Chapter 5

L moments and measures of income inequality

5.1 Introduction

As pointed out in section 2.4, the concept of L moments, introduced by Sillitto (1969), has been extensively used in reliability analysis. One of the interesting aspects of L moments is that it generally dominates the conventional moments in the sense that it provides smaller variance. Further the robustness of L moments against outliers enables the same as a potential tool when it comes to the modelling of lifetime data. Recently Nair & Vineshkumar (2010) studied L moments of residual life and has obtained characterization results for certain life distributions. They have also expressed the truncated Gini index and the celebrated Sen index for the poor in terms of the first two L moments.

Motivated by this in the current chapter we study the interrelationships of the L moments with other inequality measures such as Lorenz curve, Bonferroni curve and Zenga curve. We also look into the problem of characterizing distributions using possible functional relationships among these measures, in the truncated set up. We also address the problem of ordering of distributions based on L moments and compare the same with other types of orderings.
5.2 Relationship with other inequality measures

In this section, we look into possible functional relationships between the income inequality measures and truncated L moments. It follows easily from the definition of Lorenz curve that of the first residual quantile, $\alpha_1(p)$ is related to $L(p)$ through the equation

$$L(p) = 1 - \frac{(1 - p)\alpha_1(p)}{\mu}.$$  \hspace{1cm} (5.1)

For the line of equal distribution, the income variable $X$ is degenerate and $L(p) = p$. In this set up, (5.1) takes the form

$$\alpha_1(p) = \mu.$$  

From the above relationship, it may be observed that when there is equality of income the first residual quantile is equal to the mean income. It may also be noted that $\alpha_1(p)$ can be interpreted as the mean income among households with upper income level.

From the following definition of Pietra index

$$T = F(\mu) - \frac{1}{\mu} \int_0^{F(\mu)} Q(p)dp,$$

we get the mean income of lower income households, $\beta_1(p)$, is related to the Pietra index through the relationship

$$T = F(\mu) \left[ 1 - \frac{\beta_1(F(\mu))}{\mu} \right].$$

Also the Bonferroni curve is related to $\beta_1(p)$ as

$$B(p) = \frac{\beta_1(p)}{\mu}.$$  

$\beta_1(p)$ is the mean income of the truncated random variable $xX$ and so greater the first L moment $\beta_1(p)$ indicates more equality in the population. Thus $\beta_1(p)$ can be used to infer the inequality
of the population. Evidently as $\beta_1(p) \to \mu$, $B(p) \to 1$, the curve equality. Bonferroni curve uniquely determines the distribution through the relationship

$$Q(p) = \mu [pB'(p) + B(p)]$$

Substituting the above expression in the definition of reversed mean residual quantile function, we get

$$R(p) = \mu pB'(p).$$

Using the above equation in (2.15), we get

$$\beta_2(p) = \frac{1}{p} \int_0^p \mu u^2 B'(u) du = \mu \int_0^p \left[ p^2 B(p) - 2 \int_0^p uB(u) du \right] = \mu \left[ B(p) - \frac{2}{p^2} \int_0^p uB(u) du \right].$$

Differentiating the above expression we get

$$p^2 \beta_2(p) = \mu \left[ p^2 B(p) - 2 \int_0^p uB(u) du \right]$$

$$p^2 \beta_2'(p) + 2p \beta_2(p) = \mu \left[ p^2 B'(p) + 2pB(p) - 2pB(p) \right]$$

or

$$p \beta_2'(p) + 2 \beta_2(p) = \mu pB'(p)$$

That is,

$$B(p) = \frac{1}{\mu} \int_0^p \left( \beta_2(u) + \frac{2}{u} \beta_2(u) \right) du.$$  \hspace{1cm} (5.2)

Evidently the case of equality reflects no variability so that $\beta_2(p) = 0$ if and only if $B(p) = 1$.

