Chapter 6

Special Values of Gaussian Hypergeometric Series

6.1 Introduction

Classical hypergeometric functions are well understood. Mathematicians such as Gauss, Kummer, Pfaff, and Vandermonde deduced many special values of classical hypergeometric series at different arguments, for example see [4, 5, 17]. Since the introduction of hypergeometric functions over finite fields analogous to classical hypergeometric series, mathematicians are taking interest in finding special values of Gaussian hypergeometric functions. The Gaussian hypergeometric functions are closely related to different parameters of algebraic varieties and number theoretical objects similarly as classical hypergeometric series. However, only a few special values of the Gaussian hypergeometric series are known.

For a given elliptic curve $E$ over $\mathbb{Q}$, the trace of Frobenius endomorphism $a_p$ are important quantities. Recall that $\Delta(E)$ denotes the discriminant of $E$, and a prime $p$ is called good or bad accordingly $p \nmid \Delta(E)$ or $p \mid \Delta(E)$. In terms of the trace of Frobenius, the Hasse-Weil $L$-function of an elliptic $E$ is defined by the Euler product

\begin{footnote}{The contents of this chapter have been published in Int. J. Number Theory (2012) and J. Number Theory (2013).}

\end{footnote}
as

\[ L(E, s) := \prod_{p|\Delta(E)} (1 - a_p p^{-s})^{-1} \prod_{p \nmid \Delta(E)} (1 - a_p p^{-s} + p^{1-2s})^{-1}, \]  \quad (6.1.1)

where \( s \) is a complex number. It is known from Hasse-Weil bound that \(|a_p| < 2\sqrt{p}|\).

The Euler product (6.1.1) converges for \( \text{Re}(s) > \frac{3}{2} \) and has analytic continuation to the whole complex plane. Moreover, the Birch and Swinnerton-Dyer conjecture concerns the behavior of \( L(E, s) \) at \( s = 1 \). In fact, the conjecture predicts that \( \text{ord}_{s=1}(L(E, s)) = \text{rank}(E/\mathbb{Q}) \).

In Chapter 2, we have found general formulas for the trace of Frobenius endomorphism of certain families of elliptic curves in Weierstrass normal form in terms of Gaussian hypergeometric series. Thus, finding special values of Gaussian hypergeometric functions is an important and interesting problem. Earlier works of Greene [18], Ono [34], Ahlgren-Ono [1], and Evans-Greene \([11, 12]\) pave the way to find special values of many Gaussian hypergeometric functions. Most of them have been used to solve many old conjectures \([31, 32]\) and supercongruences \([33]\).

In this chapter, we mainly concentrate to find special values of certain Gaussian hypergeometric series using our earlier results.

## 6.2 Main results

In this section, we give a brief description of the special values of Gaussian hypergeometric series those have been already evaluated. Then we deduce some more values of hypergeometric functions over finite fields. We start with the special values of \( {2\choose 1} \) Gaussian hypergeometric series.

### 6.2.1 Values of \( {2\choose 1} \) Gaussian hypergeometric series

The story of special values of Gaussian hypergeometric series begins from its inception in [18] by Greene. After introducing hypergeometric functions over finite fields,
Greene [18] deduced certain special values of $2F_1$ Gaussian hypergeometric functions at some particular arguments.

**Theorem 6.2.1.** [18, (4.11), (4.14) & (4.15)] For any two characters $A, B$ on $\mathbb{F}_q$, we have

$$(i) \quad 2F_1\left( \begin{array}{c} A, B \\ \overline{AB} \end{array} \right| -1) = \begin{cases} 0, & \text{if } B \text{ is not a square;} \\ (C_A)^{(\phi C)_A}, & \text{if } B = C^2. \end{cases}$$

$$(ii) \quad 2F_1\left( \begin{array}{c} A, B \\ A^2 \end{array} \right| 2) = A(-1)\begin{cases} 0, & \text{if } B \text{ is not a square;} \\ (C_A)^{(\phi C)_A}, & \text{if } B = C^2. \end{cases}$$

$$(iii) \quad 2F_1\left( \begin{array}{c} A, \overline{A} \\ \overline{AB} \end{array} \right| \frac{1}{2}) = A(-2)\begin{cases} 0, & \text{if } B \text{ is not a square;} \\ (C_A)^{(\phi C)_A}, & \text{if } B = C^2. \end{cases}$$

Further, Ono worked in this direction and found the following interesting results in which he explicitly deduced special values of $2F_1$ hypergeometric series over $\mathbb{F}_p$. He used the technique of complex multiplication of elliptic curves to establish these results.

