CHAPTER 5

EXTENDED RUNGE-KUTTA LIKE FORMULA FOR
SOLVING HYBRID FUZZY DIFFERENTIAL EQUATIONS

5.1 INTRODUCTION

A new family of extended Runge-Kutta formula in which, evaluations of both $f$ and $f'$ is involved is introduced to solve HFDEs numerically. New parameters are introduced in the extended Runge-Kutta like formula in order to enhance the order of accuracy of the solutions using evaluations of both $f$ and $f'$ instead of the evaluations of $f$ only. Moreover, if $f'$ is approximated, the order of convergence can be retained. The strongly generalized derivative is defined for a larger class of fuzzy valued function than the H-derivative, and fuzzy differential equations can have solutions which have a decreasing length of their support. So, this differentiability concept is used to solve hybrid fuzzy differential equations.

In this chapter, we apply a numerical algorithm for solving HFDEs

\begin{align*}
(x'(t) &= f(t, x(t), \lambda_k(x_k)), \quad t \in [t_k, t_{k+1}], \\
(x(t_k) &= x_k)
\end{align*}

where $0 \leq t_0 < t_1 < \ldots < t_k \leq \ldots \ldots \ldots \ldots \ldots t_k \to \infty

f \in C[R_+ \times \mathbb{E} \times E, E], \quad \lambda_k \in C[E, E]

based on extended Runge-Kutta like formula of order four.
5.2 EXTENDED RUNGE KUTTA LIKE FORMULA

Consider the autonomous initial value problem \( \frac{dy}{dt} = f(y), a \leq t \leq b, y(a) = \alpha, \)

We assume that \( f(y) \) has derivatives to the desired order in a
domain \( D \) in \( \mathbb{R}^n \) and we assume that

\[ \|f(y_1) - f(y_2)\|_2 \leq L\|y_1 - y_2\|_2 \] holds for all \( y_1, y_2 \in D, \) where \( L \)
is the Lipschitz constant. Then the Runge Kutta like formula is given by

\[ y_{n+1} = y_n + h \sum_{j=1}^{m} b_j k_j^{(1)} + \sum_{j=1}^{m} c_j k_j^{(2)} \]  \hspace{1cm} (5.2)

where \( k_j^{(1)} = f\left(y_n + h \sum_{s=1}^{j-1} a_{js} k_s^{(1)}\right) \) \hspace{1cm} (5.3)

\[ k_j^{(2)} = f'(y_n + h \sum_{s=1}^{j-1} a_{js} k_s^{(1)}), \quad j = 1, 2, \ldots, m \] \hspace{1cm} (5.4)

Obviously, with \( c_j = 0 \) \( (j = 1, 2, \ldots, m) \) in equation (5.2), the
methods reduce to classical Runge-Kutta methods.

\[ y_{n+1} = y_n + h \sum_{j=1}^{m} b_j k_j \]

Where \( k_j = f\left(y_n + h \sum_{s=1}^{j-1} a_{js} k_s\right), \quad j = 1, 2, \ldots, m. \)

It is also noted that, if \( a_{js} = b_{js}, j = 1, 2, \ldots, m, s = 1, 2, \ldots, j - 1 \) in equations (5.2) to (5.4), then

\[ y_{n+1} = y_n + h \sum_{j=1}^{m} b_j k_j^{(1)} + h^2 \sum_{j=1}^{m} c_j k_j^{(2)} \]
where \( k_j^{(1)} = f(y_n + h \sum_{s=1}^{j-1} a_{js}k_s^{(1)}) , \)

\[
k_j^{(2)} = f' \left( y_n + h \sum_{s=1}^{j-1} a_{js}k_s^{(1)} \right), \quad j = 1, 2, \ldots, m
\]

### 5.3 Extended Runge Kutta Like Formula for FIVP

The solutions of equation (5.1) are piecewise differentiable in each interval for \( t \in [t_k, t_{k+1}] \) for a fixed \( x_k \in E \) and \( k = 0, 1, 2 \ldots \). Equation (5.1) can be replaced by an equivalent system when \( x(t) \) is considered as (i) – differentiable fuzzy – valued function:

\[
\begin{align*}
\dot{x}(t) &= f(t, x, \lambda_k(x_k)), \quad x(t_k) = x_k \\
\dot{\lambda}(t) &= \overline{f}(t, x, \lambda_k(x_k)), \quad \lambda(t_k) = \lambda_k \\
\end{align*}
\]

and also (5.1) is equivalent to the following system when \( x(t) \) is considered as (ii) – differentiable fuzzy – valued function:

\[
\begin{align*}
\dot{\overline{x}}(t) &= f(t, x, \lambda_k(x_k)), \quad \overline{x}(t_k) = \overline{x}_k \\
\dot{\underline{x}}(t) &= \underline{f}(t, x, \lambda_k(x_k)), \quad \underline{x}(t_k) = \underline{x}_k \\
\end{align*}
\]

That is for each \( t \), the pair \([x(t; r), \overline{x}(t; r)]\) is a fuzzy number, where \( x(t; r), \overline{x}(t; r) \) are, respectively, the solutions of the parametric form given by:

\[
\begin{align*}
\dot{x}_k(t) &= F_k \left( t, x(t; r), \lambda_k(x_k) \right), \quad x(t_k; r) = \lambda_k(r) \\
\dot{\overline{x}}_k(t) &= \overline{G}_k \left( t, \overline{x}(t; r), \lambda_k(x_k) \right), \quad \overline{x}(t_k; r) = \overline{\lambda}_k(r)
\end{align*}
\]

