CHAPTER 2

PROPAGATION CHARACTERISTICS OF A HELICALLY CLAD STEP-INDEX FIBER

2.1 INTRODUCTION

Optical waveguides have been investigated extensively during the last three decades (184-188). Though the step-index fiber is the simplest structure, many new structures have already been analyzed. Considerable work has been done on optical fibers with more general geometries and refractive index profiles. However, in all these works, the dielectric core - cladding interfaces are considered and the appropriate boundary conditions are used (189-193). It is necessary to study the different types of fibers, keeping in view that the design criteria for the optimum performance of optical fibers depend on the desired applications.

The present chapter is concerned with the theoretical analysis of a new type of optical fiber proposed by Singh and Singh (194) and Singh et. al. (195). In this study, the usual metal cladding is replaced by a helical metal cladding. First of all, although the helix has a simple geometry, it can be wound on a circular cylinder and has a periodic structure. It is employed in all low - and medium-power travelling wave tubes and in low-power backward-wave oscillators. It is used in highly directive broadband antennas and in video-frequency delay lines.

Apart from the purely electrical considerations, the helix is convenient from the mechanical view point and can be easily fabricated and accurately fitted with the rest of the structure. Attempts have been made to analyze the helical slow wave structure (SWS) by the so called exact approach (196-197) and by suitably modeling the
actual helix (198-201). The first of these methods is based on finding a proper coordinate system to define the helix surface; the results were, however, not much encouraging (196-197). As for the second method, both the sheath and the tape models are in use as shown in Fig. 2.1. In the sheath model, the actual helix is replaced by an infinitesimally thin cylindrical surface of a radius, which is equal to the mean helix radius, and conduction takes place only in the helix winding direction. In the tape model, the actual helix is replaced by a tape of finite width and infinitesimal thickness which unlike the behavior in the sheath model, conducts in all directions.

The earliest work on the helix problem seems to have been done by Pocklington (202). In his analysis, the helix wire was assumed to be very thin and a perfect conductor. An integral equation for the free modes was derived and approximate solutions were obtained which predicted a travelling wave whose axial phase velocity is near the velocity of light \( c \) (assuming the helix is immersed in free space) for low frequencies, and whose axial phase velocity is reduced to \( c \sin \psi \) (where \( \psi \) is the pitch angle of the helix) for high frequencies. The sheath helix abstraction is found in the early work of Ollendorf (203). Schelkunoff's analysis of the actual helix in the sheath model for the fundamental mode was presented by Pierce (198). Schulman and Heagy (204) discussed the sheath helix considering higher space and angular harmonic modes. Sensiper (205) dealt with the general case of the sheath helix where the azimuthal variations as well as higher space harmonics were allowed. The case of the sheath helix surrounded by a dielectric medium has been considered by a number of workers (206-208). The case of a metal shielded sheath helix has been studied by Sickak (209) giving an approximate solution for the lowest mode phase velocity along the axis, the characteristic impedance and the attenuation constant. Bryant (210) discussed the various cases of sheath helix in the presence of additional inner and outer co-axial conductors. Birdsall (211) presented the determinantal dispersion relation for relatively complicated structures in the sheath model using the concept of radial transmission lines. Loshakow (212) presented a general solution of the problem emphasizing the high frequency
Fig. 2.1 (a) Sheath - helix model

Fig. 2.1 (b) Helix made of a tape
behavior in the presence of an electron beam. The sheath model is found to be a very good approximation to the actual helix for frequencies and modes in which there are many turns per guide wavelength with almost no power going to the space harmonics (213). In the majority of the TWT cases, this condition prevails. In some special tubes which operate at very high voltages and frequencies, the helix pitch becomes comparable to the guided wavelength. This generates space harmonics and is analyzed in tape helix model (214-215). Tien (216) analyzed the helix surrounded by a dielectric medium using the tape model for dispersion and interaction impedance and found his theoretical results for the impedance in agreement with the measured values. Stark (217) analyzed the metal shielded helix using the tape model for its propagation characteristics and obtained approximately closed form expression for a few of the lower modes. Waffeins and Ash (218) showed the effects of various structure parameters on the impedance of the space harmonic modes. The helix finds numerous applications apart from being used as a slow wave structure (SWS). With suitable modification in the structure, a helix can be is used as a broadband antenna (219-222), a radio-frequency delay line (223-232), a microwave heater (226) etc. The helical structure has also been used in the measurement of the permittivity of a lossy dielectric (227-228). It is also worth mentioning that the helix finds applications in the devices other than the slow-wave device like the TWT, viz., the fast-wave gyro-devices where the coupling of the electron-beam mode with fast waves is of more relevance (229-230). Recent use of the helix in hyperthermia treatment of cancer and permittivity measurement has stimulated a renewed interest in their properties.

There has been significant interest in helical core optical fibers, mainly because of their high circular birefringence properly. This can be used, for example, as the basis for the measurement of electric current flow by measuring the rotation of the polarization of the fundamental mode caused by the Faraday effect. If the core of a single mode fiber follows a helical path, then the polarization of the fundamental mode rotates along the path (231). It is also observed that unlike a straight single-mode fiber,
the fundamental mode of the helical fiber is not strictly bound, and is leaky at all wavelength (232). However, for practical purpose, every mode of a single mode or multi mode helical fiber can be treated as being bound with an effective cut-off wavelength (233). Polarization metal for a helically wound optical fiber was also studied by Stale et al. (234). The helical cladded fiber was studied by Singh et. al. (195 ) in a simplified way. This chapter is concerned with the modal behavior of a helical cladded optical fiber using sheath helix approximation.

