Chapter 5

SU(2) Invariants of symmetric qubit states

5.1 Introduction

The problem of enumeration of local invariants of quantum state described by a density matrix \( \rho \) is important in the context of quantum entanglement also. Nonlocal correlations in quantum systems reflect entanglement between its parts. Genuine non-local properties should be described in a form invariant under local unitary operations. Two N-qubit states are said to be locally equivalent if one can be transformed into the other by local operations, i.e., \( \rho' = U\rho U^\dagger \) where \( U \in SU(2)^\otimes N \) and the two quantum states \( \rho \) and \( \rho' \) are said to be equally entangled. A general prescription to identify the invariants associated with a multiparticle system has been outlined by Linden et al. (1999). Well known algebraic methods for generating invariants already exists in literature [A.Osterloh (2010); Barnum & Linden (2001); Carteret et al. (1990); Grassl et al. (1998)]. Williamson et al. (2011) have presented a geometric approach for constructing SU(2) and SL(2,C) invariants. Makhlin (2002) has presented a complete set of 18 local polynomial invariants of two qubit mixed states and demonstrated the usefulness of these invariants to study entanglement. As the number of subsystems increases, the problem of identifying and interpreting independent invariants rapidly becomes very complicated. Usha Devi et al. (2005) have shown that a set of 6 invariants which is a subset of a more general set of 18 invariants proposed by Makhlin (2002) is sufficient to characterize the non local properties of a symmetric two qubit system.

We focus on symmetric N qubit states and in particular two, three qubit states as the problem of identifying independent invariants in these cases theoretically and experimentally
is interesting. Our approach makes use of the geometrical multiaxial representation\(^1\) of an arbitrary spin-\(j\) density matrix [Ramachandran & Ravishankar (1986)] which is completely characterized by a set of \(j(2j+1)\) axes and \(2j\) real positive scalars. Further we enumerate the total number of invariants which could be constructed out of these axes and positive scalars. As examples we explicitly calculate the invariants associated with some well known two, three qubit symmetric states.

5.2 Invariants of \(N\) qubit symmetric states

According to multiaxial representation of spin-\(j\) density matrix, any spherical tensor of rank \(k\) can be represented geometrically by a set of \(k\) vectors \(\hat{Q}_i\) on the surface of sphere of radius \(r_k\). Consequently, the state of a spin-\(j\) assembly can be represented geometrically by a set of \(2j\) spheres, one corresponding to each value of \(k\), \(k = 1...2j\), the \(k^{th}\) sphere having \(k\) vectors specified on its surface. Observe that since \((\hat{Q}_i(\theta_i, \phi_i) \otimes \hat{Q}_j(\theta_j, \phi_j))^0\) is an invariant \((i \neq j)\) under rotation, we can construct in general \(C_{2j}^{j(2j+1)}\) invariants out of \(j(2j+1)\) axes together with \(2j\) real positive scalars. Here \(C_{2}^{j(2j+1)}\) denotes binomial coefficient.

Examples:

Spin - 1 or two qubit symmetric states: 3 axes and 2 real scalars. Therefore 3+2 = 5 invariants.
Spin - \(\frac{3}{2}\) or three qubit symmetric states: 6 axes and 3 real scalars. Therefore 15+3 = 18 invariants.
Spin - 2 or four qubit symmetric states: 10 axes and 4 real scalars. Therefore 45+4 = 49 invariants.

5.2.1 Pure spin-1 state

Consider the normalized symmetric state of spin-1 system (chapter 2, Eq. 2.4) as

\[
|\psi_{12}\rangle_{sym} = \frac{\sqrt{2}}{\sqrt{1 + \cos^2 \theta}} \left[\cos^2 \frac{\theta}{2} |1\rangle \hat{Z}_0 - \sin^2 \frac{\theta}{2} |1 - 1\rangle \hat{Z}_0\right].
\] (5.1)

\(^1\)An introduction to the multiaxial representation is given in chapter 1
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The density matrix corresponding to the above state (Eq. 2.5) is given by

\[
\rho_{\text{sym}} = \left| \psi_{12} \right\rangle_{\text{sym}} \left\langle \psi_{12} \right|_{\text{sym}} = \frac{2}{(1 + \cos^2 \theta)} \begin{pmatrix}
\cos^4 \frac{\theta}{2} & 0 & -\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} \\
0 & 0 & 0 \\
-\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} & 0 & \sin^4 \frac{\theta}{2}
\end{pmatrix}.
\] (5.2)

