Appendices

Appendix A: Irreducible tensor operators

For j=1, matrix representation of irreducible tensor operators $\tau_q^{k\ell}$s are

\[
\tau_0^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_0^1 = \sqrt{\frac{3}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \tau_0^1 = -\sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\tau_1^0 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau_0^2 = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_1^2 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\tau_2^0 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \tau_2^2 = \sqrt{3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau_{-2}^2 = \sqrt{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]
For $j = \frac{3}{2}$, matrix representation of irreducible tensor operators $\tau^k_q$ are

\[
\begin{align*}
\tau^0_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\tau^1_0 &= \sqrt{\frac{2}{5}} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\tau^{-1}_0 &= \sqrt{\frac{2}{5}} \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \\
\tau^1_1 &= \sqrt{\frac{1}{5}} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \\
\tau^0_1 &= \sqrt{\frac{2}{5}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\tau^{-1}_1 &= \sqrt{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\tau^2_1 &= \sqrt{\frac{1}{5}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
\tau^2_0 &= \sqrt{\frac{2}{5}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\tau^{-2}_0 &= \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
\tau^3_0 &= \sqrt{\frac{1}{5}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
\tau^3_1 &= \sqrt{\frac{2}{5}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\tau^{-3}_1 &= \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{align*}
\]
\[
\tau_2^3 = \sqrt{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau_{-2}^3 = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \tau_3^3 = \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\tau_{-3}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]
Expresssions for $\tau_q^{k,t}$s in terms of $J_x, J_y$ and $J_z$

For $j = 1$:

When $k = 1$, $q = -1, 0, +1$.

\[
\tau_0^1 = \sqrt{\frac{3}{2}}(J_z),
\]
\[
\tau_1^1 = -\frac{\sqrt{3}}{2}(J_x + iJ_y),
\]
\[
\tau_{-1}^1 = \frac{\sqrt{3}}{2}(J_x - iJ_y).
\]

When $k = 2$, $q = -2, -1, 0, 1, 2$.

\[
\tau_0^2 = \frac{1}{\sqrt{2}}(2J_x^2 - J_x^2 - J_y^2) = \frac{1}{\sqrt{2}}(J_x^2 - J_y^2)
\]
\[
\tau_1^2 = -\frac{\sqrt{3}}{2}(J_xJ_z + J_zJ_x + i(J_yJ_z + J_zJ_y))
\]
\[
\tau_{-1}^2 = \frac{\sqrt{3}}{2}(J_xJ_z + J_zJ_x - i(J_yJ_z + J_zJ_y))
\]
\[
\tau_2^2 = \frac{\sqrt{3}}{2}(J_x^2 - J_y^2 + i(J_xJ_y + J_yJ_x))
\]
\[
\tau_{-2}^2 = \frac{\sqrt{3}}{2}(J_x^2 - J_y^2 - i(J_xJ_y + J_yJ_x))
\]
For $j = \frac{3}{2}$:

When $k = 1$, $q = -1, 0, +1$.

\[
\tau_0^1 = \frac{2}{\sqrt{5}}(J_z),
\]
\[
\tau_1^1 = -\sqrt{\frac{2}{5}}(J_x + iJ_y),
\]
\[
\tau_{-1}^1 = \sqrt{\frac{2}{5}}(J_x - iJ_y).
\]

When $k = 2$, $q = -2, -1, 0, 1, 2$.

\[
\tau_0^2 = \frac{1}{3}(2J_z^2 - J_x^2 - J_y^2) = \frac{1}{3}(3J_z^2 - J^2)
\]
\[
\tau_1^2 = -\frac{1}{\sqrt{6}}(J_xJ_z + J_zJ_x + i(J_yJ_z + J_zJ_y))
\]
\[
\tau_{-1}^2 = \frac{1}{\sqrt{6}}(J_xJ_z + J_zJ_x - i(J_yJ_z + J_zJ_y))
\]
\[
\tau_2^2 = \frac{1}{\sqrt{6}}(J_z^2 - J_y^2 + i(J_xJ_y + J_yJ_x))
\]
\[
\tau_{-2}^2 = \frac{1}{\sqrt{6}}(J_z^2 - J_y^2 - i(J_xJ_y + J_yJ_x))
\]
When $k = 3$, $q = -3, -2, -1, 0, 1, 2, 3$.

