1.1 Introduction

Though the concept of distance distills three natural conditions of the usual Euclidean distance in the physical 3-space, this could be applied to much more complex objects. We can talk about distance between curves defined on a common domain \([a,b]\) and about distance between closed and bounded subsets of a metric space and so on. Moreover some very natural properties enjoyed by the Euclidean distance such as “convexity” are ignored in the definition of a metric.

**Definition:** A metric space \((X,d)\) is called Convex if given a positive number \(\varepsilon\) and distinct points \(x,y\) with distance \(d(x,y) = r > \varepsilon\), it is possible to find a point \(z\) such that \(x \neq z \neq y\), \(d(x,y) = d(x,z) + d(z,y)\) and \(d(x,z) = \varepsilon\).

Thus Frechet [6] selected, what might have appeared to him as the three most important properties that a “distance function” must satisfy. Standard generalizations of metric space like topological space or uniform space dispense with the notion of distance altogether and study the resulting ‘fluid’ geometry as opposed to the ‘rigid’ geometry of the metric space. There is yet another way to
generalize the notion of a metric space: by retaining the notion of a metric and yet weakening the axioms imposed on a metric. Such studies have been initiated by Chittinden [8], Fisher [9], Frink[10], Wilson [11] and others and have received a stimulus from the work of computer scientists such as Pascal Hitzler [12], who found that their studies of computer languages and programs need weakened notions of a metric such as partial metric, dislocated metric, quasidislocated metric etc. This chapter is meant to study how weakening the notion of a metric affects the topology that is induced by the weak metric. The process by which a metric generates a topology often needs to be suitably modified if a weakened form of a metric is to yield a topology. Moreover we should not lose sight of the fact that topology is essentially the language best suited for the description and study of continuity. So the general thrust of any attempt at a generalization of the concept of metric should be to examine how the intuitive notion of continuity, applied to that weak metric is reflected in the associated topology. It is in this spirit that the contents of the present chapter are developed.

1.2 Topology of Distance spaces:

**Definition 1.2.1**: Contrary to the usual practice of calling a metric a distance function, we define a distance function $d$ on a nonempty set $X$ as any function $d : X \times X \rightarrow R^+$. 

In what follows, $X$ stands for a nonempty set and $d$ for a distance function on $X$, i.e $d : X \times X \rightarrow R^+$. 

The conditions that $d$ may or may not satisfy are as follows:

$d_1$ : Self distances are zero: $d(x, x) = 0 \ \forall x \in X$. 

\[ d_2 : \text{Distance is symmetric: } d(x, y) = d(y, x) \forall x, y \in X. \]

\[ d_3 : \quad d(x, y) = 0 \Rightarrow x = y. \]

\[ d_4 : \quad d(x, y) > 0 \Rightarrow x \neq y \]

\[ d_5 : \text{Triangle inequality: } d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X \]

\[ d_6 : \text{Ultrametric property: } d(x, y) \leq \max \{d(x, z), d(z, y)\} \quad \forall x, y, z \in X \]

\[ d_7 : \quad d(x, y) \leq \gamma, d(y, z) \leq \gamma \Rightarrow d(x, z) \leq \gamma \quad \forall x, y, z \in X \]

\[ d_8 : \text{Quadrilateral property: If } x, y, u, v \text{ are distinct} \]

\[ d(x, y) \leq d(x, u) + d(u, v) + d(v, y) \]

**Definition 1.2.2:** If \( x \in X \) and \( \varepsilon > 0 \), the set

a) \( B_\varepsilon^r(x) = \{ y \mid y \in X, d(x, y) < \varepsilon \} \) is called the right (open) ball centered on \( x \) with radius \( \varepsilon \).

b) \( V_\varepsilon^r(x) = B_\varepsilon^r(x) \cup \{ x \} \) is called the right inclusive (open) ball centered on \( x \) with radius \( \varepsilon \).

c) \( B_\varepsilon^l(x) = \{ y \mid y \in X, d(y, x) < \varepsilon \} \) is called the left (open) ball centered on \( x \) with radius \( \varepsilon \).

d) \( V_\varepsilon^l(x) = B_\varepsilon^l(x) \cup \{ x \} \) is called the left inclusive (open) ball centered on \( x \) with radius \( \varepsilon \).

e) \( B_\varepsilon(x) = B_\varepsilon^l(x) \cap B_\varepsilon^r(x) \) is called the bilateral (open) ball centered on \( x \) with radius \( \varepsilon \).
f) \( V_\varepsilon (x) = B_\varepsilon (x) \cup \{x\} \) is called the bilateral inclusive (open) ball with centered on \( x \) and radius \( \varepsilon \), and \( x \) is called the centre while \( \varepsilon \) is called the radius of the ball.