2.2 HELIX (Basic Geometry)

The helix is a simple 3-dimensional geometrical structure. A helical wire on a uniform cylinder becomes a straight wire when unwound by rolling the cylinder on a flat surface and when viewed from end on, it projects as a circle. Thus, a helix combines the geometric forms of a straight line, a circle, and a cylinder. The geometrical relationships between the helix radius ‘a’, the length of the each turn ‘L_t’, the spacing between the turns ‘P’ (pitch of the helix ) and the helix pitch angle $\psi$, are shown in Fig. 2.2 where the pitch angle is

$$Cot \psi = 2\pi a / P$$

Also $\psi = tan^{-1} P / 2\pi a = sin^{-1} P / L_t$

As in the case of other configurations which can support electromagnetic waves, one desires to know the characteristics of those solutions of Maxwell’s equations which match the boundary conditions prescribed by the helix and its material related devices. The properties of those solutions which are of harmonic time and space dependence $e^{j\omega t}$, $e^{j\beta z}$ (with $\beta$ real/complex) $z$ being the axial coordinates, are of interest. In particular, it is important to know the value of $\beta$ as a function of the frequency and various electrical and physical properties of the system.
Fig. 2.2 Helix Made of a Wire
A physical representation of this model can be made by winding a fine wire in which all the turns are insulated from each other. Besides the sheath-helix, an approximation regarding the cladding is also made. It is assumed that the core-cladding interface boundary condition corresponds to the idealised sheath-helix boundary and the cladding region is assumed to have a real dielectric constant. In order to analyze this work, an electromagnetic field analysis is presented, using solution of Maxwell's equations in cylindrical co-ordinates with appropriate helix boundary condition. This type of fiber is hereafter called the helically cladded fiber (HCF).

2.3 THEORETICAL ANALYSIS AND CHARACTERISTIC EQUATION

The structure here considered is divided into two region (i) $r < a$ corresponding to the core and (ii) $r > a$ corresponding to the cladding assumed to be infinitely extended.

Maxwell's equations together with the constitutive relations lead to the wave equation for $\mathbf{E}$ and $\mathbf{H}$. The field components $E_z$ and $H_z$ satisfy the scalar wave equation which is written in the cylindrical co-ordinate system. Taking the $Z$-axis as the direction of propagation of monochromatic sinusoidal electromagnetic wave, the general field expressions are given by:

$$E_z = E (r, \phi) \ e^{j(\omega t - \beta z)} \quad \text{(2.1)}$$

$$H_z = H (r, \phi) \ e^{j(\omega t - \beta z)} \quad \text{(2.2)}$$

where $\beta$ is the propagation constant.
Assuming a harmonic \(\exp(j \nu \phi)\) dependence for the fields a Bessel equation for the radial part of the wave equation is obtained. For fields remaining continuous at \(r = 0\), the following solution for \(r < a\) is chosen (184,185).

\[
E_{z1} = A \ J_\nu (k_1 r) \ e^{j(\omega t - \beta z + \phi)} \quad \text{-------- (2.3)}
\]

\[
H_{z1} = B \ J_\nu (k_1 r) \ e^{j(\omega t - \beta z + \phi)} \quad \text{-------- (2.4)}
\]

Where \(k_1^2 = \omega^2 \mu_0 \varepsilon_1 - \beta^2\) and \(\varepsilon_1 = \varepsilon_0 \varepsilon_{1r}\) is the permittivity of the core. For \(r > a\) the boundary condition at infinity demands a decaying solution and hence,

\[
E_{z2} = C \ K_\nu (k_2 r) \ e^{j(\omega t - \beta z + \phi)} \quad \text{-------- (2.5)}
\]

\[
H_{z2} = D \ K_\nu (k_2 r) \ e^{j(\omega t - \beta z + \phi)} \quad \text{-------- (2.6)}
\]

Where \(k_2^2 = \omega^2 \mu_0 \varepsilon_2 - \beta^2\) and \(\varepsilon_2 = \varepsilon_0 \varepsilon_{2r}\) is the permittivity of the cladding region.

A, B, C, D are unknown constants. \(J_\nu (k_1 r)\) denotes the Bessel’s function of the first kind and \(K_\nu (k_2 r)\) is the modified Bessel function of the second kind and \(\nu\) is the order of the respective function. The quantities \(\varepsilon_{1r}\) and \(\varepsilon_{2r}\) are the relative permittivities of the core and the cladding respectively.
Now we use Maxwell's equations for the remaining field components.

(1) For the core region $r < a$ we have

$$E_{\phi 1} = - \frac{j}{k_1^2} \left[ \frac{\beta}{r} \left( A j v J_v (k_1 r) - \mu_0 B k_1 J'_v (k_1 r) \right) e^{j(\omega t - \beta z + \phi)} \right] - (2.7)$$

$$H_{\phi 1} = - \frac{j}{k_1^2} \left[ \frac{\beta}{r} \left( B j v J_v (k_1 r) + \omega \epsilon_1 k_1 A J'_v (k_1 r) \right) e^{j(\omega t - \beta z + \phi)} \right] - (2.8)$$

(2) Again for cladding region $r > a$ we have

$$E_{\phi 2} = - \frac{j}{k_2^2} \left[ \frac{\beta}{r} \left( C j v K_v (k_2 r) - \mu_0 D k_2 K'_v (k_2 r) \right) e^{j(\omega t - \beta z + \phi)} \right] - (2.9)$$

$$H_{\phi 2} = - \frac{j}{k_2^2} \left[ \frac{\beta}{r} \left( D j v K_v (k_2 r) + \omega \epsilon_2 C k_2 K'_v (k_2 r) \right) e^{j(\omega t - \beta z + \phi)} \right] - (2.10)$$

In the usual discussion (193, 235-237), the field inside the helix is expressed in terms of the modified Bessel's function $I_v (k_1 r)$ rather than $J_v (k_1 r)$ assumed here. The phase velocity in the slow-wave structure is approximately given by $c \sin \psi$ where 'c' is the speed of light. However, in the present case, the field propagation is along the core and there is no need of slowing down the wave. This justifies the choice of $J_v (k_1 r)$ (194, 238).
2.4 BOUNDARY CONDITIONS

The requirement of the tangential component of electric and magnetic field being continuous at the interface gives

(1) \( E_t^1 = E_t^2 \) and (2) \( H_t^1 = H_t^2 \)

In light of the above requirements the boundary condition on the core-cladding interface at \( r = a \) correspond to the conditions of the helix boundary (190-193), Fig 2.3

A further approximation made is that the electric field along the direction of pitch angle is assumed to be zero i.e. \( E_z^1 = 0 \), when written in terms of field quantities they are

\[
E_{z2} \sin \psi + E_{\phi2} \cos \psi = 0 \quad \text{--- (i)}
\]

The electric field components normal to the direction of \( \psi \) are assumed to be continuous:

\[
E_{\perp1} = E_{\perp2}
\]

