Chapter 6

Copula based reliability concepts

6.1 Introduction

A brief review of the works pertaining to the extension of quantiles to higher dimensions was discussed in section 2.7. Let \( F_1(x_1) \) and \( F_2(x_2) \) be marginal distribution functions of \( X_1 \) and \( X_2 \) respectively. Suppose \( F_1(x_1) = u \) so that \( x_1 = F_1^{-1}(u) = \phi(u) \). Also \( x_2 = F_2^{-1}(v) = \psi(v) \). We have \( \partial x_1 = \phi'(u) \partial u \) and \( \partial x_2 = \psi'(v) \partial v \) where \( \phi'(u) = \frac{d\phi(u)}{du} \) and \( \psi'(v) = \frac{d\psi(v)}{dv} \).

Proceeding on the lines initiated by Belzunce et al. (2007), the probability \( F_{\varepsilon}(\phi(u), \psi(v)) \) depends only on the copula \( C \) for the direction \( \varepsilon \) as detailed below,

\[
F_{\varepsilon}(\phi(u), \psi(v)) = \begin{cases} 
C(u, v); & \varepsilon = \varepsilon_{--} \\
u - C(u, v); & \varepsilon = \varepsilon_{-+} \\
v - C(u, v); & \varepsilon = \varepsilon_{+-} \\
1 - u - v - C(u, v); & \varepsilon = \varepsilon_{++}
\end{cases}
\]

0 \leq u \leq 1; 0 \leq v \leq 1.

For the bivariate random vector \( X \), the joint distribution function is

\[
F_{\varepsilon_{--}}(\phi(u), \psi(v)) = C(u, v). \quad (6.1)
\]
Denote the joint survival function by \( \bar{F}_{x-x} (x_1, x_2) \) and the univariate survival functions by \( \bar{F}_1 (x_1) \) and \( \bar{F}_2 (x_2) \) respectively. Then consider

\[
\bar{F}_{x-x} (x_1, x_2) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2))
\]

where \( \hat{C} \) is the survival copula. That is,

\[
\bar{F}_{x-x} (x_1, x_2) = \hat{C}(1 - u, 1 - v).
\] (6.2)

Nelsen (1999) describes the relation between \( C \) and \( \hat{C} \) as

\[
\hat{C}(1 - u, 1 - v) = 1 - u - v - C(u, v).
\] (6.3)

Also \( C \) satisfies the following properties, \( C(u, 1) = u \) and \( C(1, v) = v \).

Nair & Sankaran (2009) defines the hazard quantile function of the random variables \( X_1 \) and \( X_2 \) as

\[
h_1(u) = \frac{1}{\phi'(u)du}
\]

and

\[
h_2(v) = \frac{1}{\psi'(v)dv}.
\]

Also the mean residual quantile functions of \( X_1 \) and \( X_2 \) are defined as

\[
m_1(u) = \frac{1}{1 - u} \int_u^1 \phi(p)dp - \phi(u)
\]

and

\[
m_2(v) = \frac{1}{1 - v} \int_v^1 \psi(p)dp - \psi(v).
\]
In the reversed setup, the reversed hazard quantile function and reversed mean residual quantile functions are defined for the random variables $X_1$ and $X_2$ as

$$a_1(u) = \frac{1}{u \phi'(u)},$$

$$a_2(v) = \frac{1}{v \psi'(v)},$$

$$r_1(u) = \phi(u) - \frac{1}{u} \int_0^u \phi(p) dp,$$

and

$$r_2(v) = \psi(v) - \frac{1}{v} \int_0^v \psi(p) dp.$$

In the following we provide a bivariate extension of the basic quantile based reliability concepts given in Nair & Sankaran (2009) using the copula. Also we look for possible relationships connecting the bivariate concepts and use the same to derive various characterization theorems for bivariate distributions.

The extension of univariate quantile based reliability concepts to the bivariate setup can be done only on the basis of the four directions. Here we consider the bivariate copula based definitions of reliability concepts in the direction $\varepsilon_{--}$.