Note that radii as well as centers are not necessarily unique.

For example, in \( \mathbb{N} \) with the discrete metric, \( B_3(2) = B_{100}(23) = \mathbb{N} \).

Remarks 1.2.3:

1. If \( d \) is symmetric then \( B^r_\varepsilon (x) = B^l_\varepsilon (x) = B_\varepsilon (x) \) and \( V^r_\varepsilon (x) = V^l_\varepsilon (x) = V_\varepsilon (x) \).

2. If self distances are zero then \( B^r_\varepsilon (x) = V^r_\varepsilon (x) = B^l_\varepsilon (x) = V^l_\varepsilon (x) \) and \( B_\varepsilon (x) = V_\varepsilon (x) \).

Definition 1.2.4: If \( A \subseteq X \), \( x \in A \) is called a right interior point of \( A \), and \( A \) is called a right neighborhood (simply nbd) of \( x \) if there exists \( \varepsilon > 0 \) such that \( V^r_\varepsilon (x) \subseteq A \).

Corresponding left notion is defined similarly.

Proposition 1.2.5: If \( d \) satisfies triangular inequality \((d_5)\) or ultra metric property \((d_6)\) then \( V^r_\varepsilon (x) \) is a right nbd of every point in it.

Proof: Assume \((d_5)\). If \( y \in V^r_\varepsilon (x) \) then \( \delta = \varepsilon - d(x, y) > 0 \). If \( z \in V^r_\delta (y) \) then \( d(y, z) < \delta = \varepsilon - d(x, y) \) so that \( d(x, z) \leq d(x, y) + d(y, z) < \varepsilon \).

Hence \( z \in V^r_\varepsilon (x) \). This implies \( V^r_\delta (y) \subseteq V^r_\varepsilon (x) \).
Since \((d_6) \Rightarrow (d_5)\), the proof of the proposition is complete.

**Remark 1.2.6:** If \(d\) satisfies the left triangular inequality

\[d'_5 : d(x, y) \leq d(z, x) + d(z, y)\] (or)

the “left” ultra metric property

\[d'_6 : d(x, y) \leq \max\{d(z, x) + d(z, y)\} \text{ for } z \in A\] then \(V^l(x)\) is a right nbd of every point in it.

A simple method for constructing distance functions [13]:

**Proposition 1.2.7:** Let ‘\(\rho\)’ be a metric on \(X\), \(A \subseteq X\) and \(f : \mathbb{R}^+ \to \mathbb{R}^+\). Define \(d : A \times A \to \mathbb{R}^+\) by \(d(x, y) = f(\rho(x, y))\) for \(x, y \in A\). Then

a) \(d\) is a symmetric distance function on \(A\).

b) If \(f\) is monotonically increasing and subadditive, then \(d\) satisfies triangle inequality \(d_5\).

c) If \(f^{-1}(0) = \{0\}\) then \(d(x, y) = 0 \iff x = y\).

Proof: (a) is clear.

(b) Let \(x, y, z \in A\), \(\rho(x, y) = a\), \(\rho(y, z) = b\) and \(\rho(z, x) = c\). By the triangle inequality \(c \leq a + b\). Since \(f\) is increasing and subadditive,

\[f(c) \leq f(a + b) \leq f(a) + f(b)\]

Hence \(d(x, z) = f(c) \leq f(a) + f(b) = d(x, y) + d(y, z)\).

(c) \(d(x, y) = 0 \iff f(\rho(x, y)) = 0 \iff \rho(x, y) = 0 \iff x = y\)
Examples:

I. Define $\rho(x, y) = |x - y|$ for $x, y \in \mathbb{R}$ and $f(x) = x^2$ for $x \in \mathbb{R}^+$

$$d(x, y) = (x - y)^2$$

defines a distance function that satisfies

$$d(x, y) = 0 \iff x = y.$$ 

If $x > 0$, $d(x, 0) = d(0, -x) = x^2$ and $d(x, -x) = 4x^2$.

Thus $d(x, -x) \neq d(x, 0) + d(0, -x)$ so that $d$ does not satisfy triangle inequality.