**Theorem 6.2.2.** [34, Thm. 2] Let $\lambda \in \{-1, \frac{1}{2}, 2\}$. If $p$ is an odd prime, then

$$2F_1\left( \begin{array}{c} \phi, \phi \\ \varepsilon \end{array} \right| \lambda) = \begin{cases} 0, & \text{if } p \equiv 3 \pmod{4}; \\ \frac{2x(-1)^{x+y+1}}{p}, & \text{if } x^2 + y^2 = p \equiv 1 \pmod{4}, \text{ and } x \text{ odd.} \end{cases}$$

Motivated by all these results, we have also deduced certain special values of $2F_1$ Gaussian hypergeometric series. We have mainly used the formulas of traces of Frobenius of elliptic curves and some transformation formulas of Gaussian hypergeometric series to prove the results.
Theorem 6.2.3. Let \( q = p^e, \ p > 0 \) a prime with \( q \equiv 1 \ (\text{mod} \ 4) \). Then

\[
\begin{align*}
(i) \quad 2F_1 \left( \begin{array}{c}
\chi_4, \ 
\chi_4^3 \ 
\varepsilon \n| \frac{1}{9}
\end{array} \right) & = \chi_4(-1)\phi(3) \left[ \left( \chi_4 \frac{1}{\phi} \right) + \left( \chi_4^3 \frac{3}{\phi} \right) \right]. \\
(ii) \quad 2F_1 \left( \begin{array}{c}
\chi_4, \ 
\chi_4^3 \ 
\varepsilon \n| \frac{8}{9}
\end{array} \right) & = \phi(3) \left[ \left( \chi_4 \frac{1}{\phi} \right) + \left( \chi_4^3 \frac{3}{\phi} \right) \right]. \\
(iii) \quad 2F_1 \left( \begin{array}{c}
\chi_4, \ 
\chi_4 \ 
\varepsilon \n| \frac{-1}{8}
\end{array} \right) & = \chi_4(-8) \left[ \left( \chi_4 \frac{1}{\phi} \right) + \left( \chi_4^3 \frac{3}{\phi} \right) \right]. \\
(iv) \quad 2F_1 \left( \begin{array}{c}
\chi_4, \ 
\chi_4 \ 
\varepsilon \n| -8
\end{array} \right) & = \left[ \left( \chi_4 \frac{1}{\phi} \right) + \left( \chi_4^3 \frac{3}{\phi} \right) \right].
\end{align*}
\]

where \( \chi_4 \) is a character of order 4 on \( \mathbb{F}_q \).

Proof. If we put \( A = B = \phi \) in Theorem 6.2.1 (iii) we obtain

\[
2F_1 \left( \begin{array}{c}
\phi, \ 
\phi \n| \frac{1}{2}
\end{array} \right) = \phi(-2) \left\{ \begin{array}{ll}
0, & \text{if } q \equiv 3 \ (\text{mod} \ 4);
\chi_4(-1)\phi(3) \left[ \left( \chi_4 \frac{1}{\phi} \right) + \left( \chi_4^3 \frac{3}{\phi} \right) \right], & \text{if } q \equiv 1 \ (\text{mod} \ 4),
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
0, & \text{if } q \equiv 3 \ (\text{mod} \ 4);
\phi(2) \left[ \left( \chi_4 \frac{1}{\phi} \right) + \left( \chi_4^3 \frac{3}{\phi} \right) \right], & \text{if } q \equiv 1 \ (\text{mod} \ 4),
\end{array} \right.
\]

(6.2.1)

This is because any character \( \chi \) of order \( l \) on \( \mathbb{F}_q \) is square if and only if \( \frac{q-1}{l} \) is even and hence \( \phi = \chi_4^2 \).

