Chapter-0

Prelude

It was more than a century ago, in 1906, that Maurice Frechet introduced the notion of metric space in his Thesis “Surquelques points du calcul functionnel”[1]. It was about a century ago, in 1914, that Felix Hausdorff published his study of metric and topological spaces in “Grundzuge der Mengenlehre”[2].

Metrics appear everywhere in Mathematics: Geometry, Probability, Statistics, Coding theory, Graph theory, Pattern recognition, Networks, Computer graphics, Molecular biology, Theory of information and Computer semantics are some of the fields in which metrics and/or their cousins play a significant role.

This Thesis too studies various metric-like functions. We recall that a metric on a nonempty set $X$ is a nonnegative real valued function $d$ on $X \times X$ satisfying

(i). $d(x, y) = 0 \iff x = y$

(ii). $d(x, y) = d(y, x)$

(iii). $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z$ in $X$.

If $d$ is a metric on $X$, the pair $(X, d)$ is called a metric space (ii) is called symmetry while (iii) is referred to as the triangular inequality. The set
$B_r(x) = \{ y / y \in X \text{ and } d(x, y) < r \}$ is called open ball centered at $x$ while $B_r[x] = \{ y / y \in X \text{ and } d(x, y) \leq r \}$ is called closed ball.

(i) ensures that balls are nonempty and (ii) and (iii) together ensure that open balls are open, closed balls are closed and that metric spaces are Hausdorff spaces.

We can weaken or dispense with (i) or (ii) or (iii) in the definition of a metric at the cost of these pleasant consequences. But there are metric like functions that do not enjoy (i) or (ii) or (iii) in full force, but are useful in the study of several practical problems. In addition to this utilitarian view, there is the platonic pure mathematical need to consider the logical consequences of weakening the defining conditions of a metric. As such, we are naturally motivated to study topological and geometrical aspects of various weakened forms of a metric space.

In this chapter we present a few practical and mathematical situations in which metrics or their cousins have been found to play an essential role.

(i). Let $(X, d')$ be a metric space. On the power set $P(X)$ of $X$,

Define $d(A, B) = \inf \{ d'(a, b) / a \in A, b \in B \}$. Then

$d(A, A) = 0$ for $A \in P(X)$ and

$d(A, B) = d(B, A)$ for $A \in P(X)$ and $B \in P(X)$.

Further $d(A, B) = 0$ if $(\bar{A} \cap B) \cup (A \cap \bar{B}) \neq \emptyset$.

If $A, B \in P(X), d(A, B) > 0, a \in A, b \in B$ and $C = \{a, b\}$

then $d(A, C) + d(C, B) = 0 + 0 = 0 \neq d(A, B)$. 
Hence the triangular inequality fails. But we have a weak form of the triangular inequality:

If \( x, y \in X \) and \( A \subseteq X \), then \( d(x, A) \leq d(x, y) + d(y, A) \).

(ii). If \( G = (V, E) \) is a connected graph, the path metric \( d \) on \( V \) is defined as \( d(u, v) \) = length of the shortest path between \( u \) and \( v \).

(iii). On \( \mathbb{N} \), the set of positive integers, define \( d(x, y) = \frac{\ln \text{lcm}(x, y)}{\ln \text{gcd}(x, y)} \), where \( \ln \) denotes natural logarithm. Then \( d \) is a metric on \( \mathbb{N} \).

(iv). Let \( p \) be a prime number. Any nonzero rational number \( x \) can be written \( x = p^\alpha \frac{c}{d} \) where \( c \) and \( d \) are integers not divisible by \( p \) and \( \alpha \) is a uniquely fixed integer.

Define \( |x|_p = p^{-\alpha} \) and \( d(x, y) = |x - y|_p \), if \( x \neq y \) and \( d(x, x) = 0 \). Then \( d \) is a metric on \( \mathbb{Q} \), the set of rational numbers. This metric is called the \( p \)-adic metric and it plays an important role in number theory.

(v). If \( (X, d) \) is a metric space and \( a \) is a point of \( X \) define \( d'(x, y) = d(x, a) + d(a, y) \). \( d' \) is symmetric and satisfies the triangle inequality.

If \( x \neq a \) then \( d'(x, x) > 0 \). However, \( d'(x, y) = 0 \Rightarrow d(x, a) + d(a, y) = 0 \) \( \Rightarrow x = y = a \).