So in terms of field components,

\[
E_{z1} \cos \psi - E_{\phi1} \sin \psi = E_{z2} \cos \psi - E_{\phi2} \sin \psi \quad \text{--- (ii)}
\]
Fig. 2.3 Cross-sectional view of helically cladded step-index fiber.
Again the tangential magnetic field components along $\psi$ are also continuous, i.e.

$$H_{ll}^1 = H_{ll}^2$$

Which in terms of field components gives,

$$H_{x1} \sin \psi + H_{\phi1} \cos \psi = H_{x2} \sin \psi + H_{\phi2} \cos \psi \quad \text{----- (iii)}$$

The magnetic field component normal to the direction of $\psi$ are assumed to be continuous.

$$H_{ll}^1 = H_{ll}^2$$

Which in terms of field components gives,

$$H_{x1} \cos \psi - H_{\phi1} \sin \psi = H_{x2} \cos \psi - H_{\phi2} \sin \psi \quad \text{---- (iv)}$$

Substituting the calculated field components in these boundary conditions at $r = a$, we have four equation in terms of four unknown coefficients $A$, $B$, $C$ and $D$.

Condition (i) gives,

$$C \ K_v (k_2 r) e^{j(\omega t - \beta (z + y))} \ Sin \psi + \left[ - \frac{j}{k_2^2} \right] \frac{\beta}{r} \ C \ jv \ K_v (k_2 r) - \mu \omega D k_2$$

$$K'_v (k_2 r) e^{j(\omega t - \beta (z + y))} \cos \psi = 0$$
\[ \Rightarrow C \ K_v (k_2 r) \sin \psi + \frac{\beta}{rk_2^2} C \ K_v (k_2 r) \cos \psi + \frac{j\mu_0 k_2}{k_2^2} D K'_v (k_2 r) \cos \psi = 0 \]

\[ \Rightarrow C \ K_v (k_2 r) \left[ \sin \psi + \frac{\beta}{r k_2^2} \cos \psi \right] + \frac{j\mu_0}{k_2} D K'_v (k_2 r) \cos \psi = 0 \]

\[ \Rightarrow C \quad K_v (k_2 r) \quad q_3 + \frac{j\mu_0}{k_2} D K'_v (k_2 r) \cos \psi = 0 \quad \text{---(2.11)} \]

Condition (ii) gives,

\[ A J_v (k_1 r) \cos \psi - \left[ - \frac{j}{k_1^2} \frac{\beta}{r} \left\{ A jv J_v (k_1 r) - \mu_0 B k_1 J'_v (k_1 r) \right\} \right] \sin \psi \]

\[ = C \quad K_v (k_2 r) \cos \psi - \left[ - \frac{j}{k_2^2} \frac{\beta}{r} \left\{ C jv K_v (k_2 r) - \mu_0 D k_2 K'_v (k_2 r) \right\} \right] \sin \psi \]

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\[ \Rightarrow A J_\nu (k_1 r) \cos \psi - \frac{\beta \nu}{k_1^2 r} A J_\nu (k_1 r) \sin \psi \frac{\mu j \omega}{k_1^2} B k_1 J'_\nu (k_1 r) \sin \psi \]

\[ = C K_\nu (k_2 r) \cos \psi - C \frac{\beta \nu}{k_2^2 r} K_\nu (k_2 r) \sin \psi \frac{\mu j \omega}{k_2^2} k_2 D K'_\nu (k_2 r) \sin \psi \]

\[ \Rightarrow A J_\nu (k_1 r) \left[ \cos \psi - \frac{\beta \nu}{k_1^2 r} \sin \psi \right] - \frac{\mu j \omega}{k_1} B J'_\nu (k_1 r) \sin \psi \]

\[ = C K_\nu (k_2 r) \left[ \cos \psi - \frac{\beta \nu}{k_2^2 r} \sin \psi \right] - \frac{\mu j \omega}{k_2} D K'_\nu (k_2 r) \sin \psi \]

\[ \Rightarrow A J_\nu (k_1 r) q_1 - \frac{\beta \nu}{k_1} B J'_\nu (k_1 r) \sin \psi - C K_\nu (k_2 r) q_2 \]

\[ + \frac{j \mu \omega}{k_2} D K'_\nu (k_2 r) \sin \psi = 0 \quad \quad \quad \quad \text{(2.12)} \]

Condition (iii) gives,

\[ \Rightarrow B J_\nu (k_1 r) \sin \psi + \left[ \frac{j}{k_1^2} \beta \right] B J_\nu (k_1 r) + \omega \in_1 k_1 A J'_\nu (k_1 r) \] \cos \psi

\[ = D K_\nu (k_2 r) \sin \psi + \left[ \frac{j}{k_2^2} \beta \right] j v D K_\nu (k_2 r) + \omega \in_2 C k_2 K'_\nu (k_2 r) \] \cos \psi

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\[ \Rightarrow B J_v \left( k_1 r \right) \sin \psi + \frac{\beta v}{k_1^2 r} \quad \text{B } J_v \left( k_1 r \right) \cos \psi - \frac{j \omega \in_1}{k_1} \quad \text{A } J'_v \left( k_1 r \right) \cos \psi \]

\[ = D K_v \left( k_2 r \right) \sin \psi + \frac{\beta v}{k_2^2 r} \quad \text{D} K_v \left( k_2 r \right) \cos \psi - \frac{j \omega \in_2}{k_2} \quad \text{C } K'_v \left( k_2 r \right) \cos \psi \]

\[ \Rightarrow B J_v \left( k_1 r \right) \left[ \sin \psi + \frac{\beta v}{k_1^2 r} \cos \psi \right] - \frac{\beta v}{k_1} \quad \text{A } J'_v \left( k_1 r \right) \cos \psi \]

\[ - D K_v \left( k_2 r \right) \left[ \sin \psi + \frac{\beta v}{k_2^2 r} \cos \psi \right] + \frac{j \omega \in_2}{k_2} \quad \text{C } K'_v \left( k_2 r \right) \cos \psi = 0 \]

\[ \Rightarrow - \frac{j \omega \in_1}{k_1} \quad \text{A } J'_v \left( k_1 r \right) \cos \psi + B J_v \left( k_1 r \right) q_3 + \frac{j \omega \in_2}{k_2} \quad \text{C } K'_v \left( k_2 r \right) \cos \psi \]

\[ - D K_v \left( k_2 r \right) q_4 = 0 \quad \text{(2.13)} \]