II. Define $\rho(x, y) = |x - y|$ for $x, y \in \mathbb{R}$ and $f(x) = e^x$ for $x \in \mathbb{R}^+$

$$d(x, y) = e^{|x-y|}$$

defines a distance function on $\mathbb{R}$ and $d(x, y) > 0$.

In the following theorem we present a few more methods to create distance functions.

**Proposition 1.2.8**: Let $f, g$ be nonnegative real valued functions on a set $X$.

Define $d(x, y) = f(x) + g(y)$ for $x, y \in X$.

I. $d(x, y) = d(y, x) \forall x, y$ if and only if $(f - g)$ is a constant function.

II. $d$ satisfies triangular inequality.

Proof: (l) $d(x, y) = d(y, x)$

$$\iff f(x) + g(y) = f(y) + g(x)$$

$$\iff f(x) - g(x) = f(y) - g(y)$$

$$\iff (f - g) \text{ is constant function.}$$
Proposition 1.2.9: Let $X \subseteq \mathbb{R}^+$ and $A = X - X = \{a - b \mid a \in X, b \in X\}$ and $d(x, y) = f(x - y)$ where $f : A \to \mathbb{R}^+$.

If $f^{-1}(0) = \{0\}$ and $f$ is even and subadditive then $d$ is a metric.

Proof: $d(x, y) = 0 \iff f(x - y) = 0 \iff x = y$.

$d(x, y) = d(y, x) \iff f(x - y) = f(y - x) \iff f$ is even.

$d(x, y) + d(y, z) = f(x - y) + f(y - z) \geq f(x - y + y - z) = f(x - z) = d(x, z)$.

Proposition 1.2.10: Let $f : X \to \mathbb{R}^+$ and $g : X \to \mathbb{R}^+$ be functions such that

$$\inf\{f(x) / x \in X\} \geq \sup\{g(y) / y \in X\}.$$ 

Define $d(x, y) = f(x) - g(y)$. Then $d$ satisfies the triangle inequality. Further $d$ is symmetric if and only if $f + g$ is constant.

Proof: $d(x, y) \geq 0$ by hypothesis

$d(x, y) = d(y, x) \iff f(x) - g(y) = f(y) - g(x)$

$\iff (f + g)x = (f + g)y$

$\iff f + g$ is constant.

For any $x, y, z$ in $X$,

$d(x, y) + d(y, z) - d(x, z)$
= f(x) - g(y) + f(y) - g(z) - f(x) + g(z)

= f(y) - g(y) ≥ 0 by hypothesis.

**Proposition 1.2.11:** For any \( f : X \rightarrow \mathbb{R}^+ \), \( d(x, y) = |f(x) - f(y)| \) defines a metric if and only if \( f \) is one-one.

Proof: Clearly \( d(x, y) ≥ 0 \) and \( d(x, x) = 0 \)

\[
d(x, y) = 0 \iff |f(x) - f(y)| = 0 \iff f(x) = f(y)
\]

Then if \( f \) is one-one; \( d(x, y) = 0 \Rightarrow x = y \)

Conversely if \( d \) is a metric and \( f(x) = f(y) \) then \( d(x, y) = 0 \), hence \( x = y \).

Since \( |f(x) - f(y)| ≤ |f(x) - f(z)| + |f(z) - f(y)| \), triangle inequality holds.

**Examples 1.2.12:**

**I.** \( d(x, y) = (x - a)^2 + (y - b)^2 \) for \( x, y \in \mathbb{R} \)

Clearly \( d(x, y) ≥ 0 \) for all \( x, y \) and \( d \) satisfies triangle inequality.

\( d \) is symmetric iff \( (x - a)^2 - (x - b)^2 = (y - a)^2 - (y - b)^2 \) \( \forall x, y \)

\( \iff (2x - a - b)(b - a) = (2y - a - b)(b - a) \) \( \forall x, y \)

\( \iff 2(x - y)(b - a) = 0 \) \( \forall x, y \)

\( \iff a = b \)

**II.** Define \( d(x, y) = (x - y)^2 \) for \( x, y \in \mathbb{R} \).

Clearly \( d(x, y) ≥ 0 \), \( d(x, y) = 0 \iff x = y \) and \( d(x, y) = d(y, x) \)

Triangular inequality does not hold since

\( d(-1,2) = 9 > d(-1,1) + d(1,2) \)
We shall now verify that balls are open in $(X,d)$.