(i) Replacing \( \alpha \) by 6 in Corollary 2.3.3, we have

\[
2F_1 \left( \begin{array}{c}
\chi_4, \ 
\chi_4^3 \ 
\varepsilon \n| \frac{1}{9}
\end{array} \right) = \chi_4(-1)\phi(6)2F_1 \left( \begin{array}{c}
\phi, \ 
\phi \n| \frac{1}{2}
\end{array} \right).
\]

Hence the proof follows from (6.2.1).

(ii) Putting \( x = \frac{8}{9} \) in Theorem 1.3.15 (i), we obtain

\[
2F_1 \left( \begin{array}{c}
\chi_4, \ 
\chi_4^3 \ 
\varepsilon \n| \frac{8}{9}
\end{array} \right) = \chi_4(-1)2F_1 \left( \begin{array}{c}
\chi_4, \ 
\chi_4^3 \ 
\varepsilon \n| \frac{1}{9}
\end{array} \right).
\]
Thus the result (i) completes the proof of (ii).

(iii) For $x = -\frac{1}{8}$, Theorem 1.3.15 (ii) yields

$$2F_1 \left( \frac{\chi_4, \chi_4}{\varepsilon}, | -\frac{1}{8} \right) = \chi_4^{3}(\frac{9}{8})2F_1 \left( \frac{\chi_4, \chi_4^3}{\varepsilon}, | \frac{1}{9} \right).$$

Hence the proof of (iii) follows from the proof of (i).

(iv) Finally, putting $x = -8$ in Theorem 1.3.15 (ii), we have

$$2F_1 \left( \frac{\chi_4, \chi_4}{\varepsilon}, | -8 \right) = \chi_4^{3}(9)2F_1 \left( \frac{\chi_4, \chi_4^3}{\varepsilon}, | \frac{8}{9} \right).$$

This completes the proof due to (ii).

Moreover, if we use Theorem 1.3.15 (i) in each of Theorem 6.2.3 (iii) & (iv), respectively we can deduce the following Corollary.

**Corollary 6.2.4.** Let $q = p^e$, $p > 0$ a prime and $q \equiv 1 \pmod{4}$. Then

(i) \[ 2F_1 \left( \frac{\chi_4, \chi_4}{\varepsilon}, | \frac{9}{8} \right) = \chi_4(8) \left[ \left( \frac{\chi_4}{\phi} \right) + \left( \frac{\chi_4^3}{\phi} \right) \right]. \]

(ii) \[ 2F_1 \left( \frac{\chi_4, \chi_4}{\varepsilon}, | 9 \right) = \chi_4(-1) \left[ \left( \frac{\chi_4}{\phi} \right) + \left( \frac{\chi_4^3}{\phi} \right) \right]. \]

where $\chi_4$ is a character of order 4 on $\mathbb{F}_q$.

The above special values of Gaussian hypergeometric functions are valid only for certain special characters of particular order in $\mathbb{F}_q$. we now focus on special values of hypergeometric functions over finite fields containing characters of arbitrary order.

**Theorem 6.2.5.** Let $S$ be a character on $\mathbb{F}_q$ whose order is not equal to 3. If $S$ is
square of some character on \( \mathbb{F}_q \), then

\[
(i) \quad _2F_1 \left( \frac{\sqrt{S^{-3}\phi}, \sqrt{S^{-3}}}{S^{-2}}, \frac{4}{3} \right) = \begin{cases} 
0, & \text{if } q \equiv 2 \pmod{3}; \\
\frac{S(\frac{8}{27})J(\sqrt{S^{-1}}, \sqrt{S^{3}\phi})}{J(\phi, S)} \left[ \left( \frac{S}{\chi_3} \right) + \left( \frac{S^2}{\chi_3^2} \right) \right], & \text{if } q \equiv 1 \pmod{3}.
\end{cases}
\]