In light of the fact that L moments find application in reliability theory, we look into the scope of studying the relationship between L moments of the truncated variables $X_x = X | X >
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$x$ and $x X = X | X \leq x$ with the Zenga curve. From the definitions of Zenga curve and $\beta_1(p)$, we get

$$I(p) = \frac{\mu - \beta_1(p)}{p [\mu - p \beta_1(p)]}.$$  

Now making use of (3.5) and (5.1), we get

$$I(p) = \frac{\alpha_1(p) - \mu}{p \alpha_1(p)} \quad (5.3)$$

or

$$\alpha_1(p) = \frac{\mu}{1 - p I(p)}. \quad (5.4)$$

This implies,

$$\alpha_1(p) \propto \frac{1}{1 - p I(p)}.$$

From the above equation it is evident that $I(p)$ uniquely determines the distribution up to a constant. Under the assumption that $\mu$ is known $I(p)$ uniquely determines the underlying distribution. This result provides a justification for using the $I(p)$ curve in the context of modelling income data. Recall that the $L$ moments $\alpha_2(p)$ and $\beta_2(p)$ are measures of variability of $X_x$ and $x X$. The relationship between the $I(p)$ curve and $\alpha_2(p)$ and $\beta_2(p)$ are given below.

From (2.14) and (4.4), we get

$$\alpha_2(p) = (1 - p)^{-2} \mu \int_p^1 \frac{(1 - u)^2}{1 - u I(u)} du.$$  

From section 4.2, the relationship between $I(p)$ and $B(p)$ curve is

$$I(p) = \frac{1 - B(p)}{1 - pB(p)}. \quad (5.5)$$
From (5.5) and (5.2), we have

\[ I(p) = \frac{\mu - \int_0^p \left( \beta'_2(u) + \frac{2}{u} \beta_2(u) \right) du}{\mu - p \int_0^p \left( \beta'_2(u) + \frac{2}{u} \beta_2(u) \right) du}. \]

The interrelationships discussed in this section will be used in sequel to investigate characterization results and stochastic orders in the forthcoming sections.

### 5.3 Characterization Results

In this section, we present characterization results for probability distributions based on functional relationships between the L moments and the different measures of inequality. Among the income inequality measures, Bonferroni curve is widely used as the measure of poverty since it is very sensitive to low level incomes [Giorgi & Crescenzi (2001b)]. Bonferroni curve is used to measure the variability in income distribution. Also the second L moment of reversed residual life is considered as the measure of variation. It is now of interest to investigate if the behaviour of the former can be inferred from the latter. However the behaviour of \( B(p) \) need not be necessarily similar to that of \( \beta_2(p) \). But the power distribution exhibits such a behaviour. Theorem 5.1 characterizes the power distribution using the functional relationship between \( B(p) \) and \( \beta_2(p) \).

**Theorem 5.1.** Let \( X \) be nonnegative continuous random variable with Bonferroni curve \( B(p) \) and second truncated L moment \( \beta_2(p) \). The relationship

\[ B(p) = K \beta_2(p); \quad K > 0 \]

holds if and only if \( X \) follows the quantile function specified by

\[ Q(p) = ap^b; \quad a, b > 0. \]
Proof. Using the relationship between $B(p)$ and $\beta_2(p)$ in the previous section, we get

$$B(p) = K\mu \left[ B(p) - \frac{2}{p^2} \int_0^p uB(u)du \right].$$

Differentiating the above equation with respect to $p$, we get

$$\frac{B'(p)}{B(p)} = \frac{2}{p(K\mu - 1)}.$$

The solution to the above differential equation is

$$B(p) = Cp^{\frac{2}{K\mu - 1}},$$

where $C$ is a constant. Since $B(1) = 1$, we get $C = 1$. Thus $B(p) = p^{\frac{2}{K\mu - 1}}$. The proof of the converse is straightforward and hence omitted.