(vi). Let \( B \) be a convex open balanced neighborhood of \( \overline{0} \) in \( \mathbb{R}^n \) i.e.
(1). \( \alpha \bar{x} + \beta \bar{y} \in B \) whenever \( \bar{x}, \bar{y} \in B, \alpha \geq 0, \beta \geq 0 \) and \( \alpha + \beta = 1 \)

(2). \( \bar{x} \in B \Rightarrow \alpha \bar{x} \in B \) whenever \( |\alpha| \leq 1 \)

(3). \( \bar{x} \in B \Rightarrow \exists r > 0 \in B_r(\bar{x}) \subseteq B \).

For \( \bar{x}, \bar{y} \in R^n \), define \( d(\bar{x}, \bar{y}) = \inf \{ \alpha > 0 / \bar{y} - \bar{x} \in \alpha B \} \). Then

\( d(\bar{x}, \bar{y}) = 0 \) iff \( \bar{x} = \bar{y} \), \( d \) has the triangle inequality but \( d(\bar{x}, \bar{y}) \) need not be equal to \( d(\bar{y}, \bar{x}) \). For a counter example,

let \( n = 1 \) and \( B = \left( -\frac{1}{2}, 1 \right) \). Then \( d\left( -\frac{1}{4}, -\frac{1}{2} \right) = \frac{3}{4} \) while \( d\left( \frac{1}{2}, -\frac{1}{4} \right) = \frac{3}{2} \)

(vii). Let \( \Sigma \) denote the set of all intervals \([a, b]\) with \( a \leq b \) in the real number system \( R \). For \( I \in \Sigma \) and \( J \in \Sigma \),

define \( d(I, J) = \max\{l(I), l(J)\} \) where \( l(I) = l([a, b]) = b - a \).

Then \( d \) is symmetric and satisfies the triangular inequality. But \( d(I, I) = l(I) > 0 \) for all non-degenerate closed intervals. Also \( d(I, J) = 0 \Rightarrow l(I) = l(J) = 0 \).

\( \Rightarrow I, J \) are singletons.

(viii). If \( (M, \mu) \) is a measure space, define \( d(A, B) = \mu(A \Delta B) \) for \( A, B \in M \)

where \( \Delta \) denotes symmetric difference:

\( A \Delta B = A \cup B - A \cap B \). \( d(A, A) = 0, d(A, B) = d(B, A) \) and

\( d(A, B) \leq d(A, C) + d(C, B) \)

Note that \( d(A, B) = 0 \Leftrightarrow \mu(A \Delta B) = 0 \Leftrightarrow A = B \).
(ix). **Information distance:**[3] The “Information distance” that we are about to present is not exactly a metric. But our interest is not only in metrics but metric-like functions and so it is not out of place to mention about information distance. Let $x, y$ be binary strings. Let us fix a programming language in which our programs are to be written. Let $p(x, y)$ denote the shortest program which computes both $y$ when $x$ is given as input and $x$ when $y$ is given as input. Let $k(x)$ denote the smallest input needed to construct $x$ on a computer. If $x, y$ are unrelated then we would expect that the length of the program $p(x, y)$, denoted by $|p(x, y)|$ satisfies roughly $|p(x, y)| = k(x) + k(y)$. As can be expected, there will be an additive constant $C$ which gives the exact formula $|p(x, y)| = k(x) + k(y) + C$. We can also write this as $|p(x, y)| = k(x) + k(y) + O(1)$, in view of some redundant overlap of information it turns out that we can, with few modifications, view $|p(x, y)|$ as a metric like function of $(x, y)$. If $x$ is a binary string, let $x^*$ denote the shortest program that computes $x$. If there are ties for $x^*$, break the tie by selecting the program which occurs first when all the contending programs are written in binary and arranged in lexicographic order.

Let $k\left(\frac{x}{y}\right)$ denote the shortest program that computes $x$ when $y$ is given as an auxiliary input.
Define \( d(x, y) = k \left( \frac{x}{y^*} \right) \). Then it can be proved [3] that

\[
d(x, x) = 0, \\
d(x, y) = d(y, x) \text{ and} \\
d(x, y) \leq d(x, z) + d(z, y) + C \text{ for some constant } C \geq 0.
\]

It is the presence of this constant "\( C \)" that causes \( d \) to fail to be a metric. This "\( d \)" is called the "Information distance".