\[
(ii) \quad _2F_1 \left( \frac{\sqrt{S^{-3}\phi}, \sqrt{S^{-3}}}{S^{-1}\phi}, \frac{-1}{3} \right) = \begin{cases} 
0, & \text{if } q \equiv 2 \pmod{3}; \\
\frac{S(\frac{8}{27})J(\sqrt{S^{-1}}, \sqrt{S^{3}\phi})}{\sqrt{S}\phi(-1)J(\phi, S)} \left[ \left( \frac{S}{\chi_3} \right) + \left( \frac{S}{\chi_3^2} \right) \right], & \text{if } q \equiv 1 \pmod{3}.
\end{cases}
\]

\[
(iii) \quad _2F_1 \left( \frac{\sqrt{S^{-3}\phi}, \sqrt{S^{-1}}}{S^{-2}}, \frac{4}{3} \right) = \begin{cases} 
0, & \text{if } q \equiv 2 \pmod{3}; \\
\frac{\sqrt{S}(\frac{64}{27})J(\sqrt{S^{-1}}, \sqrt{S^{3}\phi})}{\phi(-3)J(\phi, S)} \left[ \left( \frac{S}{\chi_3} \right) + \left( \frac{S^2}{\chi_3^2} \right) \right], & \text{if } q \equiv 1 \pmod{3} \text{ and } S \neq \phi.
\end{cases}
\]

\[
(iv) \quad _2F_1 \left( \frac{\sqrt{S^{-3}\phi}, \sqrt{S\phi}}{S^{-1}\phi}, \frac{1}{4} \right) = \begin{cases} 
0, & \text{if } q \equiv 2 \pmod{3}; \\
\frac{\sqrt{S}(\frac{-1}{27})J(\sqrt{S^{-1}}, \sqrt{S^{3}\phi})}{\phi(3)J(\phi, S)} \left[ \left( \frac{S}{\chi_3} \right) + \left( \frac{S^2}{\chi_3^2} \right) \right], & \text{if } q \equiv 1 \pmod{3}.
\end{cases}
\]

We need the following two corollaries to deduce the above special values.

**Lemma 6.2.6.** Let \( S \) be any character on \( \mathbb{F}_q \). For \( \lambda = \frac{1}{3} \), we have

\[
\sum_{x \in \mathbb{F}_q} S((x - 1)(x^2 + \lambda)) = \begin{cases} 
0, & \text{if } q \equiv 2 \pmod{3}; \\
qS(\frac{-8}{27}) \left[ \left( \frac{S}{\chi_3} \right) + \left( \frac{S^2}{\chi_3^2} \right) \right], & \text{if } q \equiv 1 \pmod{3},
\end{cases}
\]

where \( \chi_3 \) is a character of order 3 on \( \mathbb{F}_q \).

**Proof.** Recall that making the change of variables \((x, y) \rightarrow (\frac{x}{5} + \frac{1}{3}, y)\), and then replacing \(-\frac{x}{6}\) by \(x\) we obtain the equivalent form of

\[
y' = (x - 1)(x^2 + \frac{1}{3})
\]

as

\[
y' = -\frac{8}{27}(1 + x^3).
\]
For any multiplicative character \( A \) on \( \mathbb{F}_q \), we have the binomial theorem from [18] as

\[
A(1 + x) = \delta(x) + \frac{q}{q-1} \sum_{\chi} \binom{A}{\chi} \chi(x),
\]

where \( \delta(x) = 1 \) (resp. 0) if \( x = 0 \) (resp. \( x \neq 0 \)). Using this, we have

\[
\sum_{x \in \mathbb{F}_q} S((x - 1)(x^2 + \frac{1}{3})) = \sum_{x \in \mathbb{F}_q} S(-\frac{8}{27})S(1 + x^3)
\]

\[
= S(-\frac{8}{27}) + \frac{q}{q-1} S(-\frac{8}{27}) \sum_{x \in \mathbb{F}_q} \sum_{\chi} \binom{S}{\chi} \chi^3(x)
\]

\[
= S(-\frac{8}{27}) + \frac{q}{q-1} S(-\frac{8}{27}) \sum_{\chi} \sum_{x \in \mathbb{F}_q} \chi^3(x).
\]