We define the bivariate hazard rate(bivariate reversed hazard rate) and bivariate mean residual life(bivariate reversed mean residual life) in a copula setup for the direction $\varepsilon_{--}$ and study their relationships. We also check for the independence property of the concepts using the product copula.
6.2 Bivariate copula based reliability concepts

Different versions of bivariate hazard rate are discussed in Basu (1971), Cox (1992), Marshall (1975), Shaked & Shanthikumar (2007), Sun & Basu (1995) and Finkelstein (2003). However the most commonly used bivariate hazard rate is the vector valued hazard rate defined in Johnson & Kotz (1975). For a bivariate random vector absolutely continuous distribution function with survival function $\bar{F}(x)$, the hazard rate is defined as the vector

$$H(x) = [H_1(x), H_2(x)]$$

where

$$H_i(x) = -\frac{\partial}{\partial x_i} \log \bar{F}(x); \ i = 1, 2. \quad (6.4)$$

An analogous expression of $H(x)$ in the direction $\varepsilon_{--}$ in terms of copula is given by

$$h_{\varepsilon_{--}}(u,v) = (h_1(u,v), h_2(u,v))$$

where

$$h_1(u,v) = H_1[\phi(u), \psi(v)] = -\frac{1}{\phi'(u)} \frac{\partial}{\partial u} \log \hat{C}(1-u,1-v). \quad (6.5)$$

Similarly

$$h_2(u,v) = -\frac{1}{\psi'(v)} \frac{\partial}{\partial v} \log \hat{C}(1-u,1-v). \quad (6.6)$$

This is obtained as follows. Since $x_1 = \phi(u)$ and $x_2 = \psi(v)$, we have $\partial x_1 = \phi'(u)\partial u$. Thus $\frac{\partial x_1}{\partial u} = \phi'(u)$. Also we have $\partial x_2 = \psi'(v)\partial v$. This gives $\frac{\partial x_2}{\partial v} = \psi'(v)$.

Substituting (6.2) in the expression (6.4), we get (6.5) and (6.6)

**Remark 6.1**

When $X_1$ and $X_2$ are independent, the bivariate hazard rate reduces to a vector with components equal to the hazard rates of $X_1$ and $X_2$. As an example, when $\hat{C}(1-u,1-v) = (1-u)(1-v)$,
we have
\[ \frac{\partial}{\partial u} \log \left( \hat{C}(1-u,1-v) \right) = \frac{-1}{1-u}, \]
and from (6.5), we get
\[ h_1(u, v) = \frac{1}{(1-u)\phi'(u)} = h_1(u). \]
Also we have
\[ \frac{\partial}{\partial v} \log \left( \hat{C}(1-u,1-v) \right) = \frac{-1}{1-v}. \]
Therefore (6.6) yields
\[ h_2(u, v) = \frac{1}{(1-v)\psi'(v)} = h_2(v). \]
Thus we get,
\[ h_{\mathcal{E}_{--}}(u, v) = (h_1(u), h_2(v)) = (h_1(u,0), h_2(0,v)). \]

**Example 6.1**
Consider the Gumbel bivariate exponential distribution with survival copula specified by
\[ \hat{C}(u,v) = uve^{-\theta \ln u \ln v} \]
with univariate marginals \( \phi(u) = -\ln(1-u) \) and \( \psi(v) = -\ln(1-v) \) respectively. Direct calculations yield the bivariate hazard rate as
\[ h_{\mathcal{E}_{--}}(u, v) = (\theta \ln(1-v) - 1, \theta \ln(1-u) - 1). \]

Next theorem discusses the uniqueness property of the bivariate hazard function.

**Theorem 6.1.** For the bivariate random vector \( X \), copula \( C(u, v) \) can be expressed uniquely in
Chapter 6. Copula based reliability concepts

118

terms of \( h_{\varepsilon}(u, v) \) through

\[
C(u, v) = (1 - v)(1 - e^{\int_0^u h_1(p, v) (1 - p) h_1(p, 0) dp}) - u
\]

(6.7)

and

\[
C(u, v) = (1 - u)(1 - e^{\int_0^v h_2(u, p) (1 - p) h_2(0, p) dp}) - v.
\]

(6.8)

Proof. From (6.5), we have

\[
h_1(u, v) \phi'(u) = - \frac{\partial}{\partial u} \log \hat{C}(1 - u, 1 - v).
\]