(5.7)
in the sense of (i) differentiability and

\[
\begin{align*}
    \dot{x}_k(t) &= G_k \left( t, x(t; r), \lambda_k(x_k) \right), \quad x(t_k; r) = x_k(r) \\
    \dot{x}_k(t) &= F_k \left( t, x(t; r), \lambda_k(x_k) \right), \quad \overline{x}(t_k; r) = \overline{x}_k(r)
\end{align*}
\] (5.8)

In the sense of (ii) differentiability and for each \( r \in [0, 1] \)

The Runge-Kutta like formula of order four for the fuzzy initial value problem \( y' = f(t, y(t)), a \leq t \leq b \) and \( y(a) = y_0 \) is as follows:

\[
\begin{align*}
    y_{n+1}(t; r) &= y_n(t; r) + h k_1^{(1)}(y_n(t; r)) + \frac{1}{6} h^2 k_1^{(2)}(y_n(t; r)) \\
                     & \quad + \frac{1}{3} h^2 k_3^{(2)}(y_n(t; r))
\end{align*}
\]

\[
\begin{align*}
    \overline{y}_{n+1}(t; r) &= \overline{y}_n(t; r) + h \overline{k}_1^{(1)}(\overline{y}_n(t; r)) + \frac{1}{6} h^2 \overline{k}_1^{(2)}(\overline{y}_n(t; r)) \\
                       & \quad + \frac{1}{3} h^2 \overline{k}_3^{(2)}(\overline{y}_n(t; r))
\end{align*}
\]

Where \( [k_1^{(1)}(y(t; r))]_r = [k_1^{(1)}(y(t; r)), \overline{k}_1^{(1)}(y(t; r))] \),

\[
\begin{align*}
    [k_1^{(2)}(y(t; r))]_r &= [k_1^{(2)}(y(t; r)), \overline{k}_1^{(2)}(y(t; r))], \\
    [k_3^{(2)}(y(t; r))]_r &= [k_3^{(2)}(y(t; r)), \overline{k}_3^{(2)}(y(t; r))],
\end{align*}
\]

\[
\begin{align*}
    k_1^{(1)}(y(t; r)) &= \min \left\{ f(u) | u \in \left[ y(t; r), \overline{y}(t; r) \right] \right\}, \\
    \overline{k}_1^{(1)}(y(t; r)) &= \max \left\{ f(u) | u \in \left[ y(t; r), \overline{y}(t; r) \right] \right\},
\end{align*}
\]

and

\[
\begin{align*}
    k_1^{(2)}(y(t; r)) &= \min \left\{ f'(u) | u \in \left[ y(t; r), \overline{y}(t; r) \right] \right\},
\end{align*}
\]
$$\bar{k}_1^{(2)}(y(t; r)) = \max\{f'(u)|u \in [y(t; r), \overline{y}(t; r)]\},$$

$$k_2^{(3)}(y(t; r)) = \min\{f'(w)|w \in [z_2(y(t; r)), \overline{z}_2(y(t; r))]\},$$

$$\bar{k}_3^{(2)}(y(t; r)) = \max\{f'(w)|w \in [\overline{z}_2(y(t; r)), \overline{z}_2(y(t; r))]\},$$

where

$$z_2(y(t; r)) = y(t; r) + \frac{1}{2}h\bar{k}_2^{(1)}(y(t; r))$$

$$\overline{z}_2(y(t; r)) = \overline{y}(t; r) + \frac{1}{2}h\bar{k}_2^{(1)}(y(t; r)).$$

so that

$$[k_2^{(1)}(y(t; r))]_r = \left[k_2^{(1)}(y(t; r)), \overline{k}_2^{(1)}(y(t; r))\right],$$

$$k_2^{(1)}(y(t; r)) = \min\{f'(v)|v \in [z_1(y(t; r)), \overline{z}_1(y(t; r))]\},$$

$$\overline{k}_2^{(1)}(y(t; r)) = \max\{f'(v)|v \in [\overline{z}_1(y(t; r)), \overline{z}_1(y(t; r))]\},$$

where

$$z_1(y(t; r)) = y(t; r) + \frac{1}{4}h\bar{k}_1^{(1)}(y(t; r))$$

$$\overline{z}_1(y(t; r)) = \overline{y}(t; r) + \frac{1}{4}h\bar{k}_1^{(1)}(y(t; r)).$$

Define

$$F[y(t_n, r)] = h\bar{k}_1^{(1)}(y(t; r)) + \frac{1}{6}h^2\bar{k}_1^{(2)}(y(t; r)) + \frac{1}{3}h^2\bar{k}_3^{(2)}(y(t; r))$$

$$G[y(t_n, r)] = h\overline{k}_1^{(1)}(y(t; r)) + \frac{1}{6}h^2\bar{k}_1^{(2)}(y(t; r)) + \frac{1}{3}h^2\bar{k}_3^{(2)}(y(t; r))$$

(5.9)

(5.10)