By Lemma 1.3.7, the innermost sum in the second term is nonzero only if \( \chi^3 = \varepsilon \) at which it is \( q - 1 \). Thus \( \chi = \varepsilon \), if \( q \equiv 2 \) (mod 3); and \( \chi = \varepsilon, \chi_3, \) or \( \chi_3^2 \), if \( q \equiv 1 \) (mod 3). Hence the result follows immediately.

**Lemma 6.2.7.** If \( S \) is square of some character on \( \mathbb{F}_q \) and \( S \) is not of order 3, then

\[
\sum_{x \in \mathbb{F}_q} S((x - 1)(x^2 + \lambda)) = \frac{qJ(\phi, S)}{J(\sqrt{S^{-1}}, \sqrt{S}^3)\phi} \cdot _2F_1\left( \begin{array}{c} \sqrt{S^{-3}}\phi, \sqrt{S^{-3}} \\ S^{-2} \end{array} \right| 1 + \lambda \right).
\]

**Proof.** Putting \( A = S, B = S \) and \( x = -\frac{1}{\lambda} \) in (4.2.3), we obtain

\[
\sum_{x \in \mathbb{F}_q} S((x - 1)(x^2 + \lambda)) = S(-\lambda)g(S, S; -\frac{1}{\lambda}).
\]

Again, \( S \) is a square of some character of \( \mathbb{F}_q \). Hence applying Theorem 4.2.1, we deduce that

\[
g(S, S; -\frac{1}{\lambda}) = qS^3(2)S(-\frac{1}{\lambda})F^*(S^{-3}, S^{-2}; 1 + \lambda),
\]

and hence

\[
\sum_{x \in \mathbb{F}_q} S((x - 1)(x^2 + \lambda)) = qS^3(2)F^*(S^{-3}, S^{-2}; 1 + \lambda).
\]

Thus using Theorem 4.2.3, we complete the proof.
Following the proof of Lemma 6.2.7 and applying Proposition 4.2.4 in spite of Theorem 4.2.3 in (6.2.2), we have the following result.

**Lemma 6.2.8.** If $S$ is a character of order 3 on $\mathbb{F}_q$, then

$$\sum_{x \in \mathbb{F}_q} S((x - 1)(x^2 + \lambda)) = q \cdot 2F1 \left( \begin{array}{c} \phi, \varepsilon \\ S \end{array} \right) .$$

(6.2.3)

Now, we give the proof of Theorem 6.2.5 using Lemma 6.2.6 and Lemma 6.2.7.

**Proof of 6.2.5.**

(i) Putting $\lambda = \frac{1}{3}$ in Lemma 6.2.7, we have

$$2F1 \left( \begin{array}{c} \sqrt{S^{-3}} \phi, \sqrt{S^{-3}} \\ S^{-2} \end{array} \mid \frac{4}{3} \right) = \frac{J(\sqrt{S^{-1}}, \sqrt{S^3} \phi)}{qJ(\phi, S)} \sum_{x \in \mathbb{F}_q} S((x - 1)(x^2 + \frac{1}{3})).$$

Therefore, we complete the proof of (i) after using Lemma 6.2.6.

(ii) Taking $x = \frac{4}{3}$ in Theorem 1.3.15 (i), we obtain

$$2F1 \left( \begin{array}{c} \sqrt{S^{-3}} \phi, \sqrt{S^{-3}} \\ S^{-1} \phi \end{array} \mid -\frac{1}{3} \right) = \sqrt{S} \phi(-1)2F1 \left( \begin{array}{c} \sqrt{S^{-3}} \phi, \sqrt{S^{-3}} \\ S^{-2} \end{array} \mid \frac{4}{3} \right).$$

Now using (i), we complete the proof.

