Chapter 6

Triangular Sum Labeling of Graphs
6.1 Introduction

This chapter is intended to discuss triangular sum labeling of graphs. We will show that some classes of graph can be embedded as an induced subgraphs of a triangular sum graph. In the succeeding section we will provide brief summary of definitions which are necessary for the subsequent development.

6.2 Triangular sum labeling

6.2.1 Triangular number

A triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer \( n \). If \( n^{th} \) triangular number is denoted by \( T_n \) then \( T_n = \frac{1}{2} n(n+1) \). It is easy to observe that there does not exist consecutive integers which are triangular numbers.

6.2.2 Triangular sum graph

A triangular sum labeling of a graph \( G \) is a one-to-one function \( f : V \rightarrow N \) (where \( N \) is the set of all non-negative integers) that induces a bijection \( f^+ : E(G) \rightarrow \{T_1, T_2, \ldots, T_q\} \) of the edges of \( G \) defined by \( f^+(uv) = f(u) + f(v) \), \( \forall e = uv \in E(G) \).

The graph which admits such labeling is called a triangular sum graph.

6.2.3 Some existing results

This concept was introduced by Hegde and Shankaran [39] and they proved that

- Path \( P_n \), Star \( K_{1,n} \) are triangular sum graphs.
• Any tree obtained from the star $K_{1,n}$ by replacing each edge by a path is a triangular sum graph.

• The lobster $T$ obtained by joining the centers of $k$ copies of a star to a new vertex $w$ is a triangular sum graph.

• The complete $n$-ary tree $T_m$ of level $m$ is a triangular sum graph.

• The complete graph $K_n$ is triangular sum if and only if $n \leq 2$.

They also shown that

• If $G$ is an Eulerian $(p,q)$-graph admitting a triangular sum labeling then $q \not\equiv 1 \pmod{12}$.

• The dutch windmill DW$(n)$($n$ copies of $K_3$ sharing a common vertex) is not a triangular sum graph.

• The complete graph $K_4$ can be embedded as an induced subgraph of a triangular sum graph.

In a paper by Vaidya et al.[79] it has been shown that

• In any triangular sum graph $G$ the vertices with labels 0 and 1 are always adjacent.

• In any triangular sum graph $G$, 0 and 1 cannot be the vertex labels in the same triangle contained in $G$.

• In any triangular sum graph $G$, 1 and 2 cannot be the vertex labels of the same triangle contained in $G$.

• The helm graph $H_n$ is not a triangular sum graph.

• If every edge of a graph $G$ is an edge of a triangle then $G$ is not a triangular sum graph.
6.3 Some important results on triangular sum graphs

Theorem 6.3.1. Every cycle can be embedded as an induced subgraph of a triangular sum graph.

Proof. Let \( G = C_n \) be a cycle with \( n \) vertices. We define labeling \( f : V(G) \rightarrow N \) as follows such that the induced function \( f^+ : E(G) \rightarrow \{T_1, T_2, \ldots, T_q\} \) is bijective.

\[
\begin{align*}
    f(v_1) &= 0 \\
    f(v_2) &= 6 \\
    f(v_i) &= T_{i+2} - f(v_{i-1}); \quad 3 \leq i \leq n-1 \\
    f(v_n) &= T_{f(v_{n-1})-1}
\end{align*}
\]

Now let \( A = \{T_1, T_2, \ldots, T_r\} \) be the set of missing edge labels. That is, elements of set \( A \) are the missing triangular numbers between 1 and \( T_{f(v_{n-1})-1} \). Now add \( r \) pendant vertices which are adjacent to the vertex with label 0 and label these new vertices with labels \( T_1, T_2, \ldots, T_r \). This construction will give rise to edges with labels \( T_1, T_2, \ldots, T_r \) such that the resultant supergraph \( H \) admits triangular sum labeling. Thus we proved that every cycle can be embedded as an induced subgraph of a triangular sum graph. \( \square \)

Illustration 6.3.2. In the following Figure 6.1 embedding of \( C_5 \) as an induced subgraph of a triangular sum graph is shown.

![Figure 6.1: Embedding of \( C_5 \) as an induced subgraph of a triangular sum graph](image-url)
**Theorem 6.3.3.** Every cycle with one chord can be embedded as an induced subgraph of a triangular sum graph.

*Proof.* Let $G$ be the cycle with one chord and $e = v_1v_k$ be the chord of cycle $C_n$. We define labeling $f : V(G) \rightarrow N$ as follows such that the induced function $f^+ : E(G) \rightarrow \{T_1, T_2, \ldots, T_q\}$ is bijective.

- $f(v_1) = 0$
- $f(v_2) = 6$
- $f(v_i) = T_{i+2} - f(v_{i-1}); \quad 3 \leq i \leq k - 1$
- $f(v_k) = T_{f(v_{k-1}) - 1}$
- $f(v_{n+i-1}) = T_{f(v_{n-k-1}) + i} - f(v_{n+i-2}); \quad 2 \leq i \leq n - k$
- $f(v_n) = T_{f(v_{n-1}) - 1}$

Now following the procedure described in Theorem 6.3.1 and the resultant supergraph $H$ admits triangular sum labeling. Thus we proved that every cycle with one chord can be embedded as an induced subgraph of a triangular sum graph. $\square$

**Illustration 6.3.4.** In the following Figure 6.2 embedding of $C_4$ with one chord as an induced subgraph of a triangular sum graph is shown.

![Figure 6.2: Embedding of $C_4$ with one chord as an induced subgraph of a triangular sum graph](image)
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**Theorem 6.3.5.** Every cycle with twin chords can be embedded as an induced subgraph of a triangular sum graph.

**Proof.** Let $G$ be the cycle with twin chords and $e_1 = v_1v_k$ and $e_2 = v_1v_{k+1}$ be its chords. We define labeling $f : V(G) \rightarrow N$ such that the induced function $f^+ : E(G) \rightarrow \{T_1, T_2, \ldots, T_q\}$ is bijective.

$f(v_1) = 0$
$f(v_2) = 6$
$f(v_i) = T_{i+2} - f(v_{i-1}); \quad 3 \leq i \leq k - 1$
$f(v_k) = T_{f(v_{k-1})-1}$
$f(v_{k+1}) = T_{f(v_k)-1}$
$f(v_{k+i}) = T_{f(v_k)-1+i} - f(v_{k+i-1}); \quad 2 \leq i \leq n - k - 1$
$f(v_n) = T_{f(v_{n-1})-1}$

Now following the procedure adapted in Theorem 6.3.1 the resulting supergraph $H$ admits triangular sum labeling. That is, every cycle with twin chords can be embedded as an induced subgraph of a triangular sum graph. $\square$

**Illustration 6.3.6.** In the following Figure 6.3 embedding of $C_6$ with twin chord as an induced subgraph of a triangular sum graph is shown.

![Figure 6.3: Embedding of $C_6$ with twin chord as an induced subgraph of a triangular sum graph](image-url)
6.4 Concluding Remarks

As every graph is not a triangular sum graph it is very interesting to investigate graphs or graph families which are not triangular sum graphs but they can be embedded as an induced subgraph of a triangular sum graph. We show that cycle, cycle with one chord and cycle with twin chords can be embedded as an induced subgraph of a triangular sum graph.

The next chapter is focused on L(2,1) and Radio labeling of graphs.