(x). **Evolutionary distance:** This distance arose in two quite different scientific studies. In Biology[4], it arose in the study of proteins. A protein is specified by the sequence of amino acids that compose it since 20 different amino acids are building blocks of proteins. A protein could be thought of as a finite string made of 20 different symbols. Biologists believe that proteins have evolved over time and this evolution can be represented mathematically as a tree or trees whose terminal nodes are the proteins that we can find now in living organisms. Biologists want to understand the mechanism of evolutionary processes and also the probabilities under which two different proteins could have descended from a common ancestral protein. One approach to this study consists in studying numerically the dissimilarities in protein structures. A good number of publications concern themselves with a metrical study of protein sequences. The reader can easily appreciate that “string matching” in computer science also involved the same ideas and hence the idea of evolutionary distance is useful in the detection of plagiarism. The following abstract account of evolutionary metric, easily seen to be a mathematical model of the biological scenario is presented here.
Let $A$ be a finite set containing a distinguished element $\Delta$, called the neutral element. An infinite sequence $a_1, a_2, ...$ is called an $A$-sequence if all but finitely many of the $a_i$'s are $\Delta$ and $a_i \in A$ for each $i$.

A-sequences $a_1, a_2, ...$ and $b_1, b_2, ...$ are called equivalent if $a_i = b_i$ whenever $a_i \neq \Delta$ and $b_i \neq \Delta$. An evolutionary sequence $\bar{a} = a_1, a_2, ...$ is the set of all A-sequences equivalent to $a = a_1, a_2, ...$.

Every evolutionary sequence $\bar{a} = a_1, a_2, ...$ has a member $b_1, b_2, ...$ such that if $b_n \neq \Delta$ then $b_i \neq \Delta$ for all $i < n$.

For example, $10\Delta\Delta1\Delta\Delta0\Delta\Delta1\Delta\Delta...$ contains $100010000001\Delta\Delta...$, which has the property that $a_n \neq \Delta$ implies $a_1, a_2, ... a_{n-1}$ are all $\neq \Delta$. Let $\bar{S}$ consist of all A-sequences that have this property. $\bar{S}$ includes $\Delta\Delta...$.

We now consider a set $\tau$ of partial transformations on $\bar{S}$ as follows:

I. The identity transformation $I : \bar{S} \to \bar{S}, I(\bar{a}) = \bar{a}$ for all $\bar{a} \in \bar{S}$ is in $\tau$.

II. Each $T$ in $\tau$ has for its domain and range, subsets of $\bar{S}$.

III. Each $T$ in $\tau$ has an associated nonnegative number $w(T)$ called its weight.

Fix $j > 1$. Suppose $\bar{a} = a_1, a_2, ...$ is an A-sequence and the sequence $a_j, a_{j+1}, ... \in$ domain of $T$, where $T \in \tau$. We define $T^j$ by

$$T^j(\bar{a}) = T^j(a_1a_2...a_{j-1}T(a_ja_{j+1}...)).$$
If $j = 1$, define $T^1 = T$. Define $w(T^j) = w(T)$.

For $\bar{a}, \bar{b} \in \bar{S}$, we define

$$\{ \bar{a} \rightarrow \bar{b} \}_\tau = \left\{ T_{i_l}^{j_l} T_{i_{l-1}}^{j_{l-1}} \cdots T_{i_1}^{j_1} / T_{i_l}^{j_l} T_{i_{l-1}}^{j_{l-1}} \cdots T_{i_1}^{j_1} (\bar{a}) = \bar{b} \right\} \text{ where } T_{i_k} \in \tau.$$  

In other words $\{ \bar{a} \rightarrow \bar{b} \}_\tau$ consists of finite number of compositions of elements of $\tau$ which transform $\bar{a}$ into $\bar{b}$. The set $\{ \bar{a} \rightarrow \bar{b} \}_\tau$ may be empty. For $\bar{a}, \bar{b} \in \bar{S}$ we define

$$d(\bar{a}, \bar{b}) = \max \left\{ \min_{l_1} \sum_{k=1}^{l_1} w(T_{i_k}), \min_{l_2} \sum_{k=1}^{l_2} w(T_{i_k}) \right\} \text{ if } \{ \bar{a} \rightarrow \bar{b} \} \neq \phi$$

the above minima exist. If $\{ \bar{a} \rightarrow \bar{b} \} = \phi$ define $d(\bar{a}, \bar{b}) = +\infty$. 'd' defines a metric like function on the set of $A$-equivalence classes of $\bar{S}$. 'd' may fail to be a metric because infinite distances are possible. Except for this, all the metric axioms are satisfied.

(XI) **Scott’s Fixed Point Theorem for DCPOs:**

Domains are special kinds of partially ordered sets. In domain theory[5,6,7], if an object $x$ is less informative than an object $y$ then we say that $x$ is smaller than $y$.