Integrating the above expression from 0 to \( u \), we get

\[
\int_0^u h_1(p, v) \phi'(p) dp = \log \frac{1 - v}{C(1 - u, 1 - v)}
\]

or

\[
\hat{C}'(1 - u, 1 - v) = (1 - v)e^{-\int_0^u h_1(p, v) \phi'(p) dp}.
\]

(6.9)

Since \( \phi'(u) = \frac{1}{(1 - u)h_1(u, 0)} \), (6.9) gives

\[
\hat{C}'(1 - u, 1 - v) = (1 - v)e^{-\int_0^u \frac{h_1(p, v)}{(1 - p)h_1(p, 0)} dp}.
\]

(6.10)

Using (6.3), (6.10) can be written as

\[
C(u, v) = (1 - v)(1 - e^{\int_0^u \frac{h_1(p, v)}{(1 - p)h_1(p, 0)} dp}) - u.
\]

Also \( h_2(u, v) \) characterizes the distribution specified by the copula,

\[
\hat{C}'(1 - u, 1 - v) = (1 - u)e^{-\int_0^u \frac{h_2(u, p)}{(1 - p)h_2(0, p)} dp}.
\]
(6.8) follows from the above equation.

Alternate definitions for the bivariate mean residual life function is provided independently by Shanbhag & Kotz (1987) and Arnold & Zahedi (1988). Consider the variable

\[ X_{x_i} = X_i - x_i | X_i > x_i, X_j > x_j; i, j = 1, 2; i \neq j. \]

The bivariate mean residual life function of the variable \( X_{x_i} \) is defined as the vector

\[ \mathbf{M}(x) = \left[ M_1(x), M_2(x) \right] \]

where

\[ M_1(x) = \frac{1}{F(x)} \int_{x_1}^{\infty} \bar{F}(t, x_2) dt \]

and

\[ M_2(x) = \frac{1}{F(x)} \int_{x_2}^{\infty} F(x_1, t) dt. \]

On simplification, \( M_1(x) \) and \( M_2(x) \) becomes

\[ M_1(x) = -x_1 - \frac{1}{F(x)} \int_{x_1}^{\infty} t \frac{\partial \bar{F}(t, x_2)}{\partial t} dt \]

and

\[ M_2(x) = -x_2 - \frac{1}{F(x)} \int_{x_2}^{\infty} t \frac{\partial F(x_1, t)}{\partial t} dt. \]

Substituting \( x_1 = \phi(u), x_2 = \psi(v) \) and \( \bar{F}(x) = \hat{C}(1 - u, 1 - v) \) in the above expressions, we get the copula analogue of bivariate mean residual functions as

\[ m_{\varepsilon_{--}}(u, v) = (m_1(u, v), m_2(u, v)) \]
where

\[ m_1(u, v) = M_1[\phi(u), \psi(v)] = -\phi(u) - \frac{1}{C(1-u, 1-v)} \int_u^1 \phi(p) \frac{\partial}{\partial p} \hat{C}(1-p, 1-v) dp \quad (6.12) \]

and

\[ m_2(u, v) = M_2[\phi(u), \psi(v)] = -\psi(v) - \frac{1}{C(1-u, 1-v)} \int_v^1 \psi(p) \frac{\partial}{\partial p} \hat{C}(1-u, 1-p) dp. \quad (6.13) \]

**Theorem 6.2.** Let \( X \) be a bivariate random vector admitting an absolutely continuous distribution function and bivariate copula mean residual life function defined by (6.11). Then the bivariate copula mean residual life function determines the underlying copula uniquely.

**Proof.** Differentiating (6.12) with respect to \( u \), we get

\[ \frac{\partial}{\partial u} \hat{C}(1-u, 1-v) = \frac{-\partial m_1(u, v) - \phi'(u)}{m_1(u, v)} \quad (6.14) \]

or

\[ \frac{\partial}{\partial u} \log \hat{C}(1-u, 1-v) = \frac{-\partial m_1(u, v) - \phi'(u)}{m_1(u, v)}. \]

That is,

\[ \frac{\partial}{\partial u} \log \left( \hat{C}(1-u, 1-v) \right) = -\frac{\partial}{\partial u} \log (m_1(u, v)) - \frac{\phi'(u)}{m_1(u, v)} \quad (6.15) \]