The exact and approximated solutions at $t_n, 0 \leq n \leq N$, are denoted by $[Y(t_n)]_r = [y(t_n, r), \overline{y}(t_n, r)]$ and $[y(t_n)]_r = [y(t_n, r), \overline{y}(t_n, r)],$
respectively. The solution is calculated by the grid points $t_j = j \cdot h, \ j = 0, 1, \ldots, N$ and $h = 1/N$

Such as

$$
\overline{y}_{n+1}(t; r) = \overline{y}_n(t; r) + F[y(t_n, r)]
$$

$$
\overline{\overline{y}}_{n+1}(t; r) = \overline{\overline{y}}_n(t; r) + G[y(t_n, r)].
$$

Also

$$
\underline{y}_{n+1}(t; r) = \underline{y}_n(t; r) + F[y(t_n, r)]
$$

$$
\underline{\overline{y}}_{n+1}(t; r) = \underline{\overline{y}}_n(t; r) + G[y(t_n, r)].
$$

**Lemma 5.3.1**

Let the sequence of numbers $\{W_n\}_{n=0}^N$ satisfy

$$
|W_{n+1}| \leq A|W_n| + B, \ 0 \leq n \leq N - 1,
$$

for some given positive constants A and B.

Then $|W_0| \leq A^n|W_0| + B \frac{A^{n-1}}{A-1}, \ 0 \leq n \leq N$.

**Proof:** Similar to the proof of Ming Ma et.al (1999)

**Lemma 5.3.2**

Let the sequence of numbers $\{W_n\}_{n=0}^N$, $\{V_n\}_{n=0}^N$ satisfy
\[ |W_{n+1}| \leq |W_n| + A \max\{|W_n|, |V_n|\} + B, \]
\[ |V_{n+1}| \leq |V_n| + A \max\{|W_n|, |V_n|\} + B, \]

for some given positive constants A and B, and denote

\[ U_n = |W_n| + |V_n|, \quad 0 \leq n \leq N. \]

Then \[ |U_n| \leq \overline{A}^n |U_0| + \overline{B}^{n-1} \frac{\overline{A}}{\overline{A}-1}, \quad 0 \leq n \leq N, \]

Where \[ \overline{A} = 1 + 2A \] and \[ \overline{B} = 2B. \]

Let \[ F(u, v) = h^2 k^{(1)}_1 (u, v) + \frac{1}{6} h^2 k^{(2)}_k (u, v) + \frac{1}{3} h^2 k^{(2)}_3 (u, v), \]

\[ G(u, v) = h^2 \tilde{k}^{(1)}_1 (u, v) + \frac{1}{6} h^2 \tilde{k}^{(2)}_k (u, v) + \frac{1}{3} h^2 \tilde{k}^{(2)}_3 (u, v). \]

Obtained by substituting \[ [y(t)]_r = [u, v] \] in (5.6) and (5.7)

The domain of F and G is

\[ K = \{(u, v) | -\infty < v < \infty, -\infty < u \leq v\}, \quad 0 \leq t \leq T. \]

**Proof:** Similar to the proof of Ming Ma et.al (1999)

**Theorem 5.3.1**

Let \( F(t, u, v) \) and \( G(t, u, v) \) belong to \( C^4(K) \) and let the partial derivatives of F and G be bounded over K. Then, for arbitrary fixed \( r, 0 \leq r \leq 1 \), the approximate solutions \( \bar{y}(t, r), \tilde{y}(t, r) \) converge to the exact solutions \( \underline{Y}(t, r), \bar{Y}(t, r) \) uniformly in t.

**Proof:** Refer Ghazanfari & Shakerami (2011)
5.4  FOURTH ORDER RUNGE KUTTA LIKE FORMULA

In this section, for a hybrid fuzzy differential equation (5.1) a Runge-Kutta like formula of order four (ERK4) is developed.

For a fixed $r$, to integrate the system of equations (5.7) and (5.8) in $[t_0, t_1], [t_1, t_2], \ldots, [t_k, t_{k+1}], \ldots$, each interval is replaced by a set of $N_k + 1$ discrete equally spaced grid points at which the exact solution

$$ x(t, r) = \left( \chi(t; r), \bar{x}(t; r) \right) $$

is approximated by some

$$( y_k(t; r), \overline{y}_k(t; r) ).$$

For the chosen grid points on $[t_k, t_{k+1}]$

$$ t_{k,n} = t_k + nh_k = \frac{t_{k+1}-t_k}{N_k}, 0 \leq n \leq N_k,$$

let

$$ \left( Y_k(t; r), \overline{Y}_k(t; r) \right) \equiv \left( \chi(t; r), \bar{x}(t; r) \right).$$

$$( Y_k(t; r), \overline{Y}_k(t; r) )$$

and $$( y_k(t; r), \overline{y}_k(t; r) )$$ is denoted respectively by

$$( Y_{k,n}(r), \overline{Y}_{k,n}(r) )$$

and $$( y_{k,n}(r), \overline{y}_{k,n}(r) )$$. The $N_k'$s are allowed to vary over the $[t_k, t_{k+1}]'$s so that the $h_k'$s may be comparable.