(iii) Applying Theorem 1.3.15 (ii) for $x = \frac{4}{3}$, we have

$$2F1 \left( \begin{array}{c} \sqrt{S^{-3}} \phi, \sqrt{S^{-1}} \\ S^{-2} \end{array} \mid 4 \right) = \sqrt{S} \phi(-3)2F1 \left( \begin{array}{c} \sqrt{S^{-3}} \phi, \sqrt{S^{-3}} \\ S^{-2} \end{array} \mid \frac{4}{3} \right),$$

if $S \neq \phi$. Hence the result follows from (i).

(iv) Using Theorem 1.3.15 (ii) for $x = -\frac{1}{3}$, we find that

$$2F1 \left( \begin{array}{c} \sqrt{S^{-3}} \phi, \sqrt{S} \phi \\ S^{-1} \phi \end{array} \mid \frac{1}{4} \right) = \phi(-1)\sqrt{S} \phi(\frac{3}{4})2F1 \left( \begin{array}{c} \sqrt{S^{-3}} \phi, \sqrt{S^{-3}} \\ S^{-1} \phi \end{array} \mid -\frac{1}{3} \right)$$

and then the proof follows from the proof of (ii).

\[\square\]

**6.2.2 Values of $3F_2$ Gaussian hypergeometric series**

The value of $3F_2$ Gaussian hypergeometric series at the argument 1 is first evaluated by Greene in his famous paper [18]. The non-trivial values of $3F_2$ hypergeometric
series over \( \mathbb{F}_p \) are explicitly deduced by Ono. He used the technique of complex multiplication of elliptic curves to deduce the following special values of \( _3F_2 \) Gaussian hypergeometric series.

**Theorem 6.2.9.** [34, Thm. 6] If \( \lambda \in \{ \frac{9}{2}, 36, 8, 3, -12, \frac{63}{16}, -252 \} \), then for every odd prime \( p \) for which \( \text{ord}_p(\lambda(\lambda - 4)) = 0 \), the value of \( _3F_2(\frac{4}{1-\lambda}) \) is given by:

\[
(i) \quad _3F_2 \left( \begin{array}{c} \phi, \phi, \phi \\ \varepsilon, \varepsilon \end{array} \bigg| -8 \right) = \begin{cases} -\frac{1}{p}, & \text{if } p \equiv 3 \pmod{4}; \\ 4^{x^2-p}, & \text{if } p \equiv 1 \pmod{4}, p = x^2 + y^2, \text{ and } x \text{ odd.} \end{cases}
\]

\[
(ii) \quad _3F_2 \left( \begin{array}{c} \phi, \phi, \phi \\ \varepsilon, \varepsilon \end{array} \bigg| -\frac{1}{8} \right) = \begin{cases} -\phi(2) \pmod{p}, & \text{if } p \equiv 3 \pmod{4}; \\ \phi(2)(4x^2-p) \pmod{p^2}, & \text{if } x^2 + y^2 = p \equiv 1 \pmod{4} \text{ and } x \text{ odd.} \end{cases}
\]

\[
(iii) \quad _3F_2 \left( \begin{array}{c} \phi, \phi, \phi \\ \varepsilon, \varepsilon \end{array} \bigg| -1 \right) = \begin{cases} -\phi(2) \pmod{p}, & \text{if } p \equiv 5, 7 \pmod{8}; \\ \phi(2)(4x^2-p) \pmod{p^2}, & \text{if } p \equiv 1, 3 \pmod{8}, p = x^2 + 2y^2. \end{cases}
\]

\[
(iv) \quad _3F_2 \left( \begin{array}{c} \phi, \phi, \phi \\ \varepsilon, \varepsilon \end{array} \bigg| 4 \right) = \begin{cases} -\phi(-3) \pmod{p}, & \text{if } p \equiv 2 \pmod{3}; \\ \phi(-3)(4x^2-p) \pmod{p^2}, & \text{if } p \equiv 1 \pmod{3}, p = x^2 + 3y^2. \end{cases}
\]