$$
\tau_0^3 = \frac{1}{3\sqrt{5}} (4J_z^3 - (J_z J_x^2 + J_x J_z + J_x J_x) - (J_z J_y^2 + J_y J_z + J_y J_x))
$$

$$
\tau_1^3 = -\frac{1}{3\sqrt{15}} (4((J_x^2 J_x + J_x J_z) + i (J_y^2 J_y + J_y J_z)) - 3(J_x^2 + i J_y^2) - (J_y^2 J_x + J_x J_y))
$$

$$
\tau_{-1}^3 = \frac{1}{3\sqrt{15}} (4((J_x^2 J_x + J_x J_z) - i (J_y^2 J_y + J_y J_z)) - 3(J_x^2 - i J_y^2) - (J_y^2 J_x + J_x J_y)) + i (J_x^2 J_y + J_y J_z)
$$

$$
\tau_2^3 = \sqrt{\frac{2}{27}} ((J_x^2 J_x + J_x J_z) - (J_y^2 J_z + J_z J_y) + i (J_x J_y J_z + J_x J_y J_z + J_x J_y J_z + J_z J_y J_z))
$$

$$
\tau_{-2}^3 = \sqrt{\frac{2}{27}} ((J_x^2 J_x + J_x J_z) - (J_y^2 J_z + J_z J_y) - i (J_x J_y J_z + J_x J_y J_z + J_x J_y J_z + J_z J_y J_z))
$$

$$
\tau_3^3 = -\frac{1}{6} (2(J_x^3 - i J_y^3) - (2J_x^2 J_y + 2J_x J_y J_y)) + i (2J_x J_y + 2J_y^2 + J_x J_y J_z)
$$

$$
\tau_{-3}^3 = \frac{1}{6} (2(J_x^3 + i J_y^3) - (2J_x^2 J_y + 2J_x J_y J_y)) - i (2J_x J_y + 2J_y^2 + J_x J_y J_z))
$$

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Appendix B: Direct Product of two matrices

If the matrix $A$ is $N$ dimensional and the matrix $B$ is $n$ dimensional, then the direct product $A \otimes B$ is $N \times n$ dimensional. Eg.,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

then

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & a_{13}B \\ a_{21}B & a_{22}B & a_{23}B \\ a_{31}B & a_{32}B & a_{33}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} & a_{13}b_{11} & a_{13}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} & a_{13}b_{21} & a_{13}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} & a_{23}b_{11} & a_{23}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} & a_{23}b_{21} & a_{23}b_{22} \\ a_{31}b_{11} & a_{31}b_{12} & a_{32}b_{11} & a_{32}b_{12} & a_{33}b_{11} & a_{33}b_{12} \\ a_{31}b_{21} & a_{31}b_{22} & a_{32}b_{21} & a_{32}b_{22} & a_{33}b_{21} & a_{33}b_{22} \end{pmatrix}.$$

- If $A$ and $C$ are $m \times m$ matrices and $B$ and $D$ are $n \times n$ matrices, then $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

- The operation of direct product is associative. i.e., $(A \otimes B) \otimes C = A \otimes (B \otimes C) = A \otimes B \otimes C$.

- This operation is also distributive. i.e., $A \otimes (B + C) = A \otimes B + A \otimes C$. And

- $(A \otimes B)$ follows $tr(A \otimes B) = trA \cdot trB$. 