Comparing the above matrix with the standard representation of the density matrix in terms of Fano statistical tensor parameters \( t^k_l \) (Eq. 1.11), we obtain the non zero \( t^k_l \) as

\[
t_0^1 = \frac{\sqrt{6} \cos \theta}{1 + \cos^2 \theta}, \quad t_0^0 = \frac{1}{\sqrt{2}},
\]

\[
t_2^2 = t_2^{-2} = -\frac{\sqrt{3} \sin^2 \theta}{2(1 + \cos^2 \theta)}.
\]

Since \( t_{\pm 1}^1 = 0 \), \( \hat{z}_0 \) itself is the axis (\( \hat{Q}_1 \)) associated with \( t^1 \). Here

\[
t_0^1 = r_1 (\hat{Q}_1)_0^1,
\]

and hence

\[
r_1 = \frac{t_0^1}{(\hat{Q}_1)_0^1}.
\] (5.3)

\[
(\hat{Q}_1)_0^1 = \sqrt{\frac{4\pi}{3}} Y_0^1(0),
\]

where

\[
Y_0^1(0) = \sqrt{\frac{3}{4\pi}} \cos \theta.
\]

Therefore

\[
r_1 = t_0^1 = \frac{\sqrt{6} \cos \theta}{1 + \cos^2 \theta}
\] (5.4)

Solving the polynomial equation (1.13) for \( t^2 \), we get

\[
\sqrt{C_0^4} t_2^2 - \sqrt{C_2^4} t_0^2 Z^2 + \sqrt{C_4^4} t_2^2 Z^4 = 0,
\] (5.5)

\[
\frac{\sin^2 \theta}{2(1 + \cos^2 \theta)} Z^4 - Z^2 + \frac{\sin^2 \theta}{2(1 + \cos^2 \theta)} = 0,
\] (5.6)
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or

\[ Z = \pm \frac{(1 \pm \cos \theta)}{\sin \theta}. \]  \hspace{1cm} (5.7)

from which we get \( \phi = 0 \) and \( \phi = \pi \). In other words,

\[ (\hat{Q}_2)(\theta_2, \phi_2) = \sqrt{\frac{4\pi}{3}} Y^1_q(\theta, 0) \]

and

\[ (\hat{Q}_3)(\theta_3, \phi_3) = \sqrt{\frac{4\pi}{3}} Y^1_q(\theta, \pi). \]

We have

\[ t^k_q = r_k(\hat{Q}_2 \otimes \hat{Q}_3)_q = r_k \sum_{q_1 \text{ or } q_2} C(11k; q_1q_2q)(\hat{Q}_2)_{q_1}^1(\hat{Q}_3)_{q_2}^1. \]  \hspace{1cm} (5.8)

Hence

\[ r_2 = \frac{t^2_0}{(\hat{Q}_2 \otimes \hat{Q}_3)_0^2} = \frac{t^2_2}{(\hat{Q}_2 \otimes \hat{Q}_3)_2^2}. \]  \hspace{1cm} (5.9)

\[ (\hat{Q}_2 \otimes \hat{Q}_3)_0^2 = C(112; 1 - 10)(\hat{Q}_2)_1^1(\hat{Q}_3)_1^1 + C(112; -110)(\hat{Q}_2)_1^1(\hat{Q}_3)_{-1}^1 \]

\[ + C(112; 000)(\hat{Q}_2)_1^1(\hat{Q}_3)_0^1 \]

\[ = \frac{1}{\sqrt{6}} \frac{4\pi}{3} Y^1_q(\theta, 0)Y^1_{-1}(\theta, \pi) + \frac{1}{\sqrt{6}} \frac{4\pi}{3} Y^1_{-1}(\theta, 0)Y^1_1(\theta, \pi) \]

\[ + \sqrt{\frac{2}{3}} \frac{4\pi}{3} Y^0_q(\theta, 0)Y^0_0(\theta, \pi) \]

We have

\[ Y^1_{\pm 1}(\theta, 0) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta; \]

\[ Y^1_{\pm 1}(\theta, \pi) = \pm \sqrt{\frac{3}{8\pi}} \sin \theta; \]

and

\[ Y^0_1(\theta, 0) = Y^0_1(\theta, \pi) = \sqrt{\frac{3}{4\pi}} \cos \theta. \]

