(t; r)\right)\}. \]
For the chosen grid points on \([t_k, t_{k+1}]\)

\[ t_{k,n} = t_k + nh_k, h_k = \frac{t_{k+1} - t_k}{N_k}, 0 \leq n \leq N_k, \]

Let \[ \left\{\left(\bar{y}^+_k(t; r), \bar{y}^-_k(t; r)\right), \left(y^-_k(t; r), \bar{y}^-_k(t; r)\right)\right\} \equiv \left\{\left(x^+(t; r), x^+(t; r)\right), \left(x^-(t; r), x^-(t; r)\right)\right\}. \]

\[ \{\left(y^+_k(t; r), \bar{y}^+_k(t; r)\right), \left(y^-_k(t; r), \bar{y}^-_k(t; r)\right)\} \]
and
\[ \{\left(y^+_k(t; r), \bar{y}^+_k(t; r)\right), \left(y^-_k(t; r), \bar{y}^-_k(t; r)\right)\} \]
may be denoted respectively by \[ \left\{\left(y^+_k(t; r), \bar{y}^+_k(t; r)\right), \left(y^-_k(t; r), \bar{y}^-_k(t; r)\right)\right\} \]
and \[ \left\{\left(y^+_k(t; r), \bar{y}^+_k(t; r)\right), \left(y^-_k(t; r), \bar{y}^-_k(t; r)\right)\right\}. \] The \(N_k\)'s are
allowed to vary over the \([t_k, t_{k+1}]\)’s so that the \(h_k\)'s may be comparable. The Euler method is the first order approximation of
\( Y^+_k(t; r), \bar{Y}^+_k(t; r), Y^-_k(t; r), \bar{Y}^-_k(t; r) \)
which can be written as

\[ Y^+_{k,n+1}(r) \approx Y^+_{k,n}(r) + h_k F_k \left(t_n, Y^+_{k,n}(r), \bar{Y}^+_{k,n}(r)\right) \quad (6.10) \]

\[ \bar{Y}^+_{k,n+1}(r) \approx \bar{Y}^+_{k,n}(r) + h_k G_k \left(t_n, Y^+_{k,n}(r), \bar{Y}^+_{k,n}(r)\right) \quad (6.11) \]

\[ Y^-_{k,n+1}(r) \approx Y^-_{k,n}(r) + h_k S_k \left(t_n, Y^-_{k,n}(r), \bar{Y}^-_{k,n}(r)\right) \quad (6.12) \]

\[ \bar{Y}^-_{k,n+1}(r) \approx \bar{Y}^-_{k,n}(r) + h_k T_k \left(t_n, Y^-_{k,n}(r), \bar{Y}^-_{k,n}(r)\right) \quad (6.13) \]
Following Equations (6.10) to (6.13) we define,

\begin{align}
\underline{y}_{k,n+1}^+(r) &= \underline{y}_{k,n}^+(r) + h_k F_k(t_n, \underline{y}_{k,n}^+(r), \underline{y}_{k,n}^-(r)) \\
\underline{y}_{k,n+1}^-(r) &= \underline{y}_{k,n}^-(r) + h_k G_k(t_n, \underline{y}_{k,n}^+(r), \underline{y}_{k,n}^-(r)) \\
\overline{y}_{k,n+1}^-(r) &= \overline{y}_{k,n}^-(r) + h_k S_k(t_n, \overline{y}_{k,n}^-(r), \overline{y}_{k,n}^-(r)) \\
\overline{y}_{k,n+1}^-(r) &= \overline{y}_{k,n}^-(r) + h_k T_k(t_n, \overline{y}_{k,n}^-(r), \overline{y}_{k,n}^-(r))
\end{align}

(6.14)  
(6.15)  
(6.16)  
(6.17)

However in (6.14)-(6.17) will use \( \underline{y}_{0,0}^+(r) = \underline{x}_0^+(r), \underline{y}_{0,0}^-(r) = \underline{x}_0^-(r), \overline{y}_{0,0}^+(r) = \overline{x}_0^+(r), \overline{y}_{0,0}^-(r) = \overline{x}_0^-(r) \) and \( \underline{y}_{k,0}^+(r) = \underline{y}_{k-1,Nk-1}^-(r), \overline{y}_{k,0}^+(r) = \overline{y}_{k-1,Nk-1}^- (r), \underline{y}_{k,0}^-(r) = \underline{y}_{k-1,Nk-1}^-(r), \overline{y}_{k,0}^-(r) = \overline{y}_{k-1,Nk-1}^-(r) \) if \( k \geq 1 \).