$\forall x \in R$ and $\epsilon > 0$, $B_\epsilon(x) = (x - \sqrt{\epsilon}, x + \sqrt{\epsilon})$

so that if $|x - y| < \sqrt{\epsilon}$ and $|y - z| < \sqrt{\epsilon} - |x - y|$ then

$|x - z| \leq |x - y| + |y - z| < \sqrt{\epsilon}$

Hence $d(x, z) < \epsilon$.

Thus the triangular inequality is not necessary for balls to be open.

III. Define $d$ on $R$ by

$$d(x, y) = \begin{cases} 1 & \text{if } x = y \\ 5 & \text{if } 0 < |x - y| < 1 \\ |x - y| & \text{if } |x - y| \geq 1 \end{cases}$$

$d(x, y) = d(y, x)$

d(x, y) > 0$ for all $x, y$ and triangular inequality does not hold since

$d(4, 4.5) = 5, d(4, 5.5) = 1.5$ and $d(5.5, 4.5) = 1$

It is easy to verify that,

$$B_\epsilon(1) = \begin{cases} \emptyset & \text{if } 0 < \epsilon \leq 1 \\ [2, 1+\epsilon) \cup (1-\epsilon, 0] \cup \{1\} & \text{if } 1 < \epsilon \leq 5 \\ (1-\epsilon, 1+\epsilon) & \text{if } \epsilon > 5 \end{cases}$$

$$B_\delta(0) = \begin{cases} \emptyset & \text{if } 0 < \delta \leq 1 \\ (-\delta, -1] \cup [1, \delta) \cup \{0\} & \text{if } 1 < \delta \leq 5 \\ (-\delta, \delta) & \text{if } \delta > 5 \end{cases}$$
When $\varepsilon=1$, $B_4(1)=[2,5) \cup (-3,0] \cup \{1\}$ so that $0 \notin B_4(1)$

However there does not exist $\delta>0 \ni \varphi \neq B_\delta(0) \subseteq B_4(1)$

**Proposition 1.2.13:** Let $(X,d)$ be a distance space. If $d$ satisfies the triangle inequality the set of all right inclusive balls $\{V_\varepsilon^r(x)/x \in X, \varepsilon>0\}$ is a base for a topology on $X$.

**Proof:** Clearly every $x \in X$ belongs to $V_\varepsilon^r(x)$ for every $\varepsilon>0$.

If $x_1,x_2 \in X, \varepsilon_1, \varepsilon_2>0$ and $z \in V_{\varepsilon_1}^r(x_1) \cap V_{\varepsilon_2}^r(x_2)$

$d(x_i,z)<\varepsilon_i$ for $i=1,2$. Let $\varepsilon=\min\{\varepsilon_1-d(x_1,z),\varepsilon_2-d(x_2,z)\}$

If $y \in V_\varepsilon^r(z), d(z,y)<\varepsilon \leq \varepsilon_i - d(x_i,z)$ for $i=1,2$.

$\Rightarrow d(x_i,z)+d(z,y)<\varepsilon_i$ for $i=1,2$

$\Rightarrow d(x_i,y) \leq d(x_i,z)+d(z,y)<\varepsilon_i$ for $i=1,2$

$\Rightarrow y \in V_{\varepsilon_1}^r(x_1) \cap V_{\varepsilon_2}^r(x_2)$

$\Rightarrow V_\varepsilon^r(z) \subseteq V_{\varepsilon_1}^r(x_1) \cap V_{\varepsilon_2}^r(x_2)$

This proves that the set of right inclusive balls is a base for a topology on $X$.

**Definition 1.2.14:** The topology induced by the set of right inclusive balls is called the right topology induced by $d$ on $X$ and is denoted by $\mathcal{T}_r^d$ or simply $\mathcal{T}_r$. 

Remarks:

1. The set of left inclusive balls induces a topology. We call this , the left topology induced by $d$ on $X$ and denote this by $\mathcal{I}^d$ or simply $\mathcal{I}$.

2. The above proof suggests that the triangular inequality is not in fact essential for one sided inclusive balls to induce a topology but it is sufficient that one sided balls are open.

3. If $d$ is symmetric then $V^r_\epsilon(x) = V^l_\epsilon(x)$ and hence $\mathcal{I}_r = \mathcal{I}_l$.

**Theorem 1.2.15:** Let $(X, d)$ be a distance space, $d$ satisfy the triangle inequality. Then the distance function $D$ defined by $D(x, y) = d(x, y) + d(y, x)$ induces the topology $\mathcal{I}_D = \mathcal{I}_r \cap \mathcal{I}_l$.