Illustration

In the sequel we fit the power model to the US department, Proprietors income-Quarterly data in the state Newhampshire of the period 1948-1950 and examine the behaviour of $B(p)$ and $\beta_2(p)$. (Data source: U.S. Department of Commerce, Bureau of Economic Analysis). We fit the power model for the above mentioned data by the method of L moments. The first two L moments are given by

$$l_1 = \frac{1}{a+b} \log \left( \frac{a+b}{a} \right)$$

and

$$l_2 = \frac{a \log \left( \frac{a+b}{a} \right) + b}{b^2}.$$

The population L moments are equated to sample L moments to obtain the estimates of param-
eters as

\[ \hat{a} = 17.96, \hat{b} = 3.265. \]

The KS statistic is .156 against the table value as .361 at 5% level of significance which shows that power distribution gives suitable fit to data. For the power model,

\[ B(p) = p^{1/\hat{b}} \]

and

\[ \beta_2(p) = \frac{abp^{1/\hat{b}}}{1 + 3\hat{b} + 2\hat{b}^2}. \]

Plotting \((\beta_2(p), B(p))\) for the data we can estimate from Figure 5.1 that \(k = 1.8259\).

Another distribution characterized by the functional relationship between \(\beta_2(p)\) and \(B(p)\) is the distribution with Lorenz curve

\[ L(p) = pa^{p-1}, a > 1 \]

The application of the distribution in modelling income data is studied in detail by Kakwani & Podder (1973) and Gupta (1984).
Theorem 5.2. For the random variable $X$ considered in theorem 5.1, the relationship

$$B(p) \left[ p^2 (\log a)^2 - 2p \log a + 2 \right] = \frac{p^2 (\log a)^2}{\mu} \beta_2(p) - \frac{2}{a}$$

(5.6)

holds if and only if the quantile function is given by

$$Q(p) = \mu a^{p-2} [p(p - 1) + a].$$

(5.7)

Proof. From (5.6) and the relation between $B(p)$ and $\beta_2(p)$, we get

$$B(p) \left[ p^2 (\log a)^2 - 2p \log a + 2 \right] = p^2 (\log a)^2 \left[ B(p) - \frac{2}{p^2} \int_0^p u B(u)du \right] - \frac{2}{a}.$$

Differentiating and rearranging the terms, we get

$$\frac{B'(p)}{B(p)} = \log a.$$

This gives,

$$B(p) = C a^p$$

(5.8)

where $C$ is the constant of integration. The relationship $B(1) = 1$ gives $C = \frac{1}{a}$. From (5.8), we have

$$B(p) = a^{p-1}.$$

Bonferroni curve uniquely determines the distribution through the relationship,

$$Q(p) = \mu [p B'(p) + B(p)].$$

For the above expression of $B(p)$, we get the corresponding quantile function as given in (5.7).
The above discussions show that $\beta_2(p)$ can be used to study the variability in the lower income group and $\alpha_2(p)$ can be used to study the variability in the upper income group. Recall that $I(p)$ deals with the comparison of mean incomes of subgroups of population to measure the income inequality. When the income distribution is uniform it is intuitive to infer that a measure of inequality comparing the lower income group and higher income group is inversely proportional to the mean incomes of higher income group. This is mathematically confirmed by the following theorem.

**Theorem 5.3.** For the random variable considered in theorem 5.1, the relationship

$$I(p) = \frac{k}{\alpha_1(p)}; \quad k > 0$$

holds if and only if $X$ follows the uniform distribution specified by the quantile function

$$Q(p) = \theta p; \quad 0 \leq p \leq 1.$$ 

**Proof.** For the uniform distribution specified by $Q(p) = \theta p$, direct calculations gives

$$I(p) = \frac{1}{1 + p}$$

and

$$\alpha_1(p) = \frac{\theta}{2}(1 + p).$$

Directly we get $I(p) = \frac{k}{l_1(p)}$ with $k = \frac{\theta}{2}$. Conversely using the relation $(5.4)$, we get

$$\alpha_1(p) = \mu + kp.$$
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Corresponding quantile function is

\[ Q(p) = \mu - k + 2kp. \]

\[ Q(0) = 0 \] yields \( \mu = k. \) So the resulting quantile function is \( Q(p) = 2kp \) as claimed.

\[ \square \]

5.4 Ageing concepts

In this section we study how the L moments can be applied to study the ageing behaviour of a random variable. Concept of ageing is an important notion not only in the field of Reliability theory but also in Economics. Bhattacharjee (1993) has observed that the distribution of land holdings obey anti ageing properties like DFR, DFRA, IMRL, NWUE etc. Usually in reliability based works, the ageing properties are studied using concepts such as failure rate, mean residual life function etc. In this section we explain the problem from another point of view. The ageing properties are examined using truncated L moments which have their own economic interpretations.