Then (6.14)-(6.17) represents an approximation of \( Y_k^+(t; r), \overline{Y}_k(t; r), \underline{Y}_k(t; r), \) and \( \overline{Y}_k(t; r) \) for each of the intervals \( t_0 \leq t \leq t_1, t_1 \leq t \leq t_2, ..., t_k \leq t \leq t_{k+1}, ... \)

6.5 CONVERGENCE AND STABILITY:

For a prefixed \( k \) and \( r \in [0,1] \), proof of convergence of the approximations in (6.14)-(6.17) is

\[ \lim_{h_0, ..., h_k \to 0} \underline{y}_{k,Nk}^-(r) = \underline{x}^-(t_{k+1}; r), \quad \lim_{h_0, ..., h_k \to 0} \overline{y}_{k,Nk}^+(r) = \overline{x}^+(t_{k+1}; r) \]

\[ \lim_{h_0, ..., h_k \to 0} \underline{y}_{k,Nk}^-(r) = \underline{x}^-(t_{k+1}; r), \quad \lim_{h_0, ..., h_k \to 0} \overline{y}_{k,Nk}^-(r) = \overline{x}^-(t_{k+1}; r), \]
Lemma 6.5.1

Suppose \( i \in \mathbb{Z}^+, \varepsilon_i > 0, r \in [0,1] \) and \( \delta_i < 1 \) are fixed. Let \( \{Z_{i,n}(r)\}_{n=0}^{N_i} \) be the Euler approximation with \( N = N_i \) to the intuitionistic fuzzy IVP:

\[
\begin{align*}
(x'(t) &= f(t, x(t), \lambda_i(x_i)), \quad t \in [t_i, t_{i+1}], \\
(x(t_i) &= x_i
\end{align*}
\]

If \( \{y_{i,n}(r)\}_{n=0}^{N_i} \) denotes the result (6.14)-(6.17) from some \( y_{i,0}(r) \), then there exists a \( \delta_i > 0 \) such that

\[
\begin{align*}
|z_{i,0}^+(r) - y_{i,0}^+(r)| < \delta_i, \quad |\bar{z}_{i,0}^+(r) - \bar{y}_{i,0}^+(r)| < \delta_i \\
|z_{i,0}^-(r) - y_{i,0}^-(r)| < \delta_i, \quad |\bar{z}_{i,0}^-(r) - \bar{y}_{i,0}^-(r)| < \delta_i
\end{align*}
\]

Implies

\[
\begin{align*}
|z_{i,N_i}^+(r) - y_{i,N_i}^+(r)| < \varepsilon_i, \quad |\bar{z}_{i,N_i}^+(r) - \bar{y}_{i,N_i}^+(r)| < \varepsilon_i \\
|z_{i,N_i}^-(r) - y_{i,N_i}^-(r)| < \varepsilon_i, \quad |\bar{z}_{i,N_i}^-(r) - \bar{y}_{i,N_i}^-(r)| < \varepsilon_i
\end{align*}
\]

Proof: Similar to the proof of Pederson & Sambandham (2007)

Theorem 6.5.1

Consider the systems (6.2)-(6.5) and (6.14)-(6.17). For a fixed \( k \in \mathbb{Z}^+ \) and \( r \in [0,1] \),

\[
\lim_{h_0, \ldots, h_{k-1}} z_{k,N_k}^+(r) = x^+(t_{k+1}; r)
\]

\[
\lim_{h_0, \ldots, h_{k-1}} \bar{z}_{k,N_k}^+(r) = \bar{x}^+(t_{k+1}; r)
\]
\[
\lim_{h_0, \ldots, h_k \to 0} y_{k, Nk}^{-}(r) = x^{-}(t_{k+1}; r)
\]
\[
\lim_{h_0, \ldots, h_k \to 0} \bar{y}_{k, Nk}^{-}(r) = \bar{x}^{-}(t_{k+1}; r).
\]

**Proof:** Similar to the proof of Pederson & Sambandham (2007).

### 6.6 NUMERICAL ILLUSTRATION

To give a clear overview of our study and to illustrate the above discussed technique the following examples are considered. Java version 1.5 is used for computation and it is represented graphically using MATLAB.