Proof: Clearly $D$ is symmetric and satisfies the triangle inequality and hence induces a topology $\mathcal{I}_D$. Further if $x \in X$ and $\epsilon > 0$ then

$$V^r_\epsilon(x) \subseteq V^r_\epsilon(x) \cap V^l_\epsilon(x) \subseteq V^l_2\epsilon(x)$$

..........(*)

where $V^r_\epsilon(x)$ is the inclusive ball with centre $x$ w.r.t $D$ while $V^r_\epsilon(x)$ and $V^l_\epsilon(x)$ are respectively the right and left inclusive balls w.r.t. $d$.

It is thus sufficient to show that the collection

$$B = \{V^r_\epsilon(x) \cap V^l_\eta(y) / x, y \in X, \epsilon > 0, \eta > 0\}$$

is a base for $\mathcal{I}_r \cap \mathcal{I}_l$.

Clearly if $x \in X$ and $\epsilon > 0$, then $x \in V^r_\epsilon(x) \cap V^l_\epsilon(x)$.

Now let $x_1, y_1, x_2, y_2 \in X$ and $\epsilon_1, \eta_1, \epsilon_2, \eta_2 > R^+$
\[ U = \{ V^R_{\eta_1}(x_1) \cap V^I_{\eta_1}(y_1) \} \cap \{ V^R_{\eta_2}(x_2) \cap V^I_{\eta_2}(y_2) \} \] and \( z \in U \)

Then \( z \in V^R_{\eta_1}(x_1) \cap V^R_{\eta_2}(x_2) \) and \( z \in V^I_{\eta_1}(y_1) \cap V^I_{\eta_2}(y_2) \)

\[ \Rightarrow \exists \delta_1, \delta_2 > 0 \text{ such that } V^R_{\delta_1}(z) \subseteq V^R_{\eta_1}(x_1) \cap V^R_{\eta_2}(x_2) \] and \( V^I_{\delta_2}(z) \subseteq V^I_{\eta_1}(y_1) \cap V^I_{\eta_2}(y_2) \)

If \( \delta = \min\{\delta_1, \delta_2\} \) then \( \delta > 0 \) and \( z \in V^R_{\delta}(z) \cap V^I_{\delta}(z) \subseteq U \).

Hence \( B \) is a base for \( \mathfrak{F}_r \cap \mathfrak{F}_l \). By (*) \( \mathfrak{F}_d = \mathfrak{F}_r \cap \mathfrak{F}_l \).

Examples 1.2.16:

i. Let \( X = (0, \infty) \) and \( d(x, y) = \begin{cases} x + y & \text{if } x < y \\ x - y & \text{if } x \geq y \end{cases} \)

Let \( x \in X \) and \( \epsilon > 0 \).

If \( x < \frac{\epsilon}{2} \) then \( x + y < \epsilon \Leftrightarrow y \in (x, \epsilon - x) \)

Also if \( 0 < y \leq x \) then \( x - y < \epsilon \Leftrightarrow x - \epsilon < 0 < y \)

Thus if \( 0 < x < \frac{\epsilon}{2} \) then \( V^r(x) = (0, x] \cup (x, \epsilon - x) = (0, \epsilon - x) \) …….. (1)

If \( \frac{\epsilon}{2} \leq x < \epsilon \) then \( \epsilon - x \leq x \) so \( x < y \) then

Further if \( y \leq x \) then \( 0 \leq x - y \leq x < \epsilon \)

Thus if \( \frac{\epsilon}{2} \leq x < \epsilon \) then \( V^r(x) = (0, x] \) …….. (2)

If \( x \geq \epsilon, x + y > \epsilon \) while \( y \leq x, x - y < \epsilon \Leftrightarrow x - \epsilon < y \leq x \)

so that \( V^r(x) = (x - \epsilon, x] \) if \( x \geq \epsilon \)
Hence \( V^r_\varepsilon (x) = \begin{cases} (0, \varepsilon - x) & \text{if } x < \frac{\varepsilon}{2} \\ (0, \varepsilon] & \text{if } \frac{\varepsilon}{2} \leq x < \varepsilon \\ (x - \varepsilon, x] & \text{if } x \geq \varepsilon \end{cases} \)

Clearly \( x \in V^r_\varepsilon (x) \)

If \( 0 < a < x < b \) and \( 0 < \varepsilon < x - a \) then \( a < x - \varepsilon < x < b \)

So \( V^r_\varepsilon (x) = (x - \varepsilon, x] \subseteq (a, b) \)