\textbf{Theorem 5.4.} Let \( X \) be a random variable representing income with finite positive mean \( \mu. \) Denote by the first residual quantile \( \alpha_1(p), \) then \( X \) is IFR(DFR) if and only if

\[ \alpha_1(p) \leq (\geq) \mu [1 - \ln(1 - p)]. \]

\textit{Proof.} From theorem 4.4, we get the necessary and sufficient condition for a distribution to be IFR(DFR) in terms of \( I(p) \) curve as

\[ I_F(p) \leq (\geq) \frac{1}{p \left[ 1 - \ln(1 - p) \right]^{-1}} \] (5.9)
From (5.9) and (5.4), we get

\[
\frac{\alpha_1(p) - \mu}{p \alpha_1(p)} \leq (\geq) \frac{1}{p \left[ 1 - \ln(1 - p) \right]^{1-1} }.
\]

Rearranging the terms in the above expression, we get the result. The converse can be obtained by retracing the steps. \(\square\)

**Theorem 5.5.** For an IMRL(DMRL) distribution, if \(\alpha_1(p)\) and \(Q(p)\) respectively denote the first residual quantile and quantile function, then

\[
\alpha_1(p) - \mu \geq (\leq) Q(p).
\]

**Proof.** From theorem 4.5 we get the sufficient condition for a distribution to be IMRL(DMRL) as

\[
I(p) \geq (\leq) \frac{1}{p} \left[ \frac{Q(p)}{Q(p) + \mu} \right] \quad (5.10)
\]

From (5.10) and (5.4), we get

\[
\frac{\alpha_1(p) - \mu}{p \alpha_1(p)} \leq (\geq) \frac{1}{p \left[ Q(p) + \mu \right]}. \quad (5.11)
\]

That is,

\[
1 - \frac{\mu}{\alpha_1(p)} \geq (\leq) \frac{Q(p)}{Q(p) + \mu}.
\]

This in turn implies

\[
\alpha_1(p) - \mu \geq (\leq) Q(p).
\]

\(\square\)

**Remark 5.1**

The above theorem illustrates that, for a population if the average excess holding over a particular lower threshold increases, then the difference between upper mean incomes and total mean
incomes corresponding to each \( p \) is greater than the income corresponding to that \( p^{th} \) percentile for that population.

**Theorem 5.6.** A distribution is HNBUE(HNWUE) if and only if

\[
\alpha_1(p) \leq (\geq) Q(p) + \frac{\mu}{1 - p} e^{-\frac{Q(p)}{\mu}}.
\]

**Proof.** From (2.19), the random variable \( X \) is in HNBUE(HNWUE) class if and only if

\[
\int_0^1 (1 - u)q(u)du \leq (\geq) \mu e^{-\frac{Q(p)}{\mu}}.
\]

From the above equation, integrating by parts yields

\[
\int_0^1 Q(u)du \leq (\geq) \mu e^{-\frac{Q(p)}{\mu}} + (1 - p)Q(p).
\]

or

\[
\alpha_1(p) \leq (\geq) Q(p) + \frac{\mu}{1 - p} e^{-\frac{Q(p)}{\mu}}.
\]

The above results focuses attention on ageing concepts using \( \alpha_1(p) \). Our next result seeks a sufficient condition for a distribution to be NBUE which is given in terms of Gini’s mean difference of residual random variable \( X_t \).