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Appendix C: Majorana representation for pure N-qubit symmetric state

Since Majorana polynomial is elegantly derived by Usha Devi et al. (2011), we reproduce it here for the sake of completeness. A pure spin \( j = \frac{N}{2} \) quantum state can be represented as a symmetrized combination of N constituent spinors. i.e.,

\[
|\psi_{\text{sym}}\rangle = \mathcal{N} \sum_{P} \hat{P}\{|\epsilon_1, \epsilon_2, \ldots \epsilon_N\}\rangle,
\]

where

\[
|\epsilon_l\rangle = \cos\left(\frac{\beta_l}{2}\right)e^{-i\alpha_l/2}|0\rangle + \sin\left(\frac{\beta_l}{2}\right)e^{i\alpha_l/2}|1\rangle,
\]

and \( l = 0,1,2,\ldots N \), denote the spinors constituting the symmetric state \(|\psi_{\text{sym}}\rangle\); \( \hat{P} \) corresponds to the set of all \( N! \) permutations of the spinors and \( \mathcal{N} \) corresponds to an overall normalization factor. Also an arbitrary pure symmetric state of \( N \) qubits obeying exchange symmetry may be expressed as

\[
|\psi_{\text{sym}}\rangle = \sum_{l=0}^{N} c_l \left| \frac{N}{2}, l - \frac{N}{2} \right\rangle,
\]

and is completely specified by the \( (N+1) \) complex coefficients \( c_l \).

A symmetric pure state is transformed into another symmetric pure state under identical rotations \( R \otimes R \otimes \ldots \otimes R \) on all spinors of equation (1). Under identical rotation through \( R^{-1}(\alpha_s, \beta_s, 0) \otimes R^{-1}(\alpha_s, \beta_s, 0) \otimes \ldots \) where \( \alpha_s, \beta_s \) correspond to the orientation of any one of the spinors in equation (1), it may be identified that

\[
\langle 1, 1, \ldots 1 | R^{-1}(\alpha_s, \beta_s, 0) \otimes R^{-1}(\alpha_s, \beta_s, 0) \otimes \ldots |\psi_{\text{sym}}\rangle \equiv 0.
\]

Here the rotation \( R^{-1}_s \otimes R^{-1}_s \ldots \otimes R^{-1}_s \) takes one of the spinors \(|\epsilon_s\rangle\) with orientation angles \( \alpha_s, \beta_s \) to \(|0\rangle\) i.e., it aligns the spinor \(|\epsilon_s\rangle\) in the positive z-direction. Then every term in the superposition of the rotated state has at least one \(|0\rangle\) and so, the projection \( \langle 1, 1, \ldots, 1 | R^{-1}_s \otimes R^{-1}_s \otimes \ldots R^{-1}_s |\psi_{\text{sym}}^{(N)}\rangle \) of the rotated state in the ‘all-down’ direction vanishes.
The above equation holds good for collective rotations $R_s^{-1} = R_s^{-1} \otimes R_s^{-1} \ldots \otimes R_s^{-1}$, $s = 1, 2, \ldots, N$, which orient any one of the constituent spinors $|\epsilon_s\rangle$ in the positive z-direction. In other words there exist $N$ rotations in general which lead to the same above result.

In terms of alternate representation of the symmetric state $|\psi_{sym}\rangle$, the above equation leads to

$$\langle \frac{N}{2}, -\frac{N}{2} | R^{-1}(\alpha_s, \beta_s, 0) \otimes R^{-1}(\alpha_s, \beta_s, 0) \ldots \otimes R^{-1}(\alpha_s, \beta_s, 0) | \psi_{sym}\rangle = 0$$

$$\implies \langle \frac{N}{2}, -\frac{N}{2} | R^{-1}(\alpha_s, \beta_s, 0) \{ \sum_{l=0}^{N} c_l |\frac{N}{2}, l - \frac{N}{2}\rangle \} = 0$$