Similarly

\[ (\hat{Q}_2 \otimes \hat{Q}_3)_2^2 = C(112; 112)(\hat{Q}_2)_1^1(\hat{Q}_3)_1^1 \]

\[ = \frac{4\pi}{3} Y^1_1(\theta, 0)Y^1_1(\theta, \pi) \] 90
Therefore,

\[ r_2 = \frac{\sqrt{3}}{1 + \cos^2 \theta}, \tag{5.10} \]

The invariants associated with the most general pure spin-1 state are

\[ I_1 = r_1, \]

\[ I_2 = r_2, \]

\[ I_3 = (\hat{Q}_1 \otimes \hat{Q}_2)_0^0 \]
\[ = C(110; q_1 q_2 0)(\hat{Q}_1)_{q_1}^1 (\hat{Q}_2)_{q_2}^1 \]
\[ = C(110; 000) \sqrt{\frac{4\pi}{3}} Y_0^1(\theta, 0) \sqrt{\frac{4\pi}{3}} Y_0^1(\theta, \pi) \]
\[ = -\frac{\cos \theta}{\sqrt{3}} \]

\[ I_4 = (\hat{Q}_1 \otimes \hat{Q}_3)_0^0 \]
\[ = C(110; q_1 q_2 0)(\hat{Q}_1)_{q_1}^1 (\hat{Q}_3)_{q_2}^1 \]
\[ = C(110; 000) \sqrt{\frac{4\pi}{3}} Y_0^1(\theta, 0) \sqrt{\frac{4\pi}{3}} Y_0^1(\theta, \pi) \]
\[ = -\frac{\cos \theta}{\sqrt{3}} \]

and

\[ I_5 = (\hat{Q}_2 \otimes \hat{Q}_3)_0^0 \]
\[ = C(110; 1 - 10)(\hat{Q}_2)_{1}^1 (\hat{Q}_3)_{-1}^1 + C(110; -110)(\hat{Q}_2)_{-1}^1 (\hat{Q}_3)_{1}^1 \]
\[ + C(110; 000)(\hat{Q}_2)_{0}^1 (\hat{Q}_3)_{0}^1 \]
\[ = \frac{1}{\sqrt{3}} \frac{4\pi}{3} Y_1^1(\theta, 0) Y_{-1}^1(\theta, \pi) \]
\[ + \frac{1}{\sqrt{3}} \frac{4\pi}{3} Y_{-1}^1(\theta, 0) Y_1^1(\theta, \pi) \]
\[ + \frac{1}{\sqrt{3}} \frac{4\pi}{3} Y_0^1(\theta, 0) Y_0^1(\theta, \pi) \]
\[ = -\frac{\cos 2\theta}{\sqrt{3}} \]

Therefore the invariants are

\[ I_1 = \frac{\sqrt{6} |\cos \theta|}{1 + \cos^2 \theta}, \tag{5.11} \]
\[ I_2 = \frac{\sqrt{3}}{1 + \cos^2 \theta}, \quad (5.12) \]

\[ I_3 = I_4 = -\frac{\cos \theta}{\sqrt{3}}, \quad (5.13) \]

\[ I_5 = -\frac{\cos 2\theta}{\sqrt{3}}. \quad (5.14) \]

It is clear from equation (5.1) that the state \(|\psi_{12}\rangle_{\text{sym}}\) is separable for \(\theta = 0\) and \(\pi\). Hence the invariants in the case of pure spin-1 separable states are

\[ I_1 = \sqrt{\frac{3}{2}}, \quad I_2 = \frac{\sqrt{3}}{2}, \]

\[ I_3 = I_4 = \mp \frac{1}{\sqrt{3}}, \]

\[ I_5 = -\frac{1}{\sqrt{3}}. \]
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Some examples

Bell states: Consider

\[ |\psi_{\text{Bell}}\rangle_1 = \frac{|11\rangle + |1\rangle - |1\rangle}{\sqrt{2}} = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}. \]

Corresponding density matrix is

\[ \rho_{\text{Bell}1} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{5.15} \]

The non-zero \( t^j_q \)’s are

\[ t_0^2 = \frac{1}{\sqrt{2}}, \quad t_2^2 = \frac{\sqrt{3}}{2}, \quad \text{and} \quad t_{-2}^2 = \frac{\sqrt{3}}{2}. \]

Since \( t_q^4 = 0, (q = -1, 0, 1) \), \( \hat{Q}_1 \) is indeterminate and hence \( \mathcal{I}_1, \mathcal{I}_3, \mathcal{I}_4 \) are indeterminate.