**Theorem 5.7.** For a distribution to be NBUE,

\[
\eta(p) \leq \mu
\]

where \( \eta(p) \) is the Gini’s mean difference of residual random variable \( X_x \).
Proof. From (2.18), a distribution is NBUE if and only if

\[ M(p) \leq \mu \]  
(5.11)

where \( M(p) \) represents the mean residual quantile function. Multiplying (5.11) by \((1 - p)\) and integrating from \(p\) to 1, we get

\[ \int_p^1 (1 - u)M(u)du \leq \frac{\mu}{2}(1 - p)^2. \]  
(5.12)

Using (2.14) and (5.12), we have

\[ 2\alpha_2(p) \leq \mu. \]

Nair & Vineshkumar (2010) has shown that

\[ 2\alpha_2(p) = \eta(p). \]

Therefore the above inequality becomes,

\[ \eta(p) \leq \mu. \]

\[ \square \]

Remark 5.2

1. Equality holds when \( X \) has distribution with quantile function \( Q(p) = -\frac{1}{\lambda} \log[1 - p]; \lambda > 0 \). Note that for this distribution \( M(p) = \mu \).

2. From the above theorem, it is clear that as the sum of Gini’s mean difference of residual random variable \( X_t \) and the total mean income becomes very negligible, the corresponding income distribution is exponential. So the above theorem can form a basis of forming
a statistic using the empirical version of $\eta(p)$ as well as the sample mean income to check the exponentiality of a population. This work will be taken up elsewhere.

5.5 Stochastic orders based on L moments

Nair & Vineshkumar (2010) has shown the second L moment of residual as well as reversed residual lives, $\alpha_2(p)$ or $\beta_2(p)$ can be used in distinguishing lifetime distributions based on its monotonic behaviour. The condition for $\alpha_2(p)$ to be increasing (decreasing) is

$$\alpha_2(p) \geq (\leq) \frac{M(p)}{2}$$

and the change point of $\beta_2(p)$ will be the solution of $R(p) = 2\beta_2(p)$. L moments can be used to give alternative definitions of ageing concepts. We consider two random variables $X$ and $Y$. All the notations discussed in sequel are as defined in earlier sections corresponding to the suffixed random variable. To compare the orderings based on L moments, inequality measures and reliability concepts, we require the definitions of different kinds of orderings which are discussed in Chapter 2. We now define the following orderings based on L moments.

**Definition 5.1**

Let $X$ and $Y$ be two random variables with $r^{th}$ L moment residual quantile functions $\alpha_{rX}(p) = \lambda_{rX}(Q_X(p))$ and $\alpha_{rY}^*(p) = \lambda_{rY}(Q_X(p))$. Then $X$ is said to be smaller than $Y$ in $r^{th}$ L moment residual quantile function if and only if

$$\alpha_{rX}(p) \leq \alpha_{rY}^*(p)$$

for all $p$ in $(0, 1)$.

For $r = 1$, we get $\alpha_{1X}(p) \leq \alpha_{1Y}^*(p)$ denoted by $X \leq_{FL} Y$. Similarly we can obtain $X \leq_{SL} Y$ for $r = 2$. 
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**Definition 5.2**

For the random variables $X$ and $Y$ with $r^{th}$ L moment reversed residual quantile function $\beta_{rX}(p)$ and $\beta_{rY}^*(p)$ respectively, we can say that $X$ is smaller than $Y$ in $r^{th}$ L moment quantile function if and only if

$$\beta_{rX}(p) \leq \beta_{rY}^*(p).$$

Analogous to the Definition 5.1, here also we can define $X \leq_{FLR} Y$ and $X \leq_{SLR} Y$ for $r = 1$ and $r = 2$ respectively.

We say that $X$ dominates $Y$ by second order stochastic dominance denoted by $X \geq_{SSD} Y$ if

$$\int_0^p Q_X(u)du \leq \int_0^p Q_Y(u)du.$$

Then for the first truncated L moments of $X$ and $Y$ denoted by $\lambda_{1X}(t)$ and $\lambda_{1Y}(t)$ respectively, we have

$$X \geq_{SSD} Y \Leftrightarrow \lambda_{1X}(t) \leq \lambda_{1Y}(t).$$

**Theorem 5.8.** *The following stochastic orders are equivalent.*

1. $X \leq_{MRL} Y \Leftrightarrow X \leq_{FL} Y$.