Condition (iv) gives,

\[ \frac{j}{r} \quad \beta \]

\[ \frac{B J_v \left( k_1 r \right) \cos \psi - \left( \frac{-j v B}{k_1^2 r} \right) J_v \left( k_1 r \right) + \omega \in_1 k_1 A J'_v \left( k_1 r \right)}{D} \]

\[ \frac{j}{r} \quad \beta \]

\[ = D K_v \left( k_2 r \right) \cos \psi - \left( \frac{-j v K_v \left( k_2 r \right) + \omega \in_2 C k_2 K'_v \left( k_2 r \right)}{k_2^2 r} \right) \sin \psi \]

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\( \Rightarrow B J_v (k_1 r) \cos \psi - \frac{\beta_v}{r k_1^2} J_v(k_1 r) \sin \psi + \frac{j\omega}{k_1} A J'_v (k_1 r) \sin \psi \)

\( \Rightarrow B J_v (k_2 r) \cos \psi + \frac{\beta_v}{k_2^2 r} D K_v (k_2 r) \sin \psi - \frac{j\omega}{k_2} C K'_v (k_2 r) \sin \psi = 0 \)

\( \Rightarrow B J_v (K_1 r) [\cos \psi - \frac{\beta_v}{k_1^2 r} \sin \psi] + \frac{j\omega}{k_1} A J'_v (K_1 r) \sin \psi \)

\( \Rightarrow \frac{j\omega}{k_1} A J'_v (k_1 r) \sin \psi + B J_v (k_1 r) q_1 \frac{j\omega}{k_2} C K'_v (k_2 r) \sin \psi \)

\(- D K_v (k_2 r) q_2 = 0 \)

----------(2.14)
Eliminating these coefficients, the following characteristics equation is obtained

\[
\begin{align*}
0 & \quad 0 & K_\nu (k_2 r) q_3 & \quad j\mu_\omega & K'_\nu(k_2 r) \cos \psi \\
J_\nu(k_1 r) q_1 & \quad - \frac{j\mu_\omega}{k_1} & - J'_\nu(k_1 r) \sin \psi & - K_\nu(k_2 r) q_2 & \quad j\mu_\omega & K'_\nu(k_2 r) \sin \psi \\
\frac{j}{k_1} & \quad 0 & 0 & 0 & 0 & 0 \\
\frac{j}{k_1} & \quad j \in \omega & J'_\nu(k_1 r) \cos \psi & - K_\nu(k_2 r) q_1 & \quad \frac{j}{k_2} & \quad j \in \omega \\
\frac{j}{k_1} & \quad j \in \omega & J'_\nu(k_1 r) \sin \psi & J_\nu(k_1 r) q_1 & \quad \frac{j}{k_2} & \quad j \in \omega \\
\frac{j}{k_1} & \quad j \in \omega & 0 & K'_\nu(k_2 r) \sin \psi & - K_\nu(k_2 r) q_2 & \quad 0 \\
\frac{j}{k_1} & \quad j \in \omega & 0 & 0 & 0 & 0 \\
\frac{j}{k_1} & \quad j \in \omega & 0 & 0 & 0 & 0 \\
\frac{j}{k_1} & \quad j \in \omega & 0 & 0 & 0 & 0 \\
\frac{j}{k_1} & \quad j \in \omega & 0 & 0 & 0 & 0 \\
\frac{j}{k_1} & \quad j \in \omega & 0 & 0 & 0 & 0 \\
\end{align*}
\]

\[= 0 \quad \text{---------- (2.15)}\]

Where

\[q_1 = [\cos \psi - \frac{\beta_\nu}{k_1^2 r} \sin \psi]\]

\[q_2 = [\cos \psi - \frac{\beta_\nu}{k_2^2 r} \sin \psi]\]
\[ q_3 = \left[ \sin \psi + \frac{\beta \nu}{r k_1^2} \cos \psi \right] \]
\[ q_4 = \left[ \sin \psi + \frac{\beta \nu}{k_2^2 r} \cos \psi \right] \]

Now expanding the above equation (2.15), we get,

\[ - K_v (k_2 r) q_3 \quad J_v (k_1 r) q_1 \quad J_v (k_1 r) q_3 \quad K_v (k_2 r) q_2 \]
\[ + K_v (k_2 r) q_3 \quad J_v (k_1 r) q_1 \quad K_v (k_2 r) q_4 \quad J_v (k_1 r) q_1 \]
\[ + K_v (k_2 r) q_3 \quad \frac{j \omega}{k_1} \quad J'_v (k_1 r) \sin \psi \quad \frac{j \omega}{k_1} \quad J'_v (k_1 r) \cos \psi \cdot K_v (k_2 r) q_2 \]
\[ + K_v (k_2 r) q_3 \quad \frac{j \omega}{k_1} \quad J'_v (k_1 r) \sin \psi K_v (k_2 r) q_4 \quad \frac{j \omega}{k_1} \quad J'_v (k_1 r) \sin \psi \]
\[ - K_v (k_2 r) q_3 \quad \frac{j \mu \omega}{k_2} \quad K'_v (k_2 r) \sin \psi \quad \frac{j \omega}{k_1} \quad J'_v (k_1 r) \cos \psi J_v (k_1 r) q_1 \]
\[ - K_v (k_2 r) q_3 \quad \frac{j \mu \omega}{k_2} \quad K'_v (k_2 r) \sin \psi J_v (k_1 r) q_3 \quad \frac{j \omega}{k_1} \quad J'_v (k_1 r) \sin \psi \]
\begin{align*}
\frac{j \mu \omega}{k_2} &+ \frac{K'_v(k_2 r) \cos \psi}{k_2} J_v(k_1 r) q_1 J_v(k_1 r) q_3 \frac{j \omega}{k_2} = 2 \frac{K'_v(k_2 r) \sin \psi}{k_2}.
\frac{j \mu \omega}{k_2} &+ \frac{K'_v(k_2 r) \cos \psi}{k_1} J'_v(k_1 r) q_1 J_v(k_1 r) q_1 \frac{j \omega}{k_2} = 2 \frac{K'_v(k_2 r) \sin \psi}{k_1}.
\frac{j \mu \omega}{k_2} &- \frac{K'_v(k_2 r) \cos \psi}{k_1} J'_v(k_1 r) \sin \psi \frac{j \omega}{k_2} = 2 \frac{K'_v(k_2 r) \sin \psi}{k_1}.
\frac{j \mu \omega}{k_2} &- \frac{K'_v(k_2 r) \cos \psi}{k_1} \frac{j \omega}{k_1} = 2 \frac{K'_v(k_2 r) \sin \psi}{k_1}.
\frac{j \mu \omega}{k_2} &- \frac{K'_v(k_2 r) \cos \psi}{k_1} J'_v(k_1 r) \cos \psi J_v(k_1 r) q_2 \frac{j \omega}{k_1} = 2 J'_v(k_1 r) \sin \psi
= 0 \quad \text{--- (2.16)}
\end{align*}
Further simplification gives,