\[
(v) \quad _3F_2 \left( \begin{array}{c} \phi, \phi, \phi \\ \varepsilon, \varepsilon \end{array} \bigg| \frac{1}{4} \right) = \begin{cases} -\phi(3) \pmod{p}, & \text{if } p \equiv 2 \pmod{3}; \\ \phi(3)(4x^2-p) \pmod{p^2}, & \text{if } p \equiv 1 \pmod{3}, p = x^2 + 3y^2. \end{cases}
\]

\[
(vi) \quad _3F_2 \left( \begin{array}{c} \phi, \phi, \phi \\ \varepsilon, \varepsilon \end{array} \bigg| 64 \right) = \begin{cases} -\phi(-7) \pmod{p}, & \text{if } p \equiv 3, 5, 6 \pmod{7}; \\ \phi(-7)(4x^2-p) \pmod{p^2}, & \text{if } p \equiv 1, 2, 4 \pmod{7}, p = x^2 + 7y^2. \end{cases}
\]

\[
(vii) \quad _3F_2 \left( \begin{array}{c} \phi, \phi, \phi \\ \varepsilon, \varepsilon \end{array} \bigg| \frac{1}{64} \right) = \begin{cases} -\phi(7) \pmod{p}, & \text{if } p \equiv 3, 5, 6 \pmod{7}; \\ \phi(7)(4x^2-p) \pmod{p^2}, & \text{if } p \equiv 1, 2, 4 \pmod{7}, p = x^2 + 7y^2. \end{cases}
\]

The characters involved in the above formulas are only quadratic and trivial. In [11], Evans and Greene gave an expression for \( _3F_2(\frac{1}{4}) \) containing characters of arbitrary orders, which extend Theorem 6.2.9 (v) evaluated by Ono. To obtain the following result, Evans and Greene deduced some transformation relations between \( _3F_2 \) and \( _2F_1 \) hypergeometric functions over finite fields analogous to Clausen Theorem of classical hypergeometric series.
Theorem 6.2.10. [11, Thm. 1.3] Let $S$ be a character on $\mathbb{F}_q$ which is not trivial, cubic, or quartic. Then

$$3F_2 \left( \begin{array}{l} \overline{S}, \ S^3, \ S \ | \ 1 \\ S^2, \ S\phi \end{array} \right) = \left\{ \begin{array}{ll} \frac{\phi(-1)S(4)}{q}, & \text{if } q \equiv 2 \pmod{3}; \\ \frac{\phi(-1)S(4)}{q} (1 + \frac{J(S,\chi)}{J(S,\overline{S})} + \frac{J(S,\chi)}{J(S,\overline{S})}), & \text{if } q \equiv 1 \pmod{3}, \end{array} \right.$$  

where $\chi$ is a character of order 3 on $\mathbb{F}_q$.

Further, Evans and Greene deduced the following special value of Gaussian hypergeometric series.

Theorem 6.2.11. [12, Thm. 1.8] Suppose that $S$ is a character whose order is not equal to 1, 3 or 4 over $\mathbb{F}_q$. Then

$$3F_2 \left( \begin{array}{l} \overline{S}, \ S^3, \ S \ | \ -1 \\ S^2, \ S\phi \end{array} \right) = \left\{ \begin{array}{ll} -\frac{\phi(-1)S(-8)}{q}, & \text{if } S \text{ is not a square}; \\ \frac{\phi(-1)S(8)}{q} + \frac{\phi(-1)S(2)J(S,\overline{S})}{q^2J(S,S)} (J(S,\overline{S})^2 + J(S,D\phi)^2), & \text{if } S = D^2. \end{array} \right.$$  

In the following theorem, we evaluate the value of $3F_2(4)$ hypergeometric series over $\mathbb{F}_q$, which extends another result of Ono [34] (see Theorem 6.2.9 (vi)). The result of Ono can be obtained by putting $S = \phi$, thus solving a problem posed by M. Koike [25, p. 465].