Solving the polynomial equation for \( t^2 \), we get

\[ \sqrt{C_0^4 t_{-2}^2} + \sqrt{C_1^4 t_0^4 Z^2} + \sqrt{C_4^4 t_2^4 Z^4} = 0, \tag{5.16} \]

\[ Z^4 + 2Z^2 + 1 = 0, \]

we get

\[ Z^2 = -1 \]

or

\[ Z_{1,2} = \pm i. \tag{5.17} \]

The corresponding angles are

\[ \hat{Q}_2 \equiv (\theta_2, \phi_2) = \left( \frac{\pi}{2}, \frac{\pi}{2} \right), \]

\[ \hat{Q}_3 \equiv (\theta_3, \phi_3) = \left( \frac{\pi}{2}, \frac{3\pi}{2} \right). \]

\[ r_2 = \frac{t_0^2}{(\hat{Q}_2 \otimes \hat{Q}_3)_0^2} = \frac{t_2^2}{(\hat{Q}_2 \otimes \hat{Q}_3)_2^2}. \tag{5.18} \]
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\[
(\hat{Q}_2 \otimes \hat{Q}_3)^0_2 = C(112; 1 - 10)(\hat{Q}_2)^1_2(\hat{Q}_3)^1_1 + C(112; -110)(\hat{Q}_2)^1_1(\hat{Q}_3)^1_1 \\
+ C(112; 000)(\hat{Q}_2)^0_0(\hat{Q}_3)^0_0
\]

\[
= \frac{1}{\sqrt{6}} \frac{4\pi}{3} Y^1_1(\frac{\pi}{2}, \frac{\pi}{2})Y^1_1(\frac{3\pi}{2}, \frac{\pi}{2}) + \frac{1}{\sqrt{6}} \frac{4\pi}{3} Y^1_1(\frac{\pi}{2}, \frac{\pi}{2})Y^1_1(\frac{\pi}{2}, \frac{3\pi}{2}) \\
+ \sqrt{2} \frac{4\pi}{3} Y^1_0(\frac{\pi}{2})Y^1_0(\frac{\pi}{2}) \\
= \frac{1}{\sqrt{6}}
\]

Here

\[
Y^1_{\pm 1}(\frac{\pi}{2}, \frac{\pi}{2}) = \mp \sqrt{\frac{3}{8\pi}} \sin \frac{\pi}{2} e^{\pm \frac{\pi}{2}} , \\
Y^1_{\pm 1}(\frac{\pi}{2}, \frac{3\pi}{2}) = \mp \sqrt{\frac{3}{8\pi}} \sin \frac{\pi}{2} e^{\pm \frac{3\pi}{2}} ,
\]

and

\[
Y^1_0(\frac{\pi}{2}) = \sqrt{\frac{3}{4\pi}} \cos \left(\frac{\pi}{2}\right).
\]

Similarly

\[
(\hat{Q}_2 \otimes \hat{Q}_3)^2_2 = C(112; 112)(\hat{Q}_2)^1_1(\hat{Q}_3)^1_1 \\
= \frac{4\pi}{3} Y^1_1(\frac{\pi}{2}, \frac{\pi}{2})Y^1_1(\frac{3\pi}{2}, \frac{\pi}{2}) \\
= \frac{1}{2}
\]

Therefore,

\[
r_2 = \sqrt{3}. \tag{5.19}
\]

\[
I_5 = (\hat{Q}_2 \otimes \hat{Q}_3)^0_0 = -\frac{\cos 2\theta}{\sqrt{3}}
\]

With \( \theta = \frac{\pi}{2} \), \( I_5 = \frac{1}{\sqrt{3}} \). Therefore the only local invariants which can be determined are

\[
I_2 = \sqrt{3}, \ I_5 = \frac{1}{\sqrt{3}}.
\]
Figure 5.1: Bell state 1
2) Consider

$$|\psi_{Bell}\rangle_2 = \frac{|11\rangle - |1\rangle}{\sqrt{2}} = \frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}}.$$ 

Corresponding density matrix is

$$\rho_{Bell_2} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \quad (5.20)$$

The Non-zero $t_q^k$'s are

$$t_0^2 = \frac{1}{\sqrt{2}}, \quad t_2^2 = -\frac{\sqrt{3}}{2}, \quad \text{and} \quad t_{-2}^2 = -\frac{\sqrt{3}}{2}.$$ 

Here $t_1^q = 0$, $(q = -1, 0, 1)$, and hence $Q_1$ is indeterminate and $I_1, I_3, I_4$ are indeterminate.