Hence \((a, b) \in \mathcal{R}, \forall b > a > 0.\)

Also \((a, b) \in \mathcal{R}, \forall b > a > 0\) since

1. if \( a < x < b \) and \( 0 < \varepsilon < x - a \) then \( V^r_\varepsilon (x) \subseteq (a, b) \) while
2. if \( 0 < \varepsilon < b - a, \ a < b - \varepsilon < b \) so \( V^b_\varepsilon = (b - \varepsilon, b] \subseteq (a, b) \)

If \( 0 < y < x < \frac{\varepsilon}{2} \) then

\[ y + x < \varepsilon \Leftrightarrow 0 < y < x < \varepsilon - x \Leftrightarrow y \in (0, x) \text{ and } x < \frac{\varepsilon}{2}.\]

Further if \( 0 < x \leq y \) then \( y - x < \varepsilon \Leftrightarrow x - y < x + \varepsilon \Leftrightarrow y \in [x, x + \varepsilon].\)

Thus if \( 0 < x < \frac{\varepsilon}{2}, d(y, x) \varepsilon \Leftrightarrow y \in (0, x + \varepsilon).\)

If \( \frac{\varepsilon}{2} \leq x < \varepsilon \) then \( y < x < y + \varepsilon \Leftrightarrow 0 < y < \varepsilon - x \leq x.\)

If \( x \leq y, 0 \leq y - x < \varepsilon \Leftrightarrow y < x + \varepsilon.\)
Thus if $\frac{\varepsilon}{2} \leq x < \varepsilon$, $d(y,x) < \varepsilon \Leftrightarrow y \in (0,\varepsilon-x) \cup [x,x+\varepsilon]$.

If $x \geq \varepsilon$ then $x + y > \varepsilon$ when $0 < y < x$.

If $x \leq y$, $y - x < \varepsilon$ when $y < x + \varepsilon$.

Thus if $x \geq \varepsilon$, $d(y,x) < \varepsilon \Leftrightarrow y \in [x,x+\varepsilon)$

$$V_{\varepsilon}^l(x) = \begin{cases} 
(0,x+\varepsilon) & \text{if } 0 < x < \frac{\varepsilon}{2} \\
(0,-x] \cup [x,x+\varepsilon) & \text{if } \frac{\varepsilon}{2} \leq x < \varepsilon \\
[x,x+\varepsilon) & \text{if } x \geq \varepsilon
\end{cases}$$

Hence $V_{\varepsilon}^l(x)$ if $0 < a < b$, $(a,b) \in \mathcal{I}$ since $\forall x \in (a,b)$ and $0 < \varepsilon < \min\{b-x, \varepsilon\}$

$V_{\varepsilon}^l(x) = [x,x+\varepsilon) \subseteq (a,b)$.

However $(a,b) \notin \mathcal{I}$ since $U_{\varepsilon}^l(b) \notin (a,b)$ for any $\varepsilon > 0$.

Thus $\mathcal{I}_r \neq \mathcal{I}_l$.

Further $D(x,y) = d(x,y) + d(y,x) = 2y$ for all $x,y$

and $V_{\varepsilon}^b(x) = \{y / D(x,y) < \varepsilon\} = (0,\varepsilon)$

ii. Define $d(x,y) = x + y$ if $x + y > 0$ and $0$ if $x + y \leq 0$

Clearly $d(x,y) = 0 \Leftrightarrow x \leq -y$.

$d(x,y) = d(y,x)$
If $x > z > 0$ and $y = -x, y + z = -x + z > 0$

$d(x, y) = d(x, -x) = 0, d(y, z) = d(-x, z) = 0$

Thus $d(x, y) = d(y, z) = 0$ but $d(x, z) = x + z > 0$

so triangle inequality does not hold.

For that $B_{\varepsilon}(x) = (-\infty, -x] \cup (-x, \varepsilon - x) = (-\infty, (\varepsilon - x))$

iii. $d(x, y) = |x|$ for $x, y \in R$

Clearly $d(x, y) = d(y, x)$ if and only if $y = \pm x$

Further $d$ satisfies triangle inequality.