i.e.,

$$\sum_{l=0}^{N} c_l D_{l-N/2,-N/2}^{N/2^*}(\alpha_s, \beta_s, 0) = 0,$$

where $R^{-1}(\alpha_s, \beta_s, 0) \otimes R^{-1}(\alpha_s, \beta_s, 0) \ldots \otimes R^{-1}(\alpha_s, \beta_s, 0) = R_s^{-1}(\alpha_s, \beta_s, 0)$ in the collective $(N+1)$ dimensional symmetric subspace of $N$ qubits and

$$[D^{N/2^*}]_{-N/2,N/2} = \langle \frac{N}{2}, -\frac{N}{2} | R^{-1}_{l-N/2} | \frac{N}{2}, l - \frac{N}{2}\rangle$$

represents the collective rotation in the Wigner-D representation [Rose (1957)]. Substituting the explicit form of the D-matrix,

$$[D^{N/2^*}]_{-N/2,N/2} = D^{N/2^*}_{l-N/2,-N/2}(\alpha, \beta, 0) = \sqrt{NC_l} [\cos \left(\frac{\beta}{2}\right)]^{N-l} [-\sin \left(\frac{\beta}{2}\right)]^l e^{i(l-\frac{N}{2})\alpha},$$

and on subsequent simplication one gets,

$$A \sum_{l=0}^{N} (-1)^l \sqrt{NC_l} c_l z^l = 0$$

where $z = tan \left(\frac{\beta}{2}\right) e^{i\alpha}$ and the overall coefficient $A = \cos^N \left(\frac{\beta}{2}\right) e^{-i\alpha N/2}$.

In other words, the $N$-roots $z_l = tan \left(\frac{\beta_l}{2}\right) e^{i\alpha_l}$, $l = 1, 2, \ldots, N$ of the Majorana polynomial $P(z)$

$$P(z) = \sum_{l=0}^{N} (-1)^l \sqrt{NC_l} c_l z^l$$

determine the orientations $(\alpha_l, \beta_l)$ of the spinors constituting the $N$-qubit symmetric state, in terms of collective parameters $c_l$. 121
Appendix D: Linearly independent basis operators

$O_i'$s in qubit basis where $i = 1...16$.

The following 16 operators can be constructed from the $I, \sigma_i$'s and $\tau_i$'s, ($i = x,y,z$) where $I$ is the $2 \times 2$ identity matrix, $\sigma_i, \tau_i$'s are the Pauli spin matrices.

\[
O_1 = I \otimes I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad O_2 = \frac{\sigma_z \otimes I_2}{2} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
\]

\[
O_3 = \frac{\tau_z \otimes I_2}{2} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad O_4 = \frac{\sigma_x \otimes \tau_z}{4} = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\[
O_5 = \frac{\sigma_x \otimes I_2}{2} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad O_6 = \frac{\sigma_y \otimes I_2}{2} = \frac{i}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]

\[
O_7 = \frac{\sigma_x \otimes \tau_z}{4} = \frac{1}{4} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad O_8 = \frac{\sigma_y \otimes \tau_z}{4} = \frac{i}{4} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},
\]

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\[ O_9 = \frac{\tau_x \otimes I_2}{2} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad O_{10} = \frac{\tau_y \otimes I_2}{2} = \frac{i}{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \]

\[ O_{11} = \frac{\sigma_z \otimes \tau_x}{4} = \frac{1}{4} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad O_{12} = \frac{\sigma_z \otimes \tau_y}{4} = \frac{i}{4} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \]

\[ O_{13} = \frac{\sigma_x \otimes \tau_x}{4} = \frac{1}{4} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad O_{14} = \frac{\sigma_z \otimes \tau_y}{4} = \frac{i}{4} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \]

\[ O_{15} = \frac{\sigma_x \otimes \tau_y}{4} = \frac{i}{4} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad O_{16} = \frac{\sigma_y \otimes \tau_x}{4} = \frac{i}{4} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]
Appendix E: Gell-mann matrices

\[ \Lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ \Lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \Lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \]

\[ \Lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \Lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \]