Solving the polynomial equation for $t^2$

$$\sqrt{C_0^4 t_{-2}^2} + \sqrt{C_1^4 t_0^2} Z^2 + \sqrt{C_4^4 t_2^2} Z^4 = 0, \quad (5.21)$$

$$Z^4 - 2Z^2 + 1 = 0,$$

we get

$$Z^2 = 1$$

or

$$Z_{1,2} = \pm 1. \quad (5.22)$$

The corresponding angles are

$$\hat{Q}_2 \equiv (\theta_2, \phi_2) = (\frac{\pi}{2}, 0),$$

$$\hat{Q}_3 \equiv (\theta_3, \phi_3) = (\frac{\pi}{2}, \pi).$$

$$(\hat{Q}_2 \otimes \hat{Q}_3)_0^2 = C(112; 1-10)(\hat{Q}_2)_0^1(\hat{Q}_3)_1^{-1} + C(112; -110)(\hat{Q}_2)_1^{-1}(\hat{Q}_3)_1^1$$

$$+ C(112; 000)(\hat{Q}_2)_0^0(\hat{Q}_3)_0^0$$

$$= \frac{1}{\sqrt{6}} \frac{4\pi}{3} Y_1^1(\frac{\pi}{2}, 0) Y_1^{-1}(\frac{\pi}{2}, \pi) + \frac{1}{\sqrt{6}} \frac{4\pi}{3} Y_1^{-1}(\frac{\pi}{2}, 0) Y_1^1(\frac{\pi}{2}, \pi)$$

$$+ \sqrt{\frac{2}{3}} \frac{4\pi}{3} Y_0^1(\frac{\pi}{2}) Y_0^1(\frac{\pi}{2})$$

$$= \frac{1}{\sqrt{6}}$$
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and

\[
(\hat{Q}_2 \otimes \hat{Q}_3)^2 = C(112; 112)(\hat{Q}_2)_{11}^1(\hat{Q}_3)_{11}^1
= \frac{4\pi}{3} Y_1^1(\frac{\pi}{2}, 0) Y_1^1(\frac{\pi}{2}, \pi)
= -\frac{1}{2}
\]

Therefore,

\[
r_2 = \frac{t_0^2}{(Q_2 \otimes Q_3)^2_0} = \frac{t_2^2}{(Q_2 \otimes Q_3)^2_2} = \sqrt{3}.
\]

(5.23)

\[
I_5 = (\hat{Q}_2 \otimes \hat{Q}_3)_0^0 = -\frac{\cos 2\theta}{\sqrt{3}}
\]

With \(\theta = \frac{\pi}{2}\), \(I_5 = \frac{1}{\sqrt{3}}\). Therefore the local invariants are

\[
I_2 = \sqrt{3} , I_5 = \frac{1}{\sqrt{3}}
\]

Since the states \(|\psi_{Bell}^{1}\rangle\) and \(|\psi_{Bell}^{2}\rangle\) have the same set of invariants, the above results indicate that they belong to the same SLOCC class.
Figure 5.2: Bell state 2
5.2.2 Mixed spin-1 state

Here we consider the example of channel spin-1 system which plays an important role in nuclear physics experiments. The density matrix for spin-1 mixed system in the symmetric basis $|1⟩, |0⟩, |1−1⟩$ in special Lakin frame (chapter 2, Eq. 2.14) is

$$\rho_{sym} = \frac{1}{(3 + p^2 \cos 2\theta)} \begin{pmatrix} (1 + p\cos \theta)^2 & 0 & -p^2 \sin^2 \theta \\ 0 & 1 - p^2 & 0 \\ -p^2 \sin^2 \theta & 0 & (1 - p\cos \theta)^2 \end{pmatrix}. \quad (5.24)$$