The concept of a directed set plays a significant role in the theory. A directed subset of a poset is a subset such that each pair of elements in it has least upper bound in it. A directed set could be thought of as a set which contains pieces of information that do not contradict each other and which contains a megapiece of information $\text{Sup } A$ that contains the information contained in the separate objects...
of each finite subset $A$. Such posets in which directed sets have lub’s are called directed complete posets or dcpo’s.

**Notation:** If $S$ is a directed subset of a dcpo $P$ then we denote $\text{lub } S$ by $\vee \uparrow S$, the upper arrow serving to indicate that $S$ is directed and ‘$\vee$’ serving to indicate lub. If $x, y$ are elements of $S$, we denote $\vee \uparrow \{x, y\}$ by $x \vee y$.

We also define $\downarrow S = \{x \in P / x \leq y \text{ for some } y \in S\}$

$\uparrow S = \{x \in P / y \leq x \text{ for some } y \in S\}$

$\downarrow x = \downarrow \{x\}$ and $\uparrow x = \uparrow \{x\}$

An element $x$ of a dcpo $P$ is said to be **way below** an element $y$ of $P$ if for any directed subset $D$ of $P$ with $\text{Sup } D \geq y$, there exists an element $z$ of $D$ with $x \leq z$. If $x$ is way below $y$ then we also say that $x$ approximates $y$ and write $x \ll y$.

**Definitions:**

1) If $(P, \leq)$ is a poset, $x \in P$ and $x \ll x$, we say that $x$ is compact or isolated from below. $K(P)$ denotes the set of all compact elements of $P$.

2) A poset $(P, \leq)$ is called continuous if it satisfies the axiom of approximation:

\[ x = \uparrow \downarrow x \text{ for each } x \text{ in } P. \]

3) A continuous dcpo is called a domain.
4) A subset $B$ of a poset $P$ is called a domain theoretic base or simply a base of $P$ if for each $x$ in $P$ it is true that $\downarrow x \cap B$ has $x$ as its supremum.

If $K(P)$ is a base for $P$ then $P$ is called algebraic.

If $P$ has $K(P)$ as a countable base then we call it $w$-continuous or $w$-algebraic.

A dcpo $P$ is called bounded-complete if every nonempty subset of $P$ which is bounded above has lub.

5) A bounded-complete $w$-algebraic dcpo with a least element $\bot$ is called a Scott-domain.

6) Let $P$ be a dcpo. $U \subseteq P$ is called Scott-open if

i. $U$ is an upper set. i.e $U = \uparrow U$ and

ii. For every directed subset $D$ of $P$ such that $Sup D \in U$ it is true that $D \cap U \neq \phi$.

**Remark:** The Scott-open sets form a topology on $P$.

**Theorem:** For a function $f : S \rightarrow T$ with $S$ and $T$ dcpo’s the following are equivalent.

a) $f$ is continuous when $S$ and $T$ are equipped with Scott topologies.

b) $f$ preserves suprema of directed sets.

c) $y \ll f(x) \iff \exists z \ll x$ such that $y \ll f(z)$.

d) $f(x) = Sup\{f(y) / y \ll x\}$. 
If $S$ and $T$ are algebraic domains then the above are equivalent to (e) and (f) below.

e) If $x \in S$ and $k \in K(T)$ then $k \leq f(x)$ iff there exists $j \in K(S)$ such that $j \leq x$ and $k \leq f(j)$,

$$f(x) = \text{Sup}\{ f(j) / j \leq x$ and $j \in K(S) \} \text{ for all } x \text{ in } S.$$

We now give the fixed point theorem due to Scott.

**Theorem (Scott’s Fixed Point Theorem):** Let $D$ be a dcpo with a least element $\bot$. Then every Scott continuous function $f : D \to D$ has a least fixed point $y$ which is given by $y = \bigvee_{n \in \mathbb{N}} f^n(\bot)$.

**Definition:** Let $(P, \leq)$ be a w-continuous dcpo with a base $B = \{a_n / n \in \mathbb{N}\}$. We define the Smyth quasi-metric $d_s$ on $P$ by

$$d_s(x, y) = \text{Inf}\{ \frac{1}{2^n} / a_i \ll x \Rightarrow a_i \ll y \text{ for } 1 \leq i \leq n \}.$$

[We recall that a quasimetric 'd' on a nonempty set $X$ is a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$,
2. $d(x, x) = 0$,
3. $d(x, y) = 0 = d(y, x) \Rightarrow x = y$ and
4. $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z$ in $X$.]

The discovery of Smyth that $w$-continuous dcpo’s are quasi-metrizable triggered activity in two different spheres of mathematics and computer science:

1. Quantitative domain theory

2. Fixed point theory in metric like spaces