The Runge Kutta like formula for equation (5.1) is given by

$$ \overline{y}_{k,n+1}(r) - y_{k,n}(r) = h_k k_1(1) \left( t_{k,n}; y_{k,n}(r) \right) + \frac{1}{6} h_k^2 k_1(2) \left( t_{k,n}; y_{k,n}(r) \right) + \frac{1}{3} h_k^2 k_3(2) \left( t_{k,n}; y_{k,n}(r) \right) $$

$$ \overline{y}_{k,n+1}(r) - \overline{y}_{k,n}(r) = h_k \overline{k}_1(1) \left( t_{k,n}; \overline{y}_{k,n}(r) \right) + \frac{1}{6} h_k^2 \overline{k}_1(2) \left( t_{k,n}; \overline{y}_{k,n}(r) \right) + \frac{1}{3} h_k^2 \overline{k}_3(2) \left( t_{k,n}; \overline{y}_{k,n}(r) \right) $$
Where

$$
\begin{align*}
[k_1^{(1)}(t_{k,n}; y_{k,n}(r))] &= [k_1^{(1)}(t_{k,n}; y_{k,n}(r)), \bar{k}_1^{(1)}(t_{k,n}; y_{k,n}(r))], \\
[k_1^{(2)}(t_{k,n}; y_{k,n}(r))] &= [k_1^{(2)}(t_{k,n}; y_{k,n}(r)), \bar{k}_1^{(2)}(t_{k,n}; y_{k,n}(r))], \\
[k_3^{(2)}(t_{k,n}; y_{k,n}(r))] &= [k_3^{(2)}(t_{k,n}; y_{k,n}(r)), \bar{k}_3^{(2)}(t_{k,n}; y_{k,n}(r))],
\end{align*}
$$

$$
k_1^{(1)}(t_{k,n}; y_{k,n}(r))
= \min \left\{ f(t_{k,n}, u, \lambda_k(u_k)) / u \in [y_{k,n}(r), \bar{y}_{k,n}(r)], u_k \in [y_{k,0}(r), \bar{y}_{k,0}(r)] \right\}
$$

$$
\bar{k}_1^{(1)}(t_{k,n}; y_{k,n}(r))
= \max \left\{ f(t_{k,n}, u, \lambda_k(u_k)) / u \in [y_{k,n}(r), \bar{y}_{k,n}(r)], u_k \in [y_{k,0}(r), \bar{y}_{k,0}(r)] \right\}
$$

and

$$
\begin{align*}
[k_1^{(2)}(t_{k,n}; y_{k,n}(r))] &= \min \left\{ f'(t_{k,n}, u, \lambda_k(u_k)) / u \in [y_{k,n}(r), \bar{y}_{k,n}(r)], u_k \in [y_{k,0}(r), \bar{y}_{k,0}(r)] \right\}, \\
\bar{k}_1^{(2)}(t_{k,n}; y_{k,n}(r))
&= \max \left\{ f'(t_{k,n}, u, \lambda_k(u_k)) / u \in [y_{k,n}(r), \bar{y}_{k,n}(r)], u_k \in [y_{k,0}(r), \bar{y}_{k,0}(r)] \right\}
\end{align*}
$$
\[
\begin{align*}
\overline{k_3}^{(2)}(t_{k,n}; y_{k,n}(r)) & = \min \left\{ f'(t_{k,n} + \frac{h_k}{2}, u, \lambda_k(u_k))/u, \right. \\
& \left. \in \left[ z_{k2}(t_{k,n}; y_{k,n}(r)), \overline{z}_{k2}(t_{k,n}; y_{k,n}(r)) \right], u_k \right\} \\
\overline{k_3}^{(2)}(t_{k,n}; y_{k,n}(r)) & = \max \left\{ f'(t_{k,n} + \frac{h_k}{2}, u, \lambda_k(u_k))/u, \right. \\
& \left. \in \left[ z_{k2}(t_{k,n}; y_{k,n}(r)), \overline{z}_{k2}(t_{k,n}; y_{k,n}(r)) \right], u_k \right\} \\
\text{where} \quad z_{k2}(t_{k,n}; y_{k,n}(r)) & = \overline{y}_{k,n}(r) + \frac{1}{2} h_k \overline{k_2}^{(1)}(t_{k,n}; y_{k,n}(r)) \\
\overline{z}_{k2}(t_{k,n}; y_{k,n}(r)) & = \overline{y}_{k,n}(r) + \frac{1}{2} h_k \overline{k_2}^{(1)}(t_{k,n}; y_{k,n}(r)) \\
\text{So that} \\
\left[ k_2^{(1)}(t_{k,n}; y_{k,n}(r)) \right] = \left[ k_2^{(1)}(t_{k,n}; y_{k,n}(r)), \overline{k_2}^{(1)}(t_{k,n}; y_{k,n}(r)) \right] \\
k_2^{(1)}(t_{k,n}; y_{k,n}(r)) & = \min \left\{ f(t_{k,n} + \frac{h_k}{4}, u, \lambda_k(u_k))/u, \right. \\
& \left. \in \left[ z_{k1}(t_{k,n}; y_{k,n}(r)), \overline{z}_{k1}(t_{k,n}; y_{k,n}(r)) \right], u_k \right\} \\
& \left. \in \left[ \overline{y}_{k,0}(r), \overline{\overline{y}}_{k,0}(r) \right] \right\}
\end{align*}
\]
\( \bar{k}_2^{(1)}(t_{k,n}; y_{k,n}(r)) \)