Comparing the above density matrix with the standard form (Eq. 1.11), we get the non-zero $t_k^\prime s$ as

$$t_0^2 = \frac{2\sqrt{6}p\cos \theta}{(3 + p^2 \cos 2\theta)},$$

$$t_0^2 = \frac{\sqrt{2}p^2(1 + \cos^2 \theta)}{(3 + p^2 \cos 2\theta)},$$

$$t_2^{\pm 2} = \frac{-\sqrt{3}p^2 \sin^2 \theta}{(3 + p^2 \cos 2\theta)}.$$

Solving the polynomial equation (1.13) for $t^2$, we obtain

$$\sqrt{C_0^4 t_0^2} + \sqrt{C_2^4 t_2^2} Z^2 + \sqrt{C_4^4 t_2^2} Z^4 = 0, \quad (5.25)$$

$$\frac{p^2 \sin^2 \theta}{(3 + p^2 \cos 2\theta)} Z^4 - \frac{p^2(3 + \cos 2\theta)}{(3 + p^2 \cos 2\theta)} Z^2 + \frac{p^2 \sin^2 \theta}{(3 + p^2 \cos 2\theta)} = 0, \quad (5.26)$$

or

$$Z = \pm \frac{(1 \pm \cos \theta)}{\sin \theta}. \quad (5.27)$$

from which we get $\phi = 0$ and $\phi = \pi$. In otherwords, $t_{\pm 1}^4 = 0$ and hence

$$\hat{Q}_1 \equiv \hat{z}_0, \quad \hat{Q}_2 \equiv \hat{p}(1), \quad \text{and} \quad \hat{Q}_3 \equiv \hat{p}(2).$$
\[ r_1 = \frac{t_0^1}{Q_0^1} = \frac{t_0^1}{\sqrt{\frac{4\pi}{3}} Y_0^1(0)} = \frac{2\sqrt{6}p|\cos\theta|}{(3 + p^2\cos2\theta)} \]

\[ r_2 = \frac{t_0^2}{(\hat{Q}_2 \otimes \hat{Q}_3)^0} = \frac{t_2^2}{(\hat{Q}_2 \otimes \hat{Q}_3)^2} = \frac{2\sqrt{3}p^2}{(3 + p^2\cos2\theta)} \]

\[ \mathcal{I}_3 = (\hat{Q}_1 \otimes \hat{Q}_2)^0 = -\frac{\cos\theta}{\sqrt{3}}. \]

\[ \mathcal{I}_4 = (\hat{Q}_1 \otimes \hat{Q}_3)^0 = -\frac{\cos\theta}{\sqrt{3}}. \]

\[ \mathcal{I}_5 = (\hat{Q}_2 \otimes \hat{Q}_3)^0 = -\frac{\cos2\theta}{\sqrt{3}}. \]

Thus the invariants associated with the mixed spin-1 state (Eq. 5.24) are found to be

\[ \mathcal{I}_1 = \frac{2\sqrt{6}p|\cos\theta|}{(3 + p^2\cos2\theta)}, \]

\[ \mathcal{I}_2 = \frac{2\sqrt{3}p^2}{(3 + p^2\cos2\theta)}, \]

\[ \mathcal{I}_3 = \mathcal{I}_4 = -\frac{\cos\theta}{\sqrt{3}}, \]

\[ \mathcal{I}_5 = -\frac{\cos2\theta}{\sqrt{3}}. \]

Note that in both pure as well as mixed state, \( \mathcal{I}_3 = \mathcal{I}_4 = -\frac{\cos\theta}{\sqrt{3}}, \mathcal{I}_5 = -\frac{\cos2\theta}{\sqrt{3}} \). For \( p = 1 \) and \( \theta = 0, \pi \), the state is separable as in the case of pure state. For \( p < 1 \), the state is separable for a range of values of \( \theta \). It is observed that as \( p \) decreases, the region of \( \theta \) for which entanglement appears also decreases [Swarnamala Sirsi & Veena Adiga (2010)].
5.2.3 Spin - 3/2 or three qubit symmetric state

Here we consider GHZ and W states which belong to different SLOCC classes\(^2\).

**GHZ-State**

Consider

\[
|\psi_{GHZ}\rangle = \frac{|\frac{3}{2}, \frac{3}{2}\rangle + |\frac{3}{2}, -\frac{3}{2}\rangle}{\sqrt{2}} \equiv |\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle
\]

Corresponding density matrix in the symmetric basis \(|\frac{3}{2}, \frac{3}{2}\rangle, |\frac{3}{2}, \frac{1}{2}\rangle, |\frac{3}{2}, -\frac{1}{2}\rangle, |\frac{3}{2}, -\frac{3}{2}\rangle\) is

\[
\rho_{GHZ} = \frac{1}{2}\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]

(5.28)

\[
t^k_q = \sum_{mm'} \rho_{mm'} \sqrt{2k + 1} C(jkj; mm')
\]

(5.29)

The non-zero \(t^k_q\)’s are

\[
t^2_0 = 1 , \quad t^3_3 = -1 , \quad t^3_{-3} = 1
\]

Since \(t^1_q = 0\), (q = -1, 0, 1), \(\hat{Q}_1\) is indeterminate. Also, \(t^2_2 = t^2_{-2} = 0\), \(\hat{Q}_2\) and \(\hat{Q}_3\) are collinear and parallel to the Z-axis.