Nair & Sankaran (2009) has shown that the univariate mean residual quantile function uniquely determines the quantile function through the expression

\[ \phi(u) = \mu_1 - m_1(u) + \int_0^u \frac{m_1(p)}{1-p} dp. \quad (6.16) \]
On integration from 0 to \( u \) and using (6.16), equation (6.15) becomes

\[
\log \left( \frac{\hat{C}(1 - u, 1 - v)}{1 - v} \right) = \log \left( \frac{m_1(0, v)}{m_1(u, v)} \right) - \int_0^u \frac{\partial m_1(p, 0)}{\partial p} + \frac{m_1(p, 0)}{1-p} - m_1(0, 0) \frac{d}{d p} m_1(p, v) \, dp. \tag{6.17}
\]

That is,

\[
\hat{C}(1 - u, 1 - v) = \frac{(1 - v)m_1(0, v)}{m_1(u, v)} \exp \left[ - \int_0^u \frac{\partial m_1(p, 0)}{\partial p} + \frac{m_1(p, 0)}{1-p} - m_1(0, 0) \frac{d}{d p} m_1(p, v) \, dp \right]
\]

or

\[
C(u, v) = 1 - u - v - \frac{(1 - v)m_1(0, v)}{m_1(u, v)} \exp \left[ - \int_0^u \frac{\partial m_1(p, 0)}{\partial p} + \frac{m_1(p, 0)}{1-p} - m_1(0, 0) \frac{d}{d p} m_1(p, v) \, dp \right].
\]

Also \( m_2(u, v) \) uniquely determines the underlying copula \( C(u, v) \) as

\[
C(u, v) = 1 - u - v - \frac{(1 - u)m_2(u, 0)}{m_2(u, v)} \exp \left[ - \int_0^v \frac{\partial m_2(0, p)}{\partial p} + \frac{m_2(0, p)}{1-p} - m_2(0, 0) \frac{d}{d p} m_2(u, p) \, dp \right].
\]

The theorem is immediate from the above expression for \( C(u, v) \). \( \square \)

**Remark 6.2**

For the product copula given by

\[
C(u, v) = uv,
\]

\( \hat{C} \) takes the form

\[
\hat{C}(1 - u, 1 - v) = (1 - u)(1 - v).
\]
Using the above expression for $\hat{C}$, (6.12) and (6.13) become

$$m_1(u,v) = \frac{1}{1-u} \int_u^1 \phi(p) dp - \phi(u) = m_1(u)$$

and

$$m_2(u,v) = \frac{1}{1-v} \int_v^1 \psi(p) dp - \psi(v) = m_2(v).$$

Therefore

$$m_{\varepsilon_{-\varepsilon}}(u,v) = (m_1(u), m_2(v))$$

The implication of the above is that when we consider the product copula, the corresponding bivariate copula based mean residual function in the direction $\varepsilon_{-\varepsilon}$ becomes the vector with components, the marginal mean residual quantile functions of $X_1$ and $X_2$ respectively.

Next theorem provides a relationship connecting the bivariate copula based hazard rate and mean residual life functions.

**Theorem 6.3.** For the random vector $X$ defined in theorem 6.2, the bivariate copula hazard rate is related to bivariate copula mean residual life function through the relationship

$$h_1(u,v)m_1(u,v) = (1-u)h_1(u,0) \frac{\partial}{\partial u} m_1(u,v) + 1$$

and

$$h_2(u,v)m_2(u,v) = (1-v)h_2(0,v) \frac{\partial}{\partial v} m_2(u,v) + 1.$$

**Proof.** From (6.14) and (6.5), we get

$$-h_1(u,v)\phi'(u) = \frac{-\frac{\partial}{\partial u} m_1(u,v) - \phi'(u)}{m_1(u,v)}.$$
The above equation can be written as

$$h_1(u,v) = \frac{\frac{\partial}{\partial u} m_1(u,v) + \phi'(u)}{m_1(u,v) \phi'(u)}.$$  \hfill (6.18)

Since $\frac{1}{\phi'(u)} = (1 - u)h_1(u,0)$, the above equation becomes,

$$h_1(u,v)m_1(u,v) = (1 - u)h_1(u,0)\frac{\partial}{\partial u} m_1(u,v) + 1.$$

Proceeding on similar lines, we get

$$h_2(u,v)m_2(u,v) = (1 - v)h_2(0,v)\frac{\partial}{\partial v} m_2(u,v) + 1.$$

The expression for the joint density function of $X$ is given by

$$f(x) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x).$$

That is,

$$f(x) = \frac{\partial}{\partial x_2} \left[ \frac{\partial}{\partial x_1} F(x) \right].$$ \hfill (6.19)