\[ -K^2_v(k_2 r) J_v^2(k_1 r) q_1^2 q_1 q_2 + K^2_v(k_2 r) J_v^2(k_1 r) q_1 q_2 q_3 q_4 \]

\[ + K^2_v(k_2 r) q_2 q_3 \frac{j^2 \omega^2 \mu \in_1}{k_1^2} J'_v(k_1 r) \sin \psi \cos \psi + K^2_v(k_2 r) q_3 q_4 \frac{j^2 \omega^2 \mu \in_1}{k_1^2} \]

\[ J'_v(k_1 r) \sin^2 \psi - K_v(k_2 r) J_v(k_1 r) q_1 q_2 \frac{j^2 \omega^2 \mu \in_1}{k_1 k_2} J'_v(k_1 r) K'_v(k_2 r) \sin \psi \cos \psi \]

\[ -K_v(k_2 r) J_v(k_1 r) q_3 q_2 \frac{j^2 \omega^2 \mu \in_1}{k_1 k_2} - K'_v(k_2 r) J'_v(k_1 r) \sin^2 \psi \]

\[ + J_v^2(k_1 r) q_1 q_3 \frac{j^2 \omega^2 \mu \in_2}{k_2^2} K'^2_v(k_2 r) \sin \psi \cos \psi + J_v^2(k_1 r) q_1 q_2 \frac{j^2 \omega^2 \mu \in_2}{k_2^2} K'^2_v(k_2 r) \sin \psi \cos \psi \]

\[ \cos^2 \psi - \frac{j^2 \omega^2 \mu \in_2}{k_2^2} K'^2_v(k_2 r) \sin \psi \cos \psi - \frac{j^2 \omega^2 \mu \in_1}{k_1^2} J'_v(k_1 r) \sin \psi \cos \psi \]

\[ + \frac{j^2 \omega^2 \mu \in_2}{k_2^2} K'^2_v(k_2 r) \cos^2 \psi - \frac{j^2 \omega^2 \mu \in_1}{k_1^2} J'_v(k_1 r) \sin^2 \psi - \frac{j^2 \omega^2 \mu \in_1}{k_1 k_2} J'_v(k_1 r) \]
\[ K'(k_2 r) \cos^2 \psi J_v(k_1 r) K_v(k_2 r)q_1q_2 - \frac{j^2 \omega^2 \mu}{k_1 k_2} K'(k_2 r) J'_v(k_1 r) \]

\[ \sin \psi \cos \psi K_v(k_2 r) J_v(k_1 r)q_2q_3 = 0 \quad \text{-------- (2.17)} \]

But, \( (q_1 \cos \psi + q_3 \sin \psi) = 1 \)

Similarly

\[ q_2 \cos \psi + q_4 \sin \psi = 1 \]

And

\[ q_1q_4 - q_2q_3 = \frac{\beta v}{k_2^2 r} - \frac{\beta v}{k_1^2 r} \]

Similarly

\[ q_3 \sin \psi + q_2 \cos \psi = \left[ \frac{\beta v}{k_1^2 r} - \frac{\beta v}{k_2^2 r} \right] \sin \psi \cos \psi \]

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So equation (2.17) becomes,

\[ K_v^2 (k_2 \ r) \ J_v^2 (k_1 \ r) \ q_1 \ q_3 \ (q_1 \ q_4 - q_2 \ q_3) \]

\[ + K_v^2 (k_2 \ r) \ J_v^2 (k_1 \ r) \ q_3 \ \frac{j^2 \omega^2 \mu}{k_1^2} \ Sin \ \psi \ (q_2 \ Cos \ \psi + q_4 \ Sin \ \psi) \]

\[ - K_v (k_2 \ r) \ J_v (k_1 \ r) \ K'_v (k_2 \ r) \ J'_v (k_1 \ r) \ q_3 \ Sin \ \psi (q_1 \ Cos \ \psi + q_3 \ Sin \ \psi) \]

\[ + K_v^2 (k_2 \ r) \ J_v^2 (k_1 \ r) \ q_1 \ \frac{j^2 \omega^2 \mu}{k_2^2} \ (q_3 \ Sin \ \psi + q_1 \ Cos \ \psi) \]

\[ - \frac{j^2 \omega^2 \mu}{k_1 \ k_2} J_v (k_1 \ r) J'_v (k_1 \ r) K_v (k_2 \ r) K'_v (k_2 \ r) q_2 \ Cos \ \psi (q_1 \ Cos \ \psi + q_3 \ Sin \ \psi) \]

\[ = 0 \]

\[ \text{---- (2.18)} \]