Solving the polynomial equation for \(t^3_q\), (q = -3 to +3), we have

\[
\sqrt{C_0^6 t^3_{-3}} + \sqrt{C_0^6 t^3_3} Z^6 = 0
\]

(5.30)

we get

\[
Z^6 = 1 ,
\]

and hence

\[
Z = e^{\frac{2\pi i r}{6}} = e^{\frac{\pi i r}{3}}
\]

\(^2\text{see appendix G}\)
The three axes characterizing $t_q^3$ are
\[
\hat{Q}_4 = (\theta_4, \phi_4) = \left(\frac{\pi}{2}, 0\right), \\
\hat{Q}_5 = (\theta_5, \phi_5) = \left(\frac{\pi}{2}, \frac{\pi}{3}\right), \\
\hat{Q}_6 = (\theta_6, \phi_6) = \left(\frac{\pi}{2}, \frac{2\pi}{3}\right),
\]

\[
r_2 = \frac{t_0^2}{(\hat{Q}_2 \otimes \hat{Q}_3)_0^2},
\]

\[
(\hat{Q}_2 \otimes \hat{Q}_3)_0^2 = C(112; 1-10)(Q_2)_1^1(Q_3)_1^{-1} + C(112; -110)(Q_2)_1^{-1}(Q_3)_1^1 \\
+ C(112; 000)(\hat{Q}_2)_0^1(\hat{Q}_3)_0^1 \\
= \frac{1}{\sqrt{6}} \frac{4\pi}{3} Y_1^1(0, 0) Y_1^{-1}(0, 0) + \frac{1}{\sqrt{6}} \frac{4\pi}{3} Y_1^{-1}(0, 0) Y_1^1(0, 0) \\
+ \sqrt{\frac{2}{3}} \frac{4\pi}{3} Y_0^0(\frac{\pi}{2}) Y_0^0(\frac{\pi}{2}) \\
= \sqrt{\frac{2}{3}}.
\]

\[
(\hat{Q}_2 \otimes \hat{Q}_3)_0^2 = \sqrt{\frac{3}{2}} .
\]

Similarly,
\[
r_3 = \frac{t_0^2}{((\hat{Q}_4 \otimes \hat{Q}_5)_2^2 \otimes \hat{Q}_6)_3^3} (5.32)
\]

\[
(\hat{Q}_4 \otimes \hat{Q}_5)_2^2 = C(112; 112)(\hat{Q}_4)_1^1(\hat{Q}_5)_1^1 = \frac{(1 + \sqrt{3}i)}{4} \\
((\hat{Q}_4 \otimes \hat{Q}_5)_2^2 \otimes \hat{Q}_6)_3^3 = C(112; 112)C(213; 213)(\hat{Q}_4)_1^1(\hat{Q}_5)_1^1(\hat{Q}_6)_1^1 = \frac{1}{2\sqrt{2}}
\]
or
\[
r_3 = 2\sqrt{2} .
\]

Thus
\[
I_1 = r_1 = Indeterminate , \ I_2 = r_2 = \sqrt{\frac{3}{2}} , \ I_3 = r_3 = 2\sqrt{2}.
\]

And
\[
(\hat{Q}_1 \otimes \hat{Q}_2)_0^0 , (\hat{Q}_1 \otimes \hat{Q}_3)_0^0 , \ldots , (\hat{Q}_1 \otimes \hat{Q}_6)_0^0
\]
are indeterminate. Since \( \hat{Q}_2 \) and \( \hat{Q}_3 \) are parallel to Z-axis, \( \theta = 0 \). Therefore

\[
(\hat{Q}_2 \otimes \hat{Q}_3)_0^0 = -\frac{1}{\sqrt{3}}.
\]

Again,

\[
(\hat{Q}_2 \otimes \hat{Q}_4)_0^0, \ldots, (\hat{Q}_2 \otimes \hat{Q}_6)_0^0, (\hat{Q}_3 \otimes \hat{Q}_4)_0^0, \ldots, (\hat{Q}_3 \otimes \hat{Q}_6)_0^0 = 0.
\]