$B_{\varepsilon}^{r}(x) = \{ y / d(x, y) < \varepsilon \}$

$= \{ y / |x| < \varepsilon \}$

$= \begin{cases} \emptyset & \text{if } \varepsilon \leq |x| \\ R & \text{if } |x| < \varepsilon \end{cases}$

Thus $V_{\varepsilon}^{r}(x) = \begin{cases} \{ x \} & \text{if } \varepsilon \leq |x| \\ R & \text{if } |x| < \varepsilon \end{cases}$

Hence $\mathfrak{T}_{r} = P(X)$

$B_{\varepsilon}^{l}(x) = \{ y / d(y, x) = y | \varepsilon \} = (-\varepsilon, \varepsilon) \forall x.$

So that $V_{\varepsilon}^{l}(x) = (-\varepsilon, \varepsilon) \cup \{ x \}$.

Thus $A \subset R$ is left open if and only if $\exists \varepsilon > 0 \exists (-\varepsilon, \varepsilon) \subseteq A$

Hence $\mathfrak{T}_{l} = \{ A / \exists \varepsilon > 0 \exists (-\varepsilon, \varepsilon) \subseteq A \} \cup \{ \emptyset \}$

Clearly $\mathfrak{T}_{r} \neq \mathfrak{T}_{l}$. 
1.3 Convergence of sequences:

In a metric space \((X, d)\) sequential convergence w.r.t \(d\) plays an important role. For example

i. Constant sequences are convergent

ii. Limits are unique

iii. Accumulation points of a set are precisely the limits of nonconstant sequences from the set and

iv. Every convergent sequence is a Cauchy sequence.

One natural question is whether these hold good in arbitrary distance spaces.

**Definition 1.3.1:** Let \((X, d)\) be a distance space, \(\{x_n\}\) is a sequence in \(X\) and \(x \in X\). We say that \(\{x_n\}\) right converges to \(x\) if \(\lim d(x_n, x) = 0\) and left converges to \(x\) if \(\lim d(x, x_n) = 0\).

Clearly \(\{x_n\}\) converges to \(x\) w.r.t \(D\) iff \(\{x_n\}\) right as well as left converges to \(x\) w.r.t \(d\).

\[\lim d(x_n, x) = 0 = \lim d(x_n, y) \Rightarrow d(x, y) = 0\]

If \(x \in X\) and \(x_n = x\ \forall n\), \(\lim d(x_n, x) = 0\) if and only if \(d(x, x) = 0\).

The proof of the following theorem is routine and hence is stated without proof.
Theorem 1.3.2: Let \((X,d)\) be a distance space where \(d\) satisfies 
\(d_2, d_4\) and \(d_5\) (or \(d_7\)). Then

i. If \(\lim d(x_n, x) = 0 = \lim d(x_n, y)\) then \(d(x, y) = 0\)

ii. \(A \subseteq X\) and \(x \in X\) then \(x \in \overline{A}\) iff either \(x \in A\) or there exists a sequence \(\{x_n\}\) in \(A\) such that \(\lim d(x_n, x) = 0\)

iii. Every convergent sequence is a Cauchy sequence.

Remark: If we define \(x \theta y\) iff \(d(x, y) = 0\) then \(\theta\) is an equivalence relation and (i),(ii),(iii) hold for the metric space \(X/\theta\) of equivalence classes modulo \(\theta\).

Example: Define \(d(x, y) = |x|\) if \(x, y \in R\).

Clearly \(d\) satisfies the triangle inequality \(d_5\) and is not symmetric further 
\[\lim d(1+\frac{1}{n}, 0) = \lim (1+\frac{1}{n}) = 1 \text{ and } \lim d(0,1+\frac{1}{n}) = 0.\]

1.4 Continuity:

Having defined two topologies on a distance space \((X,d)\), it is time now to study how the notion of continuity could be described in terms of the distances involved. At the very outset, we would like to point out how nonvanishing of self distances causes insurmountable hurdles in this venture. If \((X,d_1)\) and \((Y,d_2)\) are distance spaces with right topologies \(T_{d_1}^r\) and \(T_{d_2}^r\) and 
\(f : X \rightarrow Y\) is a constant mapping with \(f(x) = y_0\) for all \(x \in X\) where \(y_0\) is an element of \(Y\) satisfying \(d_2(y_0, y_0) > 0\) then \(f\) is obviously continuous. But if continuity is interpreted as a property of \(f\) that takes close–by points of \(X\) in to
close-by points in $Y$ then the same $f$ fails to be continuous due to the nonvanishing of $d_2(y_0, y_0)$. Hence nonvanishing of self distances creates a hiatus between topological notions and distance notions. Our usual notion that a metric gives more structure to the space than the topology induced by it, is being challenged here due to the presence of points with $d(x, x) > 0$. To obviate this difficulty, we impose the condition that $d(x, x) = 0 \forall x \in X$ on all distance spaces $(X, d)$ in the rest of this chapter. We feel that the above observations proclaim the wisdom of Frechet[1] in imposing the condition $d(x, x) = 0 \forall x \in X$ in a metric space.