The copula analogue of (6.19) is given by

$$f [\phi(u), \psi(v)] = \frac{1}{\phi'(u) \psi'(v)} \frac{\partial}{\partial u \partial v} C(u,v).$$

Define

$$\mu = (\mu_1(v), \mu_2(u)).$$
where
\[
\mu_1(v) = \int_0^1 \phi(u) \frac{\partial}{\partial u} \hat{C}(1 - u, 1 - v) du
\]
and
\[
\mu_2(u) = \int_0^1 \psi(v) \frac{\partial}{\partial v} \hat{C}(1 - u, 1 - v) dv.
\]

The bivariate copula vitality function can be defined as
\[
d_{\varepsilon - \varepsilon}(u, v) = [d_1(u, v), d_2(u, v)]
\]
where
\[
d_1(u, v) = m_1(u, v) + \phi(u)
\]
and
\[
d_2(u, v) = m_2(u, v) + \psi(v).
\]

That is,
\[
d_1(u, v) = \frac{-1}{\hat{C}(1 - u, 1 - v)} \int_u^1 \phi(p) \frac{\partial}{\partial p} \hat{C}(1 - p, 1 - v) dp
\]
(6.20)
and
\[
d_2(u, v) = \frac{-1}{\hat{C}(1 - u, 1 - v)} \int_v^1 \psi(p) \frac{\partial}{\partial p} \hat{C}(1 - u, 1 - p) dp.
\]
(6.21)

Following result focuses attention on characterization of copulas using the relation between bivariate hazard function and bivariate vitality function.

**Theorem 6.4.** Let \(g(., .)\) be any positive real valued function and \(d_{\varepsilon - \varepsilon}(u, v)\) be the bivariate vitality function. Then the relationship

\[
h_1(u, v)g[\phi(u), \psi(v)] = d_1(u, v) - \mu_1(v)
\]
(6.22)
holds if
\[ \frac{\partial}{\partial u} \hat{C}(1 - u, 1 - v) = \frac{B^*(v)\phi'(u)}{g[\phi(u), \psi(v)]\phi'(0)} e^{-\int_0^u \frac{\phi'(p)[\phi(p) + \mu_1(u,v)]}{g[\phi(p), \psi(v)]} dp} \]  
(6.23)

where \( B^*(v) = g[0, \psi(v)] k(v) \) in which \( k(v) = \frac{\partial}{\partial u} \hat{C}'(1 - u, 1 - v) \) given \( u = 0 \) provided \( \phi'(0) \neq 0, g[0, \psi(v)] \neq 0 \) and \( k(v) \neq 0 \).

**Proof.** From (6.22), (6.5) and (6.20), we get
\[ \frac{\partial}{\partial u} \hat{C}(1 - u, 1 - v) \frac{g[\phi(u), \psi(v)]}{\phi'(u)} = \frac{-1}{C(1 - u, 1 - v)} \int_u^1 \phi(p) \frac{\partial}{\partial p} \hat{C}(1 - p, 1 - v) dp - \mu_1(v) \]

Differentiating the above expression w.r.t. \( u \), we get
\[ \frac{\partial^2}{\partial u^2} \hat{C}(1 - u, 1 - v) \left[ \frac{g[\phi(u), \psi(v)]}{\phi'(u)} \right] \]
\[ = \frac{\partial}{\partial u} \hat{C}(1 - u, 1 - v) \left[ -g'[\phi(u), \psi(v)] + \frac{g[\phi(u), \psi(v)] \phi''(u)}{(\phi'(u))^2} - \phi(u) - \mu_1(v) \right]. \]

That is,
\[ \frac{\partial}{\partial u} \log \left[ \frac{\partial}{\partial u} \left( \hat{C}(1 - u, 1 - v) \right) \right] = -\frac{\partial}{\partial u} \log [g(\phi(u), \psi(v))] + \frac{\partial}{\partial u} \log \phi'(u) - \frac{\phi'(u)[\phi(u) + \mu_1(u,v)]}{g(\phi(u), \psi(v))}. \]