Paul Roman representation of Hermitian matrices

\[ \rho_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \]

\[ \rho_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \rho_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \]

\[ \rho_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \rho_7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho_8 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}. \]
Appendix F: Multivariate analysis

Multivariate analysis [Anderson (1958); Eaton (1983); Rao (1973, 1974)] is an important branch of statistical theory, which is concerned with the analysis of observations on several correlated random variables for a number of individuals, the observations on different individuals being assumed to be independent. When one is interested in studying several variates simultaneously one has to consider their joint distribution and the various marginal and conditional distributions that can be obtained from it. If \( \vec{X} \) denotes the column vector of \( p \)-variates \( X_1, X_2, \ldots, X_p \) having a continuous distribution and \( f(X_1, X_2, \ldots, X_p) \equiv f(\vec{X}) \) denotes the probability density function (p.d.f) of this joint distribution, the p.d.f of the marginal distribution of one or more of the variates, say, \( X_1, X_2, \ldots, X_k, k < p \) is obtained through

\[
 f_m(X_1, X_2, \ldots, X_k) = \int dX_{k+1} \ldots \int dX_p f(X_1, X_2, \ldots, X_p),
\]

while the p.d.f of the conditional distribution of

\[
 f_c(X_{k+1}, \ldots, X_p/X_1, X_2, \ldots, X_k) = f(X_1, \ldots, X_p)/f_m(X_1, \ldots, X_k).
\]

The expectation value of \( \vec{X} \) is given by the column vector \( E(\vec{X}) \) with elements \( E(X_1), \ldots, E(X_p) \) where

\[
 E(X_1) = \int X_1 f_m(X_1) dX_1 = \mu^1(X_1),
\]

while

\[
 E(X_1^{h_1}, X_2^{h_2}, \ldots, X_p^{h_p}) = \int dX_1 \ldots \int dX_p X_1^{h_1} X_2^{h_2} \ldots X_p^{h_p} f(\vec{X}) = \mu^{h_1 h_2 \ldots h_p}(\vec{X}),
\]

defines the various joint moments associated with the multivariate distribution.

The moment generating function \( M_x(I) \), where \( \vec{I} = (I_1 \ldots I_p) \) is a real vector, is given by

\[
 M_x(I) = E(e^{\vec{I} \cdot \vec{X}}) = 1 + \sum_{h_1 h_2 \ldots h_p} (h_1! \ldots h_p!)^{-1} I_1^{h_1} I_2^{h_2} \ldots I_p^{h_p} \mu^{h_1 h_2 \ldots h_p}(\vec{X}).
\]

\(^1\)Taken from Doctoral thesis, Swarnamala Sirsi, submitted to University of Mysore, Mysore (1995)
The characteristic function $\Phi_x(\vec{I})$ is defined through

$$\Phi_x(\vec{I}) = E(e^{i\vec{I}.\vec{X}}) = 1 + \sum_{h_1 h_2 \cdots h_p} (i)^{h_1 + \cdots + h_p} (h_1! \cdots h_p!)^{-1} I_1^{h_1} \cdots I_p^{h_p} \mu^{h_1 h_2 \cdots h_p}(\vec{X}).$$

The variance-covariance matrix $\sum(\vec{X})$ of $\vec{X}$ is defined through its elements

$$\sigma_{ij} = E\{[X_i - E(X_i)][X_j - E(X_j)]\} = \mu^{11}(X_i, X_j) - \mu^1(X_i)\mu^1(X_j)$$

where the diagonal elements $i = j$ denote the variance

$$\sigma_{ii} = V_i = E(X_i^2) - E(X_i)^2 = \mu^2(X_i) - \mu^1(X_i)^2 = \sigma_i^2$$

and $\sigma_i$ denotes the standard deviation associated with $X_i$.