Theorem 6.2.12. If $S$ is a character on $\mathbb{F}_q$ with order not equal to 1, 3, or 4, then

$$3F_2 \left( \begin{array}{l} S^{-3}, \ S^{-1}, \ S^{-2}\phi \ | \ 4 \\ S^{-4}, \ S^{-2} \end{array} \right) = \left\{ \begin{array}{ll} -\frac{\phi(-3)S(16)}{q}, & \text{if } q \equiv 2 \pmod{3}; \\ S(-\frac{16}{27})J(S^{-1},S^{-1}) \left[ \frac{S}{\chi_3} + \frac{S}{\chi_2^2} \right]^2, & \text{if } q \equiv 1 \pmod{3}, \end{array} \right.$$  

where $\chi_3$ is a character of order 3 of $\mathbb{F}_q$.

We remark that in view of Theorem 1.3.14, there is a result similar to Theorem 6.2.12 in which the argument 4 is replaced by $\frac{1}{4}$. However, our result about $3F_2(\frac{1}{4})$
will be different from Theorem 6.2.10 obtained by Evans and Greene. We now prove the following lemma from which Theorem 6.2.12 will follow directly after combining with Lemma 6.2.6.

**Lemma 6.2.13.** If $S$ is a character on $\mathbb{F}_q$ whose order is not equal to 1, 3 or 4, then
\[
\begin{aligned}
3F_2 \left( \begin{array}{ccc}
S^{-3}, & S^{-1}, & S^{-2} \phi \\
S^{-4}, & S^{-2} & | \frac{1+\lambda}{\lambda}
\end{array} \right) = & \frac{J(S^{-1}, S^{-1})}{q^2 S(-4\lambda^3) J(S^{-3}, S)} \times \\
& \left[ \sum_{x \in \mathbb{F}_q} S((x - 1)(x^2 + \lambda)) \right]^2 - \frac{S^2(1+\lambda)}{q} \phi(-\lambda).
\end{aligned}
\]

**Proof.** Since $S$ is a character on $\mathbb{F}_q$ whose order is not equal to 1, 3 or 4, so applying Theorem 4.2.2 directly for $A = S^{-3}$, $C = S^{-2}$, and $x = \frac{1+\lambda}{\lambda}$, we obtain
\[
\begin{aligned}
3F_2 \left( \begin{array}{ccc}
S^{-3}, & S^{-1}, & S^{-2} \phi \\
S^{-4}, & S^{-2} & | \frac{1+\lambda}{\lambda}
\end{array} \right) = & \frac{J(S^{-1}, S^{-1})}{q^2 S(-4\lambda) J(S^{-3}, S)} g(S, S; -\frac{1}{\lambda})^2 \\
& - \frac{S^2(1+\lambda)}{q} \phi(-\lambda). \tag{6.2.4}
\end{aligned}
\]

Again, from (4.2.3), we have
\[
g(S, S; -\frac{1}{\lambda}) = \sum_{x \in \mathbb{F}_q} S^{-1}(-\lambda) S((x - 1)(x^2 + \lambda)). \tag{6.2.5}
\]

Hence combining (6.2.4) and (6.2.5), we complete the proof. \qed

**Proof of 6.2.12.** Putting $\lambda = \frac{1}{3}$ in Lemma 6.2.13, we obtain
\[
\begin{aligned}
3F_2 \left( \begin{array}{ccc}
S^{-3}, & S^{-1}, & S^{-2} \phi \\
S^{-4}, & S^{-2} & | 4
\end{array} \right) = & \frac{J(S^{-1}, S^{-1})}{q^2 S(-\frac{4}{27}) J(S^{-3}, S)} \left[ \sum_{x \in \mathbb{F}_q} S((x - 1)(x^2 + \frac{1}{3})) \right]^2 \\
& - \frac{S(16)}{q} \phi(-3).
\end{aligned}
\]

Now combining this with Lemma 6.2.6, we complete the proof of the result. \qed