Denote by \( G(u, v) = \frac{\partial}{\partial u} \left( \hat{C}(1 - u, 1 - v) \right) \), (6.24) can be written as
\[ \frac{\partial}{\partial u} \log G(u, v) = -\frac{\partial}{\partial u} \log [g(\phi(u), \psi(v))] + \frac{\partial}{\partial u} \log \phi'(u) - \frac{\phi'(u)[\phi(u) + \mu_1(u,v)]}{g(\phi(u), \psi(v))}. \]  
(6.25)

Integrating (6.25) from 0 to \( u \) and rearranging the terms, we get
\[ G(u, v) = k(v) g[0, \psi(v)] \frac{\phi'(u)}{\phi'(0) g[\phi(u), \psi(v)]} e^{-\int_0^u \frac{\phi'(p)[\phi(p) + \mu_1(v)]}{g[\phi(p), \psi(v)]} dp}. \]
The above equation can be written as

\[ \frac{\partial}{\partial u} \hat{C}(1-u, 1-v) = \frac{B^*(v)\phi'(u)}{g[\phi(u), \psi(v)]\phi'(0)} e^{-\int_0^u \frac{\phi'(p)[\phi(p)+\mu_1(v)]}{g[\phi(p), \psi(v)]} dp} \]

where \( B^*(v) = g[0, \psi(v)] k(v) \), which is same as (6.23)

Example 6.2

Consider the Gumbel type dependence with tukey lambda marginal. That is,
\[ \hat{C}(1-u, 1-v) = (1-u)(1-v)e^{-\theta \ln(1-u) \ln(1-v)} \] and \( \phi(u) = \frac{u^\lambda - (1-u)^\lambda}{\lambda} \)

Then \( h_1(u, v) \) and \( d_1(u, v) \) are related as in (6.22) where \( g[\phi(u), \psi(v)] \) can be obtained as the solution of the following differential equation,

\[
\left[ (1-u) \left( \frac{u^\lambda - (1-u)^\lambda}{\theta \ln(1-v)} \right) \right] \frac{\partial}{\partial u} g(\phi(u), \psi(v)) = \left[ \frac{1 + (1-u)(\lambda - 1)(u^\lambda - 2) + (1-u)^\lambda - 2}{\theta \ln(1-v)} \right] g(\phi(u), \psi(v))
\]

\[ = \frac{(1-u) \left( u^\lambda - (1-u)^\lambda \right) ((1-u)^\lambda - u^\lambda - \lambda \mu_1(v))}{\lambda \theta \ln(1-v)}. \]

Example 6.3

When \( \hat{C}(1-u, 1-v) = (1-u)(1-v) \) and the tukey lambda marginal as \( \phi(u) \), we get the univariate expressions for all concepts and \( g[\phi(u), \psi(v)] \) has a closed form given by

\[ g[\phi(u), \psi(v)] = \frac{[u^\lambda - (1-u)^\lambda] [u^\lambda - (1-u)^\lambda] [u^\lambda - (1-u)^\lambda] [u^\lambda - (1-u)^\lambda + 1] + \mu] \lambda \lambda - 1 \left[ u^\lambda - 2 + (1-u)^{\lambda-2} \right]. \]

6.3 Concepts in reversed time

Roy (2002) has defined the bivariate reversed hazard rate as a vector analogous to the definition of vector valued hazard rate extensively discussed in Johnson & Kotz (1975) and examined its properties. Also the author has proposed a class of bivariate distributions using this vector.
Later Sankaran & Gleeja (2006) developed a more general class of bivariate distributions which extends the result given in Roy (2002). Roy (2002) defines the bivariate reversed hazard rate as the vector
\[ \mathbf{A}(x) = \left( A_1(x), A_2(x) \right), \]
where
\[ A_i(x) = \frac{\partial}{\partial x_i} \log F(x), \quad i = 1, 2. \]
Setting \( F(x) = C(u, v) \), \( x_1 = \phi(u) \) and \( x_2 = \psi(v) \), we get the bivariate reversed hazard rate, in the copula setup as
\[
\mathbf{a}_{\epsilon \epsilon}(u, v) = (a_1(u, v), a_2(u, v))
\]
where
\[ a_1(u, v) = \frac{\partial}{\partial u} C(u, v) \frac{\phi'(u)}{C(u, v)} \] \( \tag{6.26} \)
and
\[ a_2(u, v) = \frac{\partial}{\partial u} C(u, v) \frac{\psi'(v)}{C(u, v)}. \] \( \tag{6.27} \)