Since \( \hat{Q}_4 = (\theta_4, \phi_4) = (\frac{\pi}{2}, 0) \) and \( \hat{Q}_5 = (\theta_5, \phi_5) = (\frac{\pi}{2}, \frac{\pi}{3}) \), we have

\[
(\hat{Q}_4 \otimes \hat{Q}_5)_0^0 = -\frac{1}{2\sqrt{3}}.
\]

Again, \( \hat{Q}_4 = (\theta_4, \phi_4) = (\frac{\pi}{2}, 0) \) and \( \hat{Q}_6 = (\theta_6, \phi_6) = (\frac{\pi}{2}, \frac{2\pi}{3}) \), hence

\[
(\hat{Q}_4 \otimes \hat{Q}_6)_0^0 = \frac{1}{2\sqrt{3}}.
\]

And

\[
(\hat{Q}_5 \otimes \hat{Q}_6)_0^0 = -\frac{1}{2\sqrt{3}}.
\]

Therefore the set of non-zero invariants which can be determined are

\[
I_2 = \sqrt{\frac{3}{2}}, \ I_3 = 2\sqrt{2}, \ I_9 = -\frac{1}{\sqrt{3}}, \ I_{16} = -\frac{1}{2\sqrt{3}}, \ I_{17} = \frac{1}{2\sqrt{3}}, \ I_{18} = -\frac{1}{2\sqrt{3}}.
\]
Figure 5.3: GHZ state
Similarly, consider

\[ |\psi_{GHZ}'\rangle = \frac{|\frac{3}{2}, \frac{3}{2}\rangle - |\frac{3}{2}, -\frac{3}{2}\rangle}{\sqrt{2}} = \frac{|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}} \]

Corresponding density matrix in the symmetric subspace is

\[
\rho_{GHZ}' = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 \\
\end{pmatrix}.
\] (5.33)

The non-zero \( t_q' \)'s are

\[ t_0^2 = 1 \,, \quad t_3^3 = 1 \,, \quad t_{-3}^3 = -1 \]

Solving the polynomial equation for \( t_q^3 \) (q = -3 to +3), we have

\[ \sqrt{C_6 t_{-3}^3} + \sqrt{C_6 t_3^3} Z^6 = 0 \] (5.34)

we get

\[ Z^6 = 1 \,, \]

and hence

\[ Z = e^{\frac{2\pi ir}{6}} = e^{\frac{\pi ir}{3}} \]

Note: Both the states have the same set of invariants.
Chapter 5. SU(2) Invariants of symmetric qubit states

W-State

\[ (|W\rangle)_1 = \frac{|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\down\uparrow\rangle}{\sqrt{3}} = |3/2 - 1/2\rangle \] (5.35)

The corresponding density matrix in angular momentum basis is

\[ \rho_{W_1} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \] (5.36)

The non-zero \( t^k_q \)'s are \( t^3_0 = \frac{3}{2} \) and \( t^2_0 = -1 \).

Here \( t^1_q = 0 \), \( q = -1, 0, 1 \), therefore \( \hat{Q}_1 \) is indeterminate.

Again, \( t^2_{\pm 2} = 0 \) and \( t^2_0 = -1 \), hence \( \hat{Q}_2, \hat{Q}_3 \) are along the Z-axis.

Also, \( t^3_{\pm 3} = 0 \) and therefore \( \hat{Q}_4, \hat{Q}_5, \hat{Q}_6 \) are also along the Z-axis.

Since \( \theta = 0 \), we have

\[ r_2 = \frac{t^2_0}{(\hat{Q}_2 \otimes \hat{Q}_3)_0^2} = \sqrt{\frac{3}{2}}. \]

Similarly consider

\[ (|W\rangle)_2 = \frac{|\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle}{\sqrt{3}} = |3/2 1/2\rangle \] (5.37)

The corresponding density matrix in angular momentum basis is

\[ \rho_{W_2} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \] (5.38)

The non-zero \( t^k_q \)'s are \( t^3_0 = -\frac{3}{2} \) and \( t^2_0 = -1 \).

Therefore

\[ r_2 = \frac{t^2_0}{(\hat{Q}_2 \otimes \hat{Q}_3)_0^2} = \sqrt{\frac{3}{2}}. \]

\( (|W\rangle)_1 \) and \( (|W\rangle)_2 \) have the same set of invariants.