\[ = \max \left\{ f\left(t_{k,n} + \frac{h_k}{4}, u, \lambda_k(u_k)/u \right), \bar{z}_{k1}\left(t_{k,n}; y_{k,n}(r)\right), \bar{z}_{k1}\left(t_{k,n}; y_{k,n}(r)\right), u_k \right\} \]

where

\[ \bar{z}_{k1}\left(t_{k,n}; y_{k,n}(r)\right) = \bar{y}_{k,n}(r) + \frac{1}{4} h_k \bar{k}_1^{(1)}
\]

\[ \bar{z}_{k1}\left(t_{k,n}; y_{k,n}(r)\right) = \bar{y}_{k,n}(r) + \frac{1}{4} h_k \bar{k}_1^{(1)} \]

Define

\[ S_k\left[t_{k,n}, y_{k,n}(r), \bar{y}_{k,n}(r)\right] \]

\[ = h_k \bar{k}_1^{(1)}(t_{k,n}, y_{k,n}(r)) + \frac{1}{6} h_k \bar{k}_1^{(2)}(t_{k,n}, y_{k,n}(r)) + \frac{1}{3} h_k \bar{k}_3^{(2)}(t_{k,n}, y_{k,n}(r)) \]

\[ T_k\left[t_{k,n}, y_{k,n}(r), \bar{y}_{k,n}(r)\right] \]

\[ = h_k \bar{k}_1^{(1)}(t_{k,n}, y_{k,n}(r)) + \frac{1}{6} h_k \bar{k}_1^{(2)}(t_{k,n}, y_{k,n}(r)) + \frac{1}{3} h_k \bar{k}_3^{(2)}(t_{k,n}, y_{k,n}(r)) \]

Then the exact solution at \( t_{k,n+1} \) for the case of (i) – differentiability is given by

\[ \begin{cases} Y_{k,n+1}(r) \approx Y_{k,n}(r) + S_k[t_{k,n}, y_{k,n}(r), \bar{y}_{k,n}(r)], \\ \bar{Y}_{k,n+1}(r) \approx \bar{Y}_{k,n}(r) + T_k[t_{k,n}, y_{k,n}(r), \bar{y}_{k,n}(r)] \end{cases} \]  \hspace{1cm} \text{(5.11)}

and for the case of (ii) – differentiability is given by

\[ \begin{cases} Y_{k,n+1}(r) \approx Y_{k,n}(r) + T_k[t_{k,n}, y_{k,n}(r), \bar{y}_{k,n}(r)], \\ \bar{Y}_{k,n+1}(r) \approx \bar{Y}_{k,n}(r) + S_k[t_{k,n}, y_{k,n}(r), \bar{y}_{k,n}(r)] \end{cases} \]  \hspace{1cm} \text{(5.12)}
The approximated solution based on (5.11) and (5.12) is given by

\[
\begin{align*}
\begin{cases}
  y_{k,n+1}(r) = y_{k,n}(r) + S_k [t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r)], \\
  \overline{y}_{k,n+1}(r) = \overline{y}_{k,n}(r) + T_k [t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r)]
\end{cases}
\end{align*}
\]  

(5.13)

And

\[
\begin{align*}
\begin{cases}
  y_{k,n+1}(r) = y_{k,n}(r) + T_k [t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r)], \\
  \overline{y}_{k,n+1}(r) = \overline{y}_{k,n}(r) + S_k [t_{k,n}, y_{k,n}(r), \overline{y}_{k,n}(r)]
\end{cases}
\end{align*}
\]  

(5.14)

5.5 CONVERGENCE AND STABILITY

The following result proves that convergence is point wise in \( r \) for a fixed \( k \).

**Lemma 5.5.1**

Suppose \( i \in \mathbb{Z}^+, \varepsilon_i > 0, r \in [0,1] \) and \( h_i < 1 \) are fixed. Let \( \{Z_{i,n}(r)\}_{n=0}^{N_i} \) be the Runge-Kutta like formula approximation with \( N = N_i \) to the fuzzy IVP:

\[
\begin{align*}
\begin{cases}
  x'(t) = f(t, x(t), \lambda_i(x_i)), & t \in [t_i, t_{i+1}], \\
  x(t_i) = x_i
\end{cases}
\end{align*}
\]

If \( \{y_{i,n}(r)\}_{n=0}^{N_i} \) denotes the result (5.13), or (5.14) from some \( y_{i,0}(r) \), then there exists a \( \delta_i > 0 \) such that

\[
\left| Z_{i,0}(r) - y_{i,0}(r) \right| < \delta_i, \left| \overline{Z}_{i,0}(r) - \overline{y}_{i,0}(r) \right| < \delta_i
\]

Implies

\[
\left| Z_{i,N_i}(r) - y_{i,N_i}(r) \right| < \varepsilon_i, \left| \overline{Z}_{i,N_i}(r) - \overline{y}_{i,N_i}(r) \right| < \varepsilon_i
\]