**Example 6.4** For the Gumbel bivariate logistic distribution with copula specified by
\[ C(u, v) = \frac{uv}{1 - (1 - u)(1 - v)} \]
and marginal quantile functions of \( X_1 \) and \( X_2 \) with \( \phi(u) = -\log \left[ 1 - \frac{1}{u} \right] \) and \( \psi(v) = -\log \left[ 1 - \frac{1}{v} \right] \) respectively, we calculate the bivariate copula reversed hazard rate as follows.
\[
\frac{\partial}{\partial u} \log C(u, v) = \frac{1}{u} - \frac{1 - v}{1 - (1 - u)(1 - v)}. \]
From (6.26) and (6.27), we get
\[
\mathbf{a}_{\epsilon \epsilon}(u, v) = \left[ \frac{(1 - u)v}{1 - (1 - u)(1 - v)}, \frac{v^2(1 - u)}{u [1 - (1 - u)(1 - v)]} \right]. \]
The definition and properties of the reversed mean residual life are given in Finkelstein (2003) and Nanda et al. (2003). Nair & Asha (2008) has discussed the definition and properties of the bivariate reversed mean residual life and studied the relationship between bivariate reversed hazard rate and bivariate reversed mean residual life. Nair & Asha (2008) defines the bivariate reversed mean residual life as

\[ R(x) = \left( R_1(x), R_2(x) \right) \]

where

\[ R_1(x) = \frac{1}{F(x)} \int_0^{x_1} F(t, x_2) dt \]

and

\[ R_2(x) = \frac{1}{F(x)} \int_0^{x_2} F(x_1, t) dt. \]

We define the bivariate copula reversed mean residual life denoted by \( r_{\varepsilon_{-\cdot}}(u, v) \) as

\[ r_{\varepsilon_{-\cdot}}(u, v) = (r_1(u, v), r_2(u, v)) \]

where

\[ r_1(u, v) = \phi(u) - \frac{1}{C(u, v)} \int_0^u \phi(p) \frac{\partial}{\partial p} C(p, v) dp \]  \hspace{1cm} (6.28)

and

\[ r_2(u, v) = \psi(v) - \frac{1}{C(u, v)} \int_0^v \psi(p) \frac{\partial}{\partial p} C(p, v) dp. \]  \hspace{1cm} (6.29)

Next we examine whether the bivariate copula reversed hazard rate and bivariate copula reversed mean residual life determine the copula uniquely.

**Theorem 6.5.** Bivariate copula reversed hazard rate and the bivariate copula reversed mean residual function determine the underlying copula uniquely.
Proof. From (6.26), it is observed that

\[
\frac{\partial}{\partial u} \log C(u, v) = \phi'(u) a_1(u, v).
\]

Integrating the above equation from \( u \) to 1 and rearranging the terms, we get

\[
C(u, v) = v e^{- \int_u^1 a_1(p,v) \phi'(p) dp}.
\] (6.30)

From the definition of reversed hazard quantile function, we have

\[
a_1(u, 0) = \frac{1}{u \phi'(u)}.
\]

Therefore (6.30) becomes,

\[
C(u, v) = v e^{- \int_u^1 a_1(p,v) \phi'(p) dp}.
\]

Proceeding on the similar lines, we can also get

\[
C(u, v) = u e^{- \int_v^1 a_2(u,p) \phi'(p) dp}.
\]

Further differentiating (6.28) with respect to \( u \), we get

\[
\frac{\partial}{\partial u} \log C(u, v) = \frac{\phi'(u) - \frac{\partial}{\partial u} r_1(u, v)}{r_1(u, v)}.
\] (6.31)

Integrating from \( u \) to 1 and simplifying we get

\[
C(u, v) = v e^{- \int_u^1 \frac{\phi'(p) - r_1(p,v)}{r_1(p,v)} dp}.
\] (6.32)
Also \( r_2(u, v) \) uniquely determines the copula by the relation

\[
C(u, v) = ue^{- \int_u^1 \frac{\partial}{\partial p} \left( \frac{r_2(u, p)}{r_2(u, p)} - \psi'(p) \right) dp}.
\] (6.33)