**Definition 1.4.1:** Suppose $f : X \to Y$ where $(X, d_1)$ and $(Y, d_2)$ are distance spaces in which self distances are zero. Suppose $a \in X$. We say that $f$ is right continuous at $a$ if to each $\varepsilon > 0$ there corresponds a $\delta = \delta(\varepsilon) > 0$, such that $d(fx, fa) < \varepsilon$ whenever $d(x, a) < \delta$. We say that $f$ is left continuous at $a$ if to each $\varepsilon > 0$ there corresponds a $\delta = \delta(\varepsilon) > 0$ such that $d(fa, fx) < \varepsilon$ whenever $d(a, x) < \delta$.

**Remark:** When $d(x, x) = 0 \forall x \in X$, $V^r_\varepsilon(x) = B^r_\varepsilon(x)$ and $V^l_\varepsilon(x) = B^l_\varepsilon(x)$ and for all $x \in X$ and all $\varepsilon > 0$.

**Theorem (Local Continuity) 1.4.2:** Let $(X, d_1)$ and $(Y, d_2)$ be distance spaces in which self distances vanish. Let $a \in X$ and $f : X \to Y$. Then the following are equivalent:

(a) $f$ is right continuous at $a$

(b) $x_n \xrightarrow{r} a$ in $(X, d_1)$ implies $f(x_n) \xrightarrow{r} f(a)$ in $(Y, d_2)$. 
Proof: Assume (a). Let $\varepsilon > 0$. Choose $\delta > 0$ such that $d(f(x), f(a)) < \varepsilon$ for all $x$ satisfying $d(x, a) < \delta$. Since $x_n \xrightarrow{r} a$, there exists $n_0$ such that $d(x_n, a) < \delta$ for all $n \geq n_0$. Hence $d(f(x_n), f(a)) < \varepsilon$ if $n \geq n_0$. Hence $f(x_n) \xrightarrow{r} f(a)$.

Assume (b). Suppose (a) is false. Then there exists $\varepsilon > 0$ such that for each positive integer $n$ there exists $x_n$ in $X$ such that $d(x_n, a) < \frac{1}{n}$ and $d(f(x_n), f(a)) \geq \varepsilon$.

Hence $x_n \xrightarrow{r} a$ in $(X, d_1)$ but $f(x_n) \nRightarrow f(a)$ in $(Y, d_2)$.

**Theorem (Global continuity)** 1.4.3: $f : (X, d_1) \to (Y, d_2)$ is right continuous iff $f^{-1}(G)$ is right open in $Y$ whenever $G$ is right open in $Y$.

Proof: Suppose $f$ is right continuous. Let $G$ be right open in $Y$ and let $x \in f^{-1}(G)$. Since $G$ is open and $f(x) \in G$, there exists a basic open set $V^r_\varepsilon(f(x))$ such that $f(x) \in V^r_\varepsilon(f(x)) \subseteq G$. Since $f$ is right continuous, there exists $\delta > 0$ such that $f(V^r_\delta(x)) \subseteq V^r_\varepsilon(f(x)) \subseteq G$.

Hence $V^r_\delta(x) \subseteq f^{-1}(G)$, showing that $f^{-1}(G)$ is right open in $X$.

Suppose, conversely, that $f^{-1}(G)$ is right open in $X$ whenever $G$ is right open in $Y$. Let $\varepsilon > 0$ and let $a \in X$. Consider $G = V^r_\varepsilon(f(a))$ which is obviously open. $f^{-1}(G)$ is open in $X$ and $a \in f^{-1}(G)$.

Hence there exists $\delta > 0$ such that $V^r_\delta(a) \subseteq f^{-1}(G)$.

$\Rightarrow f(V^r_\delta(a)) \subseteq G = V^r_\varepsilon(f(a))$
\[ x \in V_\delta^R(a) \Rightarrow f(x) \in f(V_\delta^R(a)) \subseteq V_\varepsilon^R(f(a)) \]

\[ d_2(f(a), f(x)) < \varepsilon \]

\[ d_1(a, x) < \delta \Rightarrow x \in V_\delta^R(a) \Rightarrow d_2(f(a), f(x)) < \varepsilon \]

This proves that \( f \) is continuous at every point of \( X \).