**Proof:** Similar to the proof of Pederson & Sambandham (2007)
**Theorem 5.5.1**

Consider the systems (5.5), and (5.13) or system (5.6) and (5.14). For a fixed \( k \in \mathbb{Z}^+ \) and \( r \in [0,1] \),

\[
\lim_{\eta_0, \ldots, \eta_{k-1} \to 0} y_{k,N_k}(r) = \bar{x}(t_{k+1}; r),
\]

\[
\lim_{\eta_0, \ldots, \eta_{k-1} \to 0} \bar{y}_{k,N_k}(r) = \bar{x}(t_{k+1}; r).
\]

**Proof:** Similar to the proof of Pederson & Sambandham (2007)

**Theorem 5.5.2:**

Consider the hybrid fuzzy differential equations in equation (5.1). For \( k = 0,1,2, \ldots \) and for each \( f_k: [t_k, t_{k+1}] \times E \to E \) is such that

(i) \( f_k(t, x, r) = \)
\[
\left[ F_k \left(t, \underline{x}(t; r), \bar{x}(t; r), \lambda_k(x_k) \right), G_k \left(t, \underline{x}(t; r), \bar{x}(t; r), \lambda_k(x_k) \right) \right]
\]

(ii) \( F_k \left(t, \underline{x}(t; r), \bar{x}(t; r), \lambda_k(x_k) \right) \) and \( G_k \left(t, \underline{x}(t; r), \bar{x}(t; r), \lambda_k(x_k) \right) \) are equicontinuous (that is, for \( \varepsilon > 0 \) there is a \( \delta_k(\varepsilon) \) such that \( |F_k(t, x, y) - G_k(t, x, y)| < \varepsilon \) for all \( r \in [0,1] \), whenever \( (t, x, y), (t_1, x_1, y_1) \in [t_k, t_{k+1}] \times \mathbb{R}^2 \) and \( |(t, x, y) - (t_1, x_1, y_1)| < \delta_k(\varepsilon) \) and uniformly bounded on any bounded set.

(iii) There exists \( l_k > 0 \) such that
\[
|F_k(t, x_2, y_2) - G_k(t, x_1, y_1)| < l_k \max\{|x_2 - x_1|, |y_2 - y_1|\}
\]

Then equation (5.1) and the hybrid system of ODEs
\[
\begin{align*}
\begin{cases}
x_k'(t; r) &= f_k \left( t, \underline{x}(t; r), \underline{x}(t; r), \lambda_k(x_k) \right) \\
\overline{x}_k'(t; r) &= G_k \left( t, \underline{x}(t; r), \underline{x}(t; r), \lambda_k(x_k) \right) \\
x_k(t_k; r) &= x_{k-1}(t_k; r) \text{ if } k > 0, \quad x_0(t_0; r) = x_0(r) \\
\overline{x}_k(t_k; r) &= \overline{x}_{k-1}(t_k; r) \text{ if } k > 0, \quad \overline{x}_0(t_0; r) = \overline{x}_0(r)
\end{cases}
\end{align*}
\]
are equivalent when \( x(t) \) is (i) – differentiable and equation (5.1) and the hybrid system of ODEs

\[
\begin{align*}
\begin{cases}
\overline{x}_k'(t; r) &= f_k \left( t, \underline{x}(t; r), \underline{x}(t; r), \lambda_k(x_k) \right) \\
x_k'(t; r) &= G_k \left( t, \underline{x}(t; r), \underline{x}(t; r), \lambda_k(x_k) \right) \\
x_k(t_k; r) &= x_{k-1}(t_k; r) \text{ if } k > 0, \quad x_0(t_0; r) = x_0(r) \\
\overline{x}_k(t_k; r) &= \overline{x}_{k-1}(t_k; r) \text{ if } k > 0, \quad \overline{x}_0(t_0; r) = \overline{x}_0(r)
\end{cases}
\end{align*}
\]
are equivalent when \( x(t) \) is (ii) – differentiable.

**Proof:** It is completely similar to the proof of Pederson & Sambandham (2009)

### 5.6 NUMERICAL ILLUSTRATION

To give a clear overview of our study and to illustrate the above discussed technique the following examples are considered. Manual Computation is very tedious hence Java version 1.5 is used to solve the examples. The obtained data are imported into MATLAB and represented graphically.

**Example 5.6.1**

Consider the following hybrid fuzzy IVP,

\[
\begin{align*}
\begin{cases}
x'(t) &= x(t) + m(t)\lambda_k(x(t_k)), \quad t \in [t_k, t_{k+1}], \\
x(0, r) &= [0.75 + 0.25r, 1.125 - 0.125r], \quad 0 \leq r \leq 1.
\end{cases}
\end{align*}
\] (5.15)
Where \( m(t) = \begin{cases} 2(t \mod 1), & \text{if } t \mod 1 \leq 0.5 \\ 2(1 - t \mod 1), & \text{if } t \mod 1 > 0.5 \end{cases} \) and

\[
\lambda_k(\mu) = \begin{cases} \hat{0}, & \text{if } k = 0 \\ \mu, & \text{if } k \in \{1, 2, \ldots \} \end{cases}
\]

The HFDE of equation (5.15) is equivalent to the following system of FDEs:

\[
\begin{cases}
x'_0(t) = x_0(t), & t \in [0, 1], \\
x_0(0; r) = [0.75 + 0.25r, 1.125 - 0.125r], & 0 \leq r \leq 1, \\
x'_i(t) = x_i(t) + m(t)x_i(t_i), & t \in [t_i, t_{i+1}], x_i(t_i) = x_{i-1}(t_i), i = 1, 2, 3 \ldots
\end{cases}
\]

In equation (5.15), \( y(t) + m(t)\lambda_k(x(t_k)) \) is a continuous function of \( t, y, \) and \( \lambda_k(x(t_k)) \).