Further simplification gives,

\[ K_v^2 (k_2 \ r) J_v^2 (k_1 \ r) q_1 \ q_2 \frac{\beta_v}{k_2^2 r} - \frac{\beta_v}{k_1^2 r} + K_v^2 (k_2 \ r) J_v^2 (k_1 \ r) \frac{j^2 \omega^2 \mu}{k_1^2} q_3 \ Sin \ \psi \]

\[ - K_v (k_2 \ r) J_v (k_1 \ r) K'_v (k_2 \ r) J'_v (k_1 \ r) q_3 \ Sin \ \psi \]

\[ + J_v^2 (k_1 \ r) K'_v (k_2 \ r) q_2 \ Cos \ \psi (q_1 \ Cos \ \psi + q_3 \ Sin \ \psi) \]

\[ = 0 \]

\[ \frac{j^2 \omega^2 \mu}{k_2^2} - K_v (k_1 \ r) J_v (k_1 \ r) K_v (k_2 \ r) K'_v (k_2 \ r) q_2 \ Cos \ \psi \frac{j^2 \omega^2 \mu}{k_1 \ k_2} = 0 \]
$$\Rightarrow K_{\nu}^2 (k_2 \ r) \ J_{\nu}^2 (k_1 \ r) \ q_1 \ q_2 \ \left[ \frac{\beta \nu}{k_2^2 \ r} - \frac{\beta \nu}{k_1^2 \ r} \right] + K_{\nu}^2 (k_2 \ r) \ J'_{\nu}^2 (k_1 \ r) \ \frac{j^2 \omega^2 \mu}{k_1^2} q_3 \ \text{Sin}\ \psi
$$

$$+ J_{\nu}^2 (k_1 \ r) \ K'_{\nu}^2 (k_2 \ r) q_1 \ \text{Cos} \ \psi \ \frac{j^2 \omega^2 \mu}{k_2^2} - K_{\nu} (k_2 \ r) \ K'_{\nu} (k_2 \ r) J_{\nu} (k_1 \ r) J'_{\nu} (k_1 \ r)$$

$$\frac{j^2 \omega^2 \mu}{k_1 \ k_2} \left[ \frac{\beta \nu}{k_1^2 \ r} - \frac{\beta \nu}{k_2^2 \ r} \right] \ \text{Sin} \ \psi \ \text{Cos} \ \psi = 0$$

$$\Rightarrow K_{\nu}^2 (k_2 \ r) \ J_{\nu}^2 (k_1 \ r) \ q_1 \ q_2 \ \left[ \frac{\beta \nu}{k_2^2 \ r} - \frac{\beta \nu}{k_1^2 \ r} \right] + K_{\nu}^2 (k_2 \ r) J'_{\nu}^2 (k_1 \ r) \ \frac{j^2 \omega^2 \mu}{k_1^2} q_3 \ \text{Sin} \ \psi
$$

$$+ J_{\nu}^2 (k_1 \ r) \ K'_{\nu}^2 (k_2 \ r) q_1 \ \text{Cos} \ \psi \ \frac{j^2 \omega^2 \mu}{k_2^2} - K_{\nu} (k_2 \ r) \ K'_{\nu} (k_2 \ r) J_{\nu} (k_1 \ r) J'_{\nu} (k_1 \ r)$$

$$\frac{j^2 \omega^2 \mu}{k_1 \ k_2} \left[ 1+ \left\{ \frac{\beta \nu}{k_1^2 \ r} - \frac{\beta \nu}{k_2^2 \ r} \right\} \ \text{Sin} \ \psi \ \text{Cos} \ \psi \right] = 0 \quad \text{(2.19)}$$
This is the general characteristics equation. For special case \( \nu = 0 \), using Bessel’s function identity, \( K'_0 = -K_1 \) and \( J'_0 = -J_1 \). Eqn. (2.19) is reduced as,

$$\Rightarrow K_0^2 \left( k_2 \ r \right) J_1^2 \left( k_1 \ r \right) \ \frac{j^2 \omega^2 \mu \in_1}{k_1^2} \ \text{Sin}^2 \ \psi + J_0^2 \left( k_1 \ r \right) K_1^2 \left( k_2 \ r \right) \ \frac{j^2 \omega^2 \mu \in_2}{k_2^2} \ \text{Cos}^2 \ \psi$$

$$+ J_0 \left( k_1 \ r \right) J_1 \left( k_1 \ r \right) K_1 \left( k_2 \ r \right) K_0 \left( k_2 \ r \right) \ \frac{j^2 \omega^2 \mu}{k_1 k_2} = 0 \ \ \ \ ---- (2.20)$$

In terms of \( U \) and \( V \) Eqn. (2.20) may be written as,

$$K_0^2 \left( \sqrt{V^2-U^2} \right) J_1^2 \left( U \right) \ \frac{4 \pi^2 a^2 \mu_1 \in_1 r}{\lambda^2 U^2} \ \text{Sin}^2 \ \psi + J_0^2 \left( U \right) K_1^2 \left( \sqrt{V^2-U^2} \right) \ \frac{4 \pi^2 a^2 \mu_2 \in_2 r}{\lambda^2 \left( V^2-U^2 \right)}$$

$$= \text{Cos}^2 \ \psi + J_0 \left( U \right) J_1 \left( U \right) K_1 \left( \sqrt{V^2-U^2} \right) K_0 \left( \sqrt{V^2-U^2} \right) \ \frac{4 \pi^2 a^2 \mu_1}{\lambda^2 U \left( \sqrt{V^2-U^2} \right)} = 0 \ \ \ \ ---- (2.21)$$

This characteristic equation (2.21) is now used in determining the mode cut-off condition and propagation characteristics.
2.5 MODE CUT-OFF CONDITION

For the case $\nu = 0$ with the limit as $(k_2 r) \to 0$ we get

\[ J_0 (k_c a) = 0 \]

(2.22)

The cut off parameter $(k_c a)$ is thus given by the roots of Eqn. (2.22). It is observed from Eqn. (2.22) that the cut-off condition is independent of $\psi$ and in general for any $\nu$, it can be shown that this property holds good. We see that as in case of a step index fiber, the TE and TM modes are supported in this structure. This is in contrast with the results of Singh et.al. (195) where they have reported that TE and TM modes cannot be supported in the structure. In the step-index fiber, the lowest order mode is $HE_{11}$. In step-index fiber the $TE_{01}$ and $TM_{01}$ mode give the cut-off parameter, $V = 2.405$. The value of the $V$-parameter for the lowest order mode is calculated to be 2.405 in agreement with the step index fiber.