2. $X \leq_{RMRL} Y \Leftrightarrow X \leq_{FLR} Y$.

**Proof.**

1. We have,

$$M_X(p) \leq (\geq) M_Y^*(p) \Leftrightarrow \frac{1}{1-p} \int_p^1 Q_X(u)du - Q_X(p) \leq (\geq) \frac{1}{F_Y(Q_X(p))} \int_{Q_X(p)}^{1} \bar{F}_Y(x)dx$$

$$\Leftrightarrow X \leq (\geq)_{FL} Y$$

2. can be proved on similar lines.

$\square$

The $\leq_{FL}$ order may not imply $\leq_{st}$ order. This can be shown using the following example.
Example 5.1

Let the random variable $X$ be distributed as exponential with p.d.f.

$$f_X(x) = \frac{1}{2} e^{-\frac{x}{2}}$$

and $Y$ follows gamma distribution with density function

$$f_Y(x) = xe^{-x}.$$

We have the mean residual life functions $M_X(x) = (x+2)e^{-x}$ and $M_Y(x) = 2$. Here $X \leq_{MRL} Y$ or $X \leq_{FL} Y$

However $X$ and $Y$ are not in usual stochastic order as is evidenced from the figure 5.2 below.

![Figure 5.2:](image)

For the example given above, we have the hazard functions, $h_X(x) = \frac{x}{2}$ and $h_Y(x) = \frac{x}{1+x}$. It may be observed from Figure 5.3 that $h_X(x) \geq h_Y(x)$. 

The following theorem provides a sufficient condition for two random variables \( X \) and \( Y \) to have \( \leq_{hr} \) order.

**Theorem 5.9.** If \( \int_{p}^{1} Q_{X(u)} du \) is increasing in \( p \) and \( X \leq_{FL} Y \), then \( X \leq_{hr} Y \).

**Proof.** We have,

\[
X \leq_{FL} Y \Rightarrow \frac{\int_{p}^{1} Q_{X(u)} du}{\int_{p}^{1} Q_{Y(u)} du} \leq \frac{\bar{F}_{X}(x)}{\bar{F}_{Y}(x)}.
\]

(5.13)

If \( \int_{p}^{1} Q_{X(u)} du \) is increasing in \( p \) we have

\[
\frac{\int_{p}^{1} Q_{X(p)} dp}{\int_{p}^{1} Q_{Y(p)} dp} \geq \frac{1}{q_{X}(p)f_{Y}(Q_{X}(p))}.
\]

(5.14)

(5.13) and (5.14) give,

\[
\frac{1}{q_{X}(p)f_{Y}(Q_{X}(p))} \leq \frac{\bar{F}_{X}(x)}{\bar{F}_{Y}(x)}
\]

or

\[
\frac{f_{X}(x)}{\bar{F}_{X}(x)} \leq \frac{f_{Y}(x)}{\bar{F}_{Y}(x)}
\]

or

\[
X \leq_{hr} Y.
\]
Theorem 5.10. If $X \leq_{NBRURH} Y$ and $X \leq_{FLR} Y$, then $X \geq_{L} Y$.

Proof. Assume that

$$\frac{\int_0^p Q_X(u)du}{\int_0^p Q_Y(u)du} \leq \frac{F_X(x)}{F_Y(x)}. \quad (5.15)$$

Since $X \leq_{NBRURH} Y$, we have $\frac{F_X(x)}{F_Y(x)} \leq \frac{\mu_X}{\mu_Y}$. Therefore (5.15) becomes

$$\frac{1}{\mu_X} \int_0^p Q_X(u)du \leq \frac{1}{\mu_Y} \int_0^p Q_Y(u)du \Rightarrow X \geq_{L} Y.$$