For each \( k = 0, 1, 2, \ldots \), the fuzzy IVP

\[
\begin{cases}
y'(t) = y(t) + m(t)\lambda_k(y(t_k)), & t \in [t_k, t_{k+1}], \quad t_k = k \\
y(t_k) = y_{tk},
\end{cases}
\]

has a unique solution on \([t_k, t_{k+1}]\).

The exact solution of (5.15) for \( t \in [0, 1] \) in the sense of (i) differentiability is

\[
x(t; r) = [(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t]
\]

For \( t \in [1, 1.5] \), the exact solution of (5.15) satisfies

\[
x(t; r) = x(1, r)(3e^{t-1} - 2t)
\]

For \( t \in [1.5, 2] \), the exact solution of (5.15) satisfies

\[
x(t; r) = x(1, r)[2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)] \quad \text{in the sense of (i) differentiability.}
\]
Table 5.1 Comparison of exact and ERK4 at lower end points with different h

<table>
<thead>
<tr>
<th>r</th>
<th>Exact $y_{k,n+1}$</th>
<th>h=0.1 $y_{k,n+1}$</th>
<th>h=0.01 $y_{k,n+1}$</th>
<th>h=0.001 $y_{k,n+1}$</th>
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</table>

Table 5.2 Comparison of exact and ERK4 solutions at upper end points with different h

<table>
<thead>
<tr>
<th>r</th>
<th>Exact $\overline{y}_{k,n+1}$</th>
<th>h=0.1 $\overline{y}_{k,n+1}$</th>
<th>h=0.01 $\overline{y}_{k,n+1}$</th>
<th>h=0.001 $\overline{y}_{k,n+1}$</th>
</tr>
</thead>
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<tr>
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</table>
The results of exact solution with approximated solution ERK4 for N=10, 100, 1000 is tabulated in Table 5.1 and Table 5.2.

![Diagram comparing exact solution and ERK4 solution](image)

**Figure 5.1 Comparison of ERK4 with exact solution**

The solution \( y(2, r) \) is compared at different \( r \) values with exact, Euler and ERK4 at N=10. It is evident from the Figure 5.1 that ERK4 better approximates with the exact solution.

**Example 5.6.2**

Consider the following hybrid fuzzy IVP,

\[
\begin{aligned}
\dot{x}(t) &= x(t) + m(t)\lambda_k(x(t_k)),  t \in [t_k, t_{k+1}], t_k = k, k = 0, 1, 2, \ldots \\
x(0, r) &= [0.75 + 0.25r, 1.125 - 0.125r], 0 \leq r \leq 1. 
\end{aligned}
\]  

(5.16)

where \( m(t) = |\sin(\pi t)|, k = 0, 1, 2, \ldots \) and
\[ \lambda_k(\mu) = \begin{cases} \hat{0}, & \text{if } k = 0 \\ \mu, & \text{if } k \in \{1, 2, \ldots \} \end{cases} \]

The hybrid FDE (5.16) is equivalent to the following system of FDEs:

\[
\begin{cases}
x_0'(t) = x_0(t), t \in [0, 1], \\
x_0(0; r) = [0.75 + 0.25r, 1.125 - 0.125r], 0 \leq r \leq 1, \\
x_i'(t) = x_i(t) + m(t) x_i(t_i), t \in [t_i, t_{i+1}], x_i(t_i) = x_{i-1}(t_i), i = 1, 2, 3, \ldots
\end{cases}
\]

In (5.16), \( y(t) + m(t) \lambda_k(x(t_k)) \) is a continuous function of \( t, y, \) and \( \lambda_k(x(t_k)) \).

For each \( k = 0, 1, 2, \ldots \), the fuzzy IVP

\[
\begin{cases}
y'(t) = y(t) + m(t) \lambda_k(y(t_k)), t \in [t_k, t_{k+1}], t_k = k \\
y(t_k) = y_{tk}
\end{cases}
\]

has a unique solution on \([t_k, t_{k+1}]\).

The exact solution of (5.16) for \( t \in [0, 1] \) in the sense of (i) differentiability is

\[
x(t; r) = [(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t]
\]

For \( t \in [1, 2] \), the exact solution of (5.16) satisfies

\[
x(t; r) = x(1, r) \frac{\pi \cos(\pi t) + \sin(\pi t)}{\pi^2 + 1} + \frac{e^t}{e} x(1, r) \left( 1 + \frac{\pi}{\pi^2 + 1} \right) \quad \text{in the sense of (i) differentiability.}
\]
### Table 5.3 Comparison of exact and ERK4 solutions at lower end points with different N

<table>
<thead>
<tr>
<th>r</th>
<th>Exact $Y_{k,n+1}$</th>
<th>h=0.1 $y_{k,n+1}$</th>
<th>h=0.01 $y_{k,n+1}$</th>
<th>h=0.001 $y_{k,n+1}$</th>
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<td>10.60476037</td>
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### Table 5.4 Comparison of exact and ERK4 solutions at upper end points with different N

<table>
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<tr>
<th>r</th>
<th>Exact $\overline{Y}_{k,N}$</th>
<th>h=0.1 $\overline{y}_{k,N}$</th>
<th>h=0.01 $\overline{y}_{k,N}$</th>
<th>h=0.001 $\overline{y}_{k,N}$</th>
</tr>
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<tbody>
<tr>
<td>1.0</td>
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<td>10.60476037</td>
<td>10.34281253</td>
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</table>
The results of exact solution with approximated solution ERK4 for N=10, 100, 1000 is tabulated in Table 5.3 and Table 5.4.