In (6.32) and (6.33), \( \phi(u) \) and \( \psi(v) \) can be replaced by the following expressions given in Nair & Sankaran (2009),

\[
\phi(u) = r_1(u) + \int_0^u p^{-1} r_1(p) dp
\]

and

\[
\psi(v) = r_2(v) + \int_0^v p^{-1} r_2(p) dp.
\]

The following theorem discusses the relation between reversed hazard rate and reversed mean residual life in copula setup.

**Theorem 6.6.** The bivariate copula reversed hazard rate is related to bivariate copula reversed mean residual life by the expression

\[
r_1(u, v)a_1(u, v) = 1 - ua_1(u, 0) \frac{\partial}{\partial u} r_1(u, v)
\]

and

\[
r_2(u, v)a_2(u, v) = 1 - va_2(0, v) \frac{\partial}{\partial v} r_2(u, v).
\]

**Proof.** Using (6.26) and (6.31), we get

\[
a_1(u, v) = \frac{\phi'(u) - \frac{\partial}{\partial u} r_1(u, v)}{r_1(u, v) \phi'(u)}
\]

That is,

\[
a_1(u, v) = \frac{1 - ua_1(u, 0) \frac{\partial}{\partial u} r_1(u, v)}{r_1(u, v)}
\]
This gives,
\[ r_1(u, v)a_1(u, v) = 1 - ua_1(u, 0) \frac{\partial}{\partial u} r_1(u, v). \]
Proceeding on the similar lines we can also get
\[ r_2(u, v)a_2(u, v) = 1 - va_2(0, v) \frac{\partial}{\partial v} r_2(u, v). \]

When the component variates are independent, the above mentioned bivariate properties will be reduce to the corresponding univariate concepts.
For example, when \( C(u, v) = uv \), we have
\[ a_{\varepsilon - -}(u, v) = (a_1(u), a_2(v)) \]
and
\[ r_{\varepsilon - -}(u, v) = (r_1(u), r_2(v)). \]

**Theorem 6.7.** The relationship
\[ A_1(u, v)g_r[\phi(u), \psi(v)] = D_1(u, v) - \mu_1(v) \]
holds if
\[ \frac{\partial}{\partial u} C(u, v) = \frac{b^*(v)\phi'(u)}{g_r[\phi(u), \psi(v)]} \int_0^u e^{f_0^u \frac{\phi'(p)\phi'(u) - \mu_1(\psi)}{\psi'(u)}} dp \]
where \( b^*(v) = g_r[0, \psi(v)]K(v) \) in which \( K(v) = \frac{\partial}{\partial u} C(u, v) \) given \( u = 0 \) provided \( \phi'(0) \neq 0, g_r[0, \psi(v)] \neq 0 \) and \( K(v) \neq 0 \) and \( D_1(u, v) = r_1(u, v) + \phi(u) \)

The proof is similar to that of Theorem 6.4 and hence omitted.
Guidelines for further research

In the present work, we have examined the potential of Zenga curve as an alternate measure of inequality. In addition to examining the connection between the measure and other existing inequality measures, the relationship of the concept with certain reliability concepts are exploited to obtain characterization results for probability distributions. Further some results on a stochastic order using Zenga curve are also established. Instead of using the conventional distribution functional approach, the definitions and concepts are reformulated using quantiles.

During the course of present study, we are able to identify the following problems which require further investigation.

1. Since incomes are measured at specific points of time and a detailed study on the inequality measures in discrete time is to be undertaken.

2. Inference procedures such as estimation of Zenga index based on observed income data, formulation of tests for exponentiality using the truncated measures of inequality is yet to be studied.

3. Only very little work seems to have been done on bivariate copula in higher dimensions. Developing these ideas considering the same in four directions shall pave way for theoretical foundations in higher dimensions.

4. Several other quantile function based models for income data can be developed in varying situations and this may help to model income data.

5. The implication of stochastic orders based on inequality measures shall be studied in detail for other existing orders also.

We hope that the problems mentioned above shall be sorted out in a future work.