In the case of discrete distributions, probability density is replaced by probability mass function and the various integrations are correspondingly replaced by the appropriate summations. In contrast to the probabilistic descriptions of observables like position or momentum in quantum theory, spin $\vec{J}$ (which may be described in a deterministic manner in classical physics by specifying simultaneously the three components $J_x, J_y, J_z$ whose values can lie anywhere on the continuous real line) undergoes more drastic changes in its quantum descriptions:

- $J^2 = \hat{J}.\hat{J}$ is restricted to discrete values $j(j+1), j = 1, 1/2, 1, \ldots.$

- The three components cannot simultaneously assume eigen values, but only one amongst them say $\hat{J}_z$ along with $\hat{J}^2$.

- this component can assume only $(2j+1)$ discrete values $m = j, j-1, \ldots, -j$.

Therefore the probability distribution $p(m)$, which is realized in the case of oriented systems [Blin-Stoyle & Grace (1957)] is the closest to statistical theory, when we identify $m$ as the
discrete set of values that \( \mathbf{J}_z \) considered as a random variable [Parthasarathy (1988)] can take. Although it is not possible to identify a set of values simultaneously for all or even any two amongst \( \mathbf{J}_x \), \( \mathbf{J}_y \), \( \mathbf{J}_z \) when they appear to get featured as random variables in the case of non-oriented systems [Ramachandran & Murthy (1979)], it is heartening to realize that the moments \( \mu^{h_1,h_2,h_3}(\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z) \) can be determined directly from experiment since the joint moments can be uniquely determined in terms of the Fano statistical tensors \( t^k_q \) and vice-versa for all \( j \).

Thus the problem of finding a quantum mechanical probability distribution \( f \) reduces to finding the quantum mechanical equivalent of the classical quantity \( e^{i(\theta q + \tau p)} \) or \( q^m p^n \). Owing to the non-commutative nature of the quantum mechanical observables, one can realize several quantum mechanical definitions of the characteristic function, based on different operator orderings [Cohen (1966); Hillery et al. (1984); Misra & Shankara (1968)] and each of these definitions correspond to different probability distributions. Rules which associate classical quantities to quantum mechanical operators are called correspondence rules or rules of association. Cohen (1966) has proposed five such rules. They are

- Dirac’s rule of associating commutators with Poisson brackets
- Von Neumann’s rules
- Weyl’s rule
- Symmetrization rule of Margenau and Hill[1961]
- Rule of Born and Jordan[1925],

of which we are interested in Weyl’s rule which corresponds to Wigner-Weyl distributions.

**Wigner-Weyl operator ordering:**

The Weyl correspondence for any quantum mechanical operator \( \mathbf{A} \) corresponding to the
classical function $A(q,p)$ can be stated as

$$Tr(\rho A) = \int dq \int dp f_w(q,p) A(q,p).$$

Here it has been assumed that the distribution function $f_w(q,p)$ describing the system corresponds to $\rho$. Instead of $A$ if we consider the operator $e^{i(\theta q + \tau p)}$ then

$$Tr(\rho e^{i(\theta q + \tau p)}) = \Phi(\theta, \tau)$$

which is the characteristic function as given by Moyal (1949). It is well known that in classical probability theory, the mixed moments are given by

$$\mu^{mn}(q,p) = E\left[ \frac{1}{(i)^{m+n}} \frac{\partial^m}{\partial \theta^m} \frac{\partial^n}{\partial \tau^n} e^{i(\theta q + \tau p)} \right]_{\theta = \tau = 0}$$

$$= \int dq \int dp f_w(q,p) q^m p^n.$$

In quantum mechanics, we have

$$\mu^{mn}_Q(q,p) = Tr[\rho \frac{1}{(i)^{m+n}} \frac{\partial^m}{\partial \theta^m} \frac{\partial^n}{\partial \tau^n} e^{i(\theta q + \tau p)}]_{\theta = \tau = 0}$$

$$= Tr[\rho \left\{ \frac{1}{2^n} \sum_{l=0}^{n} \binom{n}{l} q^{n-l} p^m q^l \right\}].$$