![Graph](image)

**Figure 5.2 Comparison of ERK4 with exact at different N values**

The solution $y(2, r)$ is compared at different r values with exact, Euler and ERK4 at N=10, 100, 1000. It is evident from the figure that ERK4 gives better solution for this example at N=100, 1000 than N=10.

**Example 5.6.3**

Consider the following hybrid fuzzy IVP,

$$\begin{align*}
    \left\{ x'(t) &= -x(t) + m(t)\lambda_k(x(t_k)), \quad t \in [t_k, t_{k+1}], \\
    x(0, r) &= [0.75 + 0.25r, 1.125 - 0.125r], \quad 0 \leq r \leq 1.
\end{align*} \tag{5.15}$$

where $m(t) = \begin{cases} 
    2(t \mod 1), & \text{if } t \mod 1 \leq 0.5 \\
    2(1 - t \mod 1), & \text{if } t \mod 1 > 0.5,
\end{cases}$
\[
\lambda_k(\mu) = \begin{cases} 
0, & \text{if } k = 0 \\
\mu, & \text{if } k \in \{1,2, \ldots \}
\end{cases}
\]

In this example (ii) differentiability is applicable and hence HFDE in equation (5.17) is solved under (ii) differentiability.

The HFDE (5.17) is equivalent to the following system of FDEs:

\[
\begin{cases}
x'_0(t) = -x_0(t), t \in [0,1], \\
x_0(0; r) = [0.75 + 0.25r, 1.125 - 0.125r], 0 \leq r \leq 1, \\
x'_i(t) = -x_i(t) + m(t)x_i(t_i), t \in [t_i, t_{i+1}], x_i(t_i) = x_{i-1}(t_i), i = 1,2,3 \ldots
\end{cases}
\]

In (5.17), \(-y(t) + m(t)\lambda_k(x(t_k))\) is a continuous function of \(t, y,\) and \(\lambda_k(x(t_k))\). For each \(k = 0,1,2 \ldots\) the fuzzy IVP

\[
\begin{cases}
y'(t) = -y(t) + m(t)\lambda_k(y(t_k)), t \in [t_k, t_{k+1}], t_k = k \\
y(t_k) = y_{tk},
\end{cases}
\]

has a unique solution on \([t_k, t_{k+1}]\).

The exact solution of (5.17) for \(t \in [0,1]\) in the sense of (ii) differentiability we get

\[
x(t; r) = [(0.75 + 0.25r)e^{-t}, (1.125 - 0.125r)e^{-t}]
\]

For \(t \in [1,1.5]\), the exact solution of (5.17) satisfies

\[
x(t; r) = x(1; r)(3e^{1-t} - 2t)
\]

and for \(t \in [1.5,2]\), the exact solution of (5.17) satisfies

\[
x(t; r) = x(1; r)[2t - 2 + e^{1.5-t}\left(\frac{3}{\sqrt{e}} - 4\right)] \text{ in the sense of (ii) differentiability.}
\]
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<th>$\frac{Y_{k,n+1}}{Y_{k,n+1}}$</th>
<th>$\frac{Y_{k,n+1}}{Y_{k,n+1}}$</th>
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Table 5.6 Results of exact and ERK4 solutions at upper end points with different N

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<tr>
<th>r</th>
<th>$Y_{k,n+1}$</th>
<th>$\frac{Y_{k,n+1}}{Y_{k,n+1}}$</th>
<th>$\frac{Y_{k,n+1}}{Y_{k,n+1}}$</th>
<th>$\frac{Y_{k,n+1}}{Y_{k,n+1}}$</th>
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</table>
The results of exact solution with approximated solution ERK4 for N=10, 100, 1000 is tabulated in Table 5.5 and Table 5.6.

![Graph comparing ERK4 with exact solution](image)

**Figure 5.3 Comparison of ERK4 with exact solution at different N**

The solution \( y(2, r) \) is compared at different \( r \) values with exact, Euler and ERK4 at \( N=10, 100 \). It is evident from the figure that ERK4 better approximates the exact solution under generalized differentiability.

### 5.7 CONCLUSION

A family of extended Runge Kutta like formula is applied in this chapter. The formula exploit the use of \( f' \). This procedure consists of using an extended Runge-Kutta like formula method with local truncation error of order four with only three evaluations. A clear advantage to this technique is that only three evaluations of \( f \) are required per step where as arbitrary classical Runge Kutta methods of order three and four used together would require six evaluations of \( f \) per step. In this work, the fuzzy Runge–Kutta-like formula of order four is introduced for approximation solution of a hybrid fuzzy initial value problem using both (i) differentiability and (ii) differentiability and it is illustrated by solving some numerical examples.