2.6 POWER DISTRIBUTION

The total power carried by a particular fiber mode along the Z-direction is given by the:

\[ P_1 = \frac{1}{2} \text{Re} \int_0^\infty \int_0^{2\pi} [\overrightarrow{E} \times \overrightarrow{H}^*]. \overrightarrow{l}_z \cdot \text{r.} \, \text{ dr.} \, d\phi \]

(2.23)

Where $\text{Re}$ indicates the real part, and the asterisk (*) indicates complex conjugates. Further $\overrightarrow{l}_z$ is the unit vector in the Z-direction. Using the expression for $\overrightarrow{E}$ and $\overrightarrow{H}$, one can integrate to obtain the total power, including the power transmitted along the core and the power lost in the cladding. The ratio of the fractional power carried by the fiber core to the total average power and the ratio of the power carried by the cladding to the total
average power can be obtained by changing the limit of the first integral from 0 to $\infty$ and from 0 to a respectively.

$$\frac{P_c}{P_t} = 1 - \left(\frac{U}{V}\right)^2 \left[1 - \left(\frac{K_0 (W)}{K_1 (W)}\right)^2\right]$$

--------(2.24)

and

$$\frac{P_c}{P_t} = \left(\frac{U}{V}\right)^2 \left[1 - \left(\frac{K_0 (W)}{K_1 (W)}\right)^2\right]$$

--------(2.25)

Where $U = k_1 a$, $W = k_2 a$ and $V^2 = (U^2 + W^2)$

2.7 RESULTS AND DISCUSSION

The characteristic equation (2.20) or (2.21) defines the propagation modes in a helical cladded fiber. The solutions to the equation are extremely complicated and are obtained numerically. The dispersion characteristics, normalised propagation constant and fractional power in the core are calculated numerically using the following parameters

Core radius $a$: 5 $\mu$m, 25 $\mu$m, 50 $\mu$m
Operating wavelengths $\lambda$: 0.86 $\mu$m, 1.33 $\mu$m, 1.55 $\mu$m.
Helix pitch angle $\psi$: 0°, 3°, 6°, 9°, 15°, 30°, 45°.
Core refractive index $n_1$: 1.450
Cladding refractive index $n_2$: 1.447
**DISPERSION CHARACTERISTICS**

The dispersion characteristics was obtained by solving Eqn. (2.21) numerically for the core parameter \( U \) and the normalized frequency \( V \) for the operating wavelengths 0.86\( \mu m \), 1.33\( \mu m \) and 1.55\( \mu m \). The results are shown in Table 2.1. It was found that the dispersion relation is independent of operating wavelength. Hence, only one table is given for all the operating wavelengths. It is further observed that the dispersion characteristics is not dependent on core parameter. The results are in contrast to those of Singh et. al. (195), where they have found the dependence of dispersion curve on wavelength \( \lambda \) and core radius \( a \). Table (2.2) lists the cut off parameter for various modes in a step-index fiber. The result of Table (2.1) are shown in Fig. 2.4-2.8 which show the plot of core parameter \( U \) against the normalised frequency \( V \).

The dependence of dispersion curve on \( \psi \) is very small for pitch angle \( \psi \) below 15\(^0\). Singh et. al. (195) have reported significant dispersion even for small angle 3\(^0\), 6\(^0\) and 9\(^0\), but our results show that the dispersion is very small and appearing only in the fourth and fifth decimal places. The dispersion is significant for pitch angle higher than 15\(^0\), Fig. 2.8. This is true for all modes (not shown). The modes having cut-off parameter equal to 2.405 and 5.52 are shown by Singh et al. (195), not having any dispersion dependence on pitch angle \( \psi \). But our results show that there is significant dependence of cut-off parameter on pitch for angle greater than 15\(^0\). A perusal of Table 2.1 and its comparison with Table 2.2 show the existence of TE, TM, as well as, hybrid modes. However, the number of hybrid modes are drastically reduced.

**PROPAGATION CONSTANT b.**

The parameters \( V \) and \( b \) are plotted in Fig. 2.9-2.12. The behaviour is similar to that in a step-index fiber. Further as in the case of \( U \) Vs \( V \), the \( b \) Vs \( V \) curve is independent of operating wavelength \( \lambda \) and core parameter \( a \) but it depends on pitch angle \( \psi \). The curve is free from any anomalies and is in contrast to that of Singh et. al.
## TABLE 2.1 NORMILIZED FREQUENCY V Vs CORE PARAMETER U

CORE RADIUS = 5µm, 25µm, 50µm and OPERATING WAVE LENGTH=0.86µm, 1.33µm, 1.55µm

<table>
<thead>
<tr>
<th>ANGLE</th>
<th>STARTING POINT</th>
<th>CUT-OFF POINT</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>2.4048</td>
<td>2.4048</td>
</tr>
<tr>
<td></td>
<td>3.4413</td>
<td>2.5058</td>
</tr>
<tr>
<td></td>
<td>5.5201</td>
<td>5.5201</td>
</tr>
<tr>
<td></td>
<td>6.2512</td>
<td>5.6027</td>
</tr>
<tr>
<td></td>
<td>8.6537</td>
<td>8.6537</td>
</tr>
<tr>
<td></td>
<td>8.8158</td>
<td>8.7136</td>
</tr>
<tr>
<td>15°</td>
<td>2.6160</td>
<td>2.4116</td>
</tr>
<tr>
<td></td>
<td>3.4414</td>
<td>2.5058</td>
</tr>
<tr>
<td></td>
<td>5.6037</td>
<td>5.5241</td>
</tr>
<tr>
<td></td>
<td>6.2513</td>
<td>5.6027</td>
</tr>
<tr>
<td></td>
<td>8.6705</td>
<td>8.6584</td>
</tr>
<tr>
<td></td>
<td>8.8159</td>
<td>8.7136</td>
</tr>
<tr>
<td>30°</td>
<td>3.0541</td>
<td>2.4400</td>
</tr>
<tr>
<td></td>
<td>3.4419</td>
<td>2.5059</td>
</tr>
<tr>
<td></td>
<td>5.8580</td>
<td>5.5427</td>
</tr>
<tr>
<td></td>
<td>6.2520</td>
<td>5.6027</td>
</tr>
<tr>
<td></td>
<td>8.7233</td>
<td>8.6858</td>
</tr>
<tr>
<td></td>
<td>8.8160</td>
<td>8.7137</td>
</tr>
<tr>
<td>45°</td>
<td>3.4212</td>
<td>2.5082</td>
</tr>
<tr>
<td></td>
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<td>2.5082</td>
</tr>
<tr>
<td></td>
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<td>5.6050</td>
</tr>
<tr>
<td></td>
<td>6.2763</td>
<td>5.6050</td>
</tr>
<tr>
<td></td>
<td>8.8090</td>
<td>8.7154</td>
</tr>
<tr>
<td></td>
<td>8.8223</td>
<td>8.7154</td>
</tr>
</tbody>
</table>
(195), where some anomalies was reported for some critical value of $V$. Fig. 2.13 shows the dependence of propagation constant on pitch angle $\psi$.