**Example 6.6.1**

Consider the following hybrid intuitionistic fuzzy IVP,

\[
\begin{align*}
x'(t) &= x(t) + m(t)\lambda_k(x(t_k)), t \in [t_k, t_{k+1}], \quad t_k = k, k = 0, 1, 2, \ldots, \\
x(0; r) &= \begin{cases} (0.75 + 0.25r, 1.125 - 0.125r), \\ (0.70 + 0.25(1 - r), 1.075 - 0.125(1 - r)) \end{cases}, \quad 0 \leq r \leq 1, \\
\end{align*}
\]

where \( m(t) = \begin{cases} 2(t mod 1), & \text{if } t \text{ mod } 1 \leq 0.5 \\ 2(1 - t \text{ mod } 1), & \text{if } t \text{ mod } 1 > 0.5, \end{cases} \)

\[
\lambda_k(\mu) = \begin{cases} 0, & \text{if } k = 0 \\ \mu, & \text{if } k \epsilon \{1, 2, \ldots\} \end{cases}
\]

This is equivalent to

\[
\begin{align*}
\begin{cases}
\bar{x}^+(t; r) &= \bar{x}^+(t; r) + m(t)\bar{x}^+(1, r) \\
x^+(0; r) &= 0.75 + 0.25r, \quad 0 \leq r \leq 1
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\bar{x}^+(t; r) &= \bar{x}^+(t; r) + m(t)\bar{x}^+(1, r) \\
x^+(0; r) &= 1.125 - 0.125r, \quad 0 \leq r \leq 1
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\{ & x^{-}(t; r) = x^{-}(t; r) + m(t)x^{-}(1, r) \\
& x^{-}(0; r) = 0.70 + 0.25(1 - r), \quad 0 \leq r \leq 1,
\end{align*}
\]

\[
\begin{align*}
\{ & x'^{-}(t; r) = x^{-}(t; r) + m(t)x^{-}(1, r) \\
& x'^{-}(0; r) = 1.075 - 0.125(1 - r), \quad 0 \leq r \leq 1,
\end{align*}
\]

The exact solution of equation (6.18) at \( t = 1 \) is given by

\[ x^{+}(1; r) = [(0.75 + 0.25r)e,(1.125 - 0.125r)e], 0 \leq r \leq 1 \text{ and} \]

\[ x^{-}(1; r) = [(0.70 + 0.25(1 - r))e,(1.150 - 0.125(1 - r))e], 0 \leq r \leq 1 \]

For \( t \in [1, 1.5] \), the exact solution of (6.18) satisfies

\[ x^{+}(t; r) = x^{+}(1, r)(3e^{t-1} - 2t) \]

\[ x^{-}(t; r) = x^{-}(1, r)(3e^{t-1} - 2t) \]

For \( t \in [1.5, 2] \), the exact solution of (6.18) satisfies

\[ x^{+}(t; r) = x^{+}(1, r)[2t - 2 + e^{t-1.5}(3\sqrt{3} - 4)]. \]

\[ x^{-}(t; r) = x^{-}(1, r)[2t - 2 + e^{t-1.5}(3\sqrt{3} - 4)]. \]

The exact and approximate solutions when \( h = 0.1 \), are given in Table 6.1 and Table 6.2.

The exact and approximate solutions when \( h = 0.01 \), are given in Table 6.3 and Table 6.4.
Table 6.1 Comparison of exact and Euler method for membership functions

<table>
<thead>
<tr>
<th>r</th>
<th>Exact</th>
<th>Euler h=0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \bar{Y}^+ (1; r) )</td>
<td>( \bar{Y}^+(1; r) )</td>
</tr>
<tr>
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Table 6.2 Comparison of exact and Euler method for non membership functions

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### Table 6.4 Comparison of exact and Euler method for non membership functions at N=100

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The results of exact solution with approximated solution using Euler method for \( N=10, 100 \) is tabulated in Table 6.1 to Table 6.4.

![Graph](image)

**Figure 6.1 Comparison of exact and Euler method for \( N=100 \)**

Solid Line : Exact, Dotted Line: Euler.

It is evident from Figure 6.1 that an approximated solution can be obtained for HIFDE using Euler method, by increasing the step size the solution better approximates the exact solution.