Thus we have the Wigner-Weyl correspondence rule

$$q^m p^n \rightarrow \frac{1}{2^n} \sum_{l=0}^{n} \binom{n}{l} q^{n-l} p^m q^l.$$
Appendix G: SLOCC classification of symmetric multiqubit pure states

For N-qubit permutation symmetric pure states $|\phi\rangle$ and $|\psi\rangle$, if $|\phi\rangle = A \otimes A \otimes \ldots \otimes N$ times $|\psi\rangle$ where $A$ is an invertible local operation (ILO), then $|\phi\rangle$ and $|\psi\rangle$ are said to be equivalent under stochastic local operation and classical communication (SLOCC). Bastin et al. (2009) have solved the entanglement classification under SLOCC for all multipartite symmetric states in the general N-qubit case. For this purpose they introduce two parameters playing a crucial role, namely the diversity degree and the degeneracy configuration of a symmetric state. The roots of Majorana polynomial can be degenerate and hence not all the N-constituent spinors of a pure symmetric N-qubit states are distinct. Let $|\epsilon_1\rangle, |\epsilon_2\rangle, \ldots, |\epsilon_d\rangle$, $d \leq N$ be the number of distinct spinors in a N-qubit symmetric state, then the numbers

$$\{n_1, n_2, \ldots, n_d; n_1 \geq n_2 \geq \ldots \geq n_d; n_1 + n_2 + \ldots + n_d = N\}$$

correspond respectively to the number of times the independent spinors $|\epsilon_i\rangle, (i = 1, 2, \ldots, d \leq N)$ appear in the symmetric state under consideration. The number $d \leq N$, which is called the diversity degree and the list of numbers $\{n_1, n_2, \ldots, n_d; n_1 \geq n_2 \geq \ldots \geq n_d; \sum_{i=1}^{d} n_i = N\}$ which is called the degeneracy configuration, form the key elements in the classification of pure symmetric states. The different classes (based on the number of distinct spinors and their arrangement in a given N-qubit symmetric state) are denoted by $\{D_{n_1, n_2, \ldots, n_d}\}$. An identical ILO $A^\otimes N$ transforms a symmetric state belonging to the class $\{D_{n_1, n_2, \ldots, n_d}\}$ to another state of the same class. More explicitly,

$$|D_{n_1, n_2, \ldots, n_d}\rangle \xrightarrow{ILO} |D'_{n_1, n_2, \ldots, n_d}\rangle = A^\otimes N |D_{n_1, n_2, \ldots, n_d}\rangle$$

with the constituent spinors transforming as $|\epsilon'\rangle = A|\epsilon\rangle$, $i = 1, 2, \ldots, d$. This forms the main basis of the SLOCC classification of symmetric pure states.

\(^2\)refer appendix C
1. \(\{D_N\}\): When all the N solutions of the Majorana polynomial are identically equal, the corresponding class of symmetric states is given by

\[|D_N\rangle = |\epsilon, \epsilon, ..., \epsilon\rangle,\]

where the diversity degree \(d = 1\). The states belonging to this family of separable symmetric states is denoted by \(|D_N\rangle\).

2. \(\{D_{n_1,n_2}; n_1 = N - k, n_2 = k = 1, 2, ..., [N/2]\}\). The states with distinct spinors have the form,

\[|D_{N-k,k}\rangle = \mathcal{N}[|\epsilon_1, \epsilon_1, ..., \epsilon_2, \epsilon_2, ...\rangle + \text{Permutations}]\]

where \(k = 1, 2, ..., N/2\). Dicke states \(|N/2, k - N/2\rangle\) are the representative states of the entanglement class \(\{D_{N-k,k}\}\) with two independent spinors and clearly, they are all inequivalent under SLOCC.

3. \(\{D_{1,1,1,...}\}\): When the N roots of the Majorana polynomial are all distinct, the pure symmetric states constitute the class \(\{D_{1,1,1,...}\}\) with diversity degree \(d = N\). The N qubit GHZ state is a representative of this entanglement class.