Table – 2.2 *

<table>
<thead>
<tr>
<th>Mode</th>
<th>Cutoff parameter, $k_c a$</th>
<th>Mode</th>
<th>Cutoff parameter, $k_c a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HE$_{11}$</td>
<td>0.0</td>
<td>EH$_{31}$</td>
<td>6.38</td>
</tr>
<tr>
<td>TE$<em>{01}$ TM$</em>{01}$</td>
<td>2.405</td>
<td>E$_{51}$</td>
<td>6.41</td>
</tr>
<tr>
<td>HE$_{21}$</td>
<td>2.42</td>
<td>HE$<em>{13}$ EH$</em>{12}$</td>
<td>7.02</td>
</tr>
<tr>
<td>HE$<em>{12}$ EH$</em>{11}$</td>
<td>3.83</td>
<td>HE$_{32}$</td>
<td>7.02</td>
</tr>
<tr>
<td>HE$_{31}$</td>
<td>3.86</td>
<td>EH$_{41}$</td>
<td>7.59</td>
</tr>
<tr>
<td>EH$_{21}$</td>
<td>5.14</td>
<td>HE$_{61}$</td>
<td>7.61</td>
</tr>
<tr>
<td>HE$_{41}$</td>
<td>5.16</td>
<td>EH$_{22}$</td>
<td>8.42</td>
</tr>
<tr>
<td>TE$<em>{02}$ TM$</em>{02}$</td>
<td>5.52</td>
<td>HE$_{52}$</td>
<td>8.43</td>
</tr>
<tr>
<td>HE$_{22}$</td>
<td>5.53</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


**POWER DISTRIBUTION**

The variation of fractional power $\eta$ in the core with normalized frequency $V$ parameter is plotted in Fig. 2.14-2.18. The fractional power in the core shows normal behaviour unlike that reported by Singh et.al. (195) and is found similar to that for step-index fiber.
Fig. 2.4 - Normalized Frequency Vs Core Parameter

\[ a = 5, 25, 50 \, \mu m \]
\[ \lambda = 0.86, 1.33, 1.55 \, \mu m \]
\[ \psi = 0^0 \]
Fig. 2.5 - Normalized Frequency Vs Core Parameter

\[ a = 5, 25, 50 \, \mu m \]
\[ \lambda = 0.86, 1.33, 1.55 \, \mu m \]
\[ \psi = 15^0 \]
Fig. 2.6 - Normalized Frequency Vs Core Parameter

\( a = 5, 25, 50 \mu m \)
\( \lambda = 0.86, 1.33, 1.55 \mu m \)
\( \psi = 30^0 \)
Fig. 2.7 - Normalized Frequency Vs Core Parameter

\[ a = 5, 25, 50 \, \mu m \]
\[ \lambda = 0.86, 1.33, 1.55 \, \mu m \]
\[ \psi = 45^0 \]
Fig. 2.8 Normalized Frequency Vs Core Parameter

\[ a = 5, 25, 50 \, \mu m \]
\[ \lambda = 0.86, 1.33, 1.55 \mu m \]
\[ \psi = 45^0, 30^0, 15^0, 0^0 \]
Fig. 2.9 - Normalized Frequency Vs Propagation Constant

\[ a = 5, 25, 50 \, \mu m \]
\[ \lambda = 0.86, 1.33, 1.55 \, \mu m \]
\[ \psi = 0^0 \]
Fig. 2.10 - Normalized Frequency Vs Propagation Constant

\( a = 5, 25, 50 \, \mu \text{m} \)
\( \lambda = 0.86, 1.33, 1.55 \mu \text{m} \)
\( \psi = 15^0 \)
Fig. 2.11 - Normalized Frequency Vs Propagation Constant

\[ a = 5, 25, 50 \, \mu m \]
\[ \lambda = 0.86, 1.33, 1.55 \, \mu m \]
\[ \psi = 30^0 \]
Fig. 2.12 - Normalized Frequency Vs Propagation Constant

\( a = 5, 25, 50 \, \mu m \)
\( \lambda = 0.86, 1.33, 1.55 \mu m \)
\( \psi = 45^0 \)
Fig. 2.13 Normalized Frequency Vs Normalized Propagation Constant

- $a = 5, 25, 50 \, \mu m$
- $\lambda = 0.86, 1.33, 1.55 \, \mu m$
- $\psi = 45^\circ, 30^\circ, 15^\circ, 0^\circ$
Fig. 2.14 - Normalized Frequency Vs Fractional Power

\[ a = 5, 25, 50 \, \mu m \]
\[ \lambda = 0.86, 1.33, 1.55 \, \mu m \]
\[ \psi = 0^0 \]
Fig. 2.15 - Normalized Frequency Vs Fractional Power

\[ a = 5, 25, 50 \, \mu m \]
\[ \lambda = 0.86, 1.33, 1.55 \, \mu m \]
\[ \psi = 15^0 \]
Fig. 2.16 - Normalized Frequency Vs Fractional Power

\[ a = 5, 25, 50 \, \mu m \]
\[ \lambda = 0.86, 1.33, 1.55 \mu m \]
\[ \psi = 30^\circ \]
Fig. 2.17 - Normalized Frequency Vs Fractional Power

\( a = 5, 25, 50 \, \mu m \)
\( \lambda = 0.86, 1.33, 1.55 \mu m \)
\( \psi = 45^0 \)
Fig. 2.18 Normalized Frequency Vs Fractional Power

\[ a = 5, 25, 50 \, \mu m \]
\[ \lambda = 0.86, 1.33, 1.55 \, \mu m \]
\[ \psi = 45^\circ, 30^\circ, 15^\circ, 0^\circ \]