**Example 6.6.2.** Consider the hybrid intuitionistic fuzzy initial value problem,

\[
\begin{align*}
  x'(t) &= x(t) + m(t)\lambda_k(x(t_k)), \quad t \in [t_k, t_{k+1}], t_k = k, k = 0,1,2, \ldots \nonumber \\
  x(0; r) &= (0.75 + 0.25r, 1.125 - 0.125r), \quad 0 \leq r \leq 1, \nonumber \\
  \lambda_k(\mu) &= \begin{cases} 
    \hat{\mu}, & \text{if } k = 0 \\
    \mu, & \text{if } k \in \{1,2,\ldots\}
  \end{cases} \tag{6.19}
\end{align*}
\]

where \( m(t) = |\sin(\pi t)|, k = 0,1,2, \ldots \).
This is equivalent to

\[
\begin{cases}
  x^+ (t; r) = x^+ (t; r) + m(t)x^+ (1, r) \\
  x^+ (0; r) = 0.75 + 0.25r, \quad 0 \leq r \leq 1,
\end{cases}
\]

\[
\begin{cases}
  x^+ (t; r) = x^+ (t; r) + m(t)x^+ (1, r) \\
  x^+ (0; r) = 1.125 - 0.125r, \quad 0 \leq r \leq 1
\end{cases}
\]

\[
\begin{cases}
  x^- (t; r) = x^- (t; r) + m(t)x^- (1, r) \\
  x^- (0; r) = 0.70 + 0.25(1 - r), \quad 0 \leq r \leq 1
\end{cases}
\]

\[
\begin{cases}
  x^- (t; r) = x^- (t; r) + m(t)x^- (1, r) \\
  x^- (0; r) = 1.075 - 0.125(1 - r), \quad 0 \leq r \leq 1
\end{cases}
\]

The exact solution of equation (6.19) at \( t = 1 \) is given by

\[
x^+ (1; r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e], \quad 0 \leq r \leq 1 \text{ and}
\]

\[
x^- (1; r) = [(0.70 + 0.25(1 - r))e, (1.150 - 0.125(1 - r))e], \quad 0 \leq r \leq 1
\]

For \( t \in [1,2] \), the exact solution of (6.19) satisfies

\[
x^+ (t; r) = x^+ (1, r) \frac{\pi \cos (\pi t) + \sin (\pi t)}{\pi^2 + 1} + \frac{e^t}{\pi} x(1, r) \left( 1 + \frac{\pi}{\pi^2 + 1} \right)
\]

\[
x^- (t; r) = x^- (1, r) \frac{\pi \cos (\pi t) + \sin (\pi t)}{\pi^2 + 1} + \frac{e^t}{\pi} x(1, r) \left( 1 + \frac{\pi}{\pi^2 + 1} \right)
\]

The exact and approximate solutions when \( h = 0.1 \), are given in Table 6.5 and Table 6.6

The exact and approximate solutions when \( h = 0.01 \), are given in Table 6.7 and Table 6.8.
Table 6.5 Comparison of exact and approximated solutions of membership functions with N=10

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<td>$Y^+(1;r)$</td>
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Table 6.6 Comparison of exact and approximated solutions of non membership functions with N=10

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### Table 6.7 Comparison of exact and approximated solutions of membership functions with N=100

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### Table 6.8 Comparison of exact and approximated solutions of non-membership functions with N=100

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The results of exact solution with approximated solution using Euler method for N=10, 100 is tabulated in Table 6.5 to Table 6.8.

**Figure 6.2 Comparison of Exact and Euler method with N=100**

Solid Line : Exact, Dotted Line: Euler.

It is evident from Figure 6.2 that an approximated solution can be obtained for HIFDE using Euler method, by increasing the step size the solution better approximates the exact solution.

### 6.7 LIMITATIONS

- Euler method is used to solve HIFDEs whereas RK & Predictor-Corrector methods gives better solution than Euler method in solving HIFDEs.

- The generalized differentiability can be applied to solve HIFDEs than the H-derivative.
6.8 CONCLUSION

In this chapter, HIFDE is framed in which the original initial value problem is replaced by four parametric ordinary differential equations containing both membership and non membership functions. So far numerical algorithms for HIFDEs are not available so an algorithm is developed using Euler’s method to solve HIFDEs numerically. This theory is illustrated by solving a first order hybrid intuitionistic fuzzy differential equation and it is evident from the numerical examples that the result better approximates the exact solution by increasing the step size.