Kelvin-Helmholtz instability in weakly coupled dusty plasmas: 2-D studies

In the previous Chapters (3,4), we have carried out 1-D studies of collective phenomena in weakly and strongly coupled dusty plasma medium where we discussed the existence, formation and evolution of coherent structures (e.g. solitons, shocks, cusps etc.) in such systems. The 1-D simulations presented in previous Chapters constrain the system wherein only longitudinal modes can be excited. In this Chapter and subsequent Chapters, we will study some nonlinear collective phenomena in 2-D. This adds another degree of freedom making way to study collective phenomena where variations along the transverse direction too are necessary. In this Chapter, we extensively explore the Kelvin-Helmholtz instability in the context of weakly coupled dusty plasmas. The linear and nonlinear (perturbative and exact nonlinear simulations) studies carried out and a comparative study of growth rate and nonlinear evolution of this sheared flow instability has been made with hydrodynamic fluids.

5.1 Introduction

The classical Kelvin-Helmholtz (KH) instability has been extensively investigated in a variety of fluid systems and applied to many physical scenarios ever since the first enunciation of its physical mechanism by Helmholtz in 1868 [67] and its mathematical formulation by Kelvin in 1871 [66]. While much of the earliest work is devoted to the excitation of this instability in neutral hydrodynamic flu-
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ids [126,127], the KH instability is also important in plasmas for understanding a variety of astrophysical phenomena involving sheared plasma flows [16,128,129]. In some of these applications the plasma can also have a significant dust component (e.g. in cometary tails, planetary ring systems, plasma torches in industrial applications etc.) [13,15,18–20] and it is important to study the characteristics of the KH instability in such a plasma.

Motivated by such considerations, in the present Chapter, we have carried out a basic investigation to look at the linear stability of the KH mode in a weakly coupled dusty plasma fluid. In particular, we have looked at the effect of compressibility and dispersion due to coupling with dust acoustic waves on the threshold and growth rate of the instability. While similar effects have been studied in the past in the context of neutral fluids [130] their manifestation in a dusty fluid can be quite distinct and different. In a dusty plasma compressibility arises through two mechanisms, namely, due to a finite dust temperature and also via the interaction energies of the dust fluid with the electron and ion species. In general the magnitude of compressibility depends on the temperature and density of these (electrons and ions) species. For this reason the variation of any of these parameters (ion density/dust charge, ion temperature/dust temperature, ion temperature/electron temperature, dust charge density, etc.) can cause large variations in the compressibility parameter. Thus, the dusty plasma can exhibit behaviour which can correspond to being totally in the incompressible regime to an extremely compressible one. This can be observed from Table - 5.1, where we show the typical range of the dust acoustic speed $c_{DA}$ and the flow velocities and the resultant Mach number for various systems where dusty plasma is prevalent.

<table>
<thead>
<tr>
<th>Physical Systems</th>
<th>$c_{DA}$ (mt/sec)</th>
<th>$V_{d0}$ (mt/sec)</th>
<th>$M_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lab. Dusty Plasmas ( [4,16,71,131,132] )</td>
<td>0.02 - 3.16</td>
<td>0.04 - 1</td>
<td>0.4 - 2</td>
</tr>
<tr>
<td>Comets and cometary tails ( [14,133] )</td>
<td>0.02 - 10</td>
<td>~ 10 - 4000</td>
<td>~ 1 - 400</td>
</tr>
<tr>
<td>Saturn Rings ( [17,19,109] )</td>
<td>3.1</td>
<td>0.5 - 100</td>
<td>0.15 - 32</td>
</tr>
<tr>
<td>Protoplanetary accretion disks ( [134] )</td>
<td>~ 200</td>
<td>~ 10000</td>
<td>~ 5</td>
</tr>
</tbody>
</table>

Furthermore, the compressible dust perturbations can also be dispersive, which is quite unlike the sound waves in a neutral hydrodynamic fluid. For a detailed
characterization of the influence of compressibility and dispersion on the KH mode we have carried out a non-local stability investigation for different shear flow profiles using both analytic and numerical approaches. Three distinct, simple and specific flow profiles, viz., step, piecewise linear and tangent hyperbolic profiles have been analysed. For the step/tangent hyperbolic profiles exact value of the growth rate have been obtained analytically/numerically respectively. A perturbative scheme for the evaluation of the growth rate and the threshold wave-number for the excitation of instability in terms of various orders of the compressibility parameter (Mach number) has been put forth. This scheme has been applied to the piecewise linear and the tangent hyperbolic cases. It has been shown that the analytic perturbative approximation is quite good when compressible effects are weak. We have also provided comparisons in suitable limits with the known results of neutral hydrodynamic fluid, all throughout our analysis, thereby putting our study in proper perspective. We find that the presence of a compressible mode reduces the growth rate and also diminishes the range of unstable wave-numbers. The threshold value of the wave-number beyond which the growth rate vanishes, is smaller in the presence of compressibility. We also find that when the dispersive effects are taken into account, the growth rate of the KH mode reduces further. The eigen functions of the unstable modes are broader in the presence of compressible effects and dispersion causes further broadening of its shape.

The nonlinear stage of the instability has also been investigated by numerically simulating the governing set of dusty plasma equations. The role of compressibility on saturation of the instability, formation of vortex structure in saturation state and process of mergers of vorticity patches have been illustrated by extensive simulations.

The Chapter is organized as follows. In section 5.2, we present the governing fluid equations for the weakly coupled dusty plasma system. Section 5.3 contains the linearized equations for the KH instability for a sheared equilibrium flow profile. In section 5.4 we provide a perturbative treatment to account for effects arising from weak compressibility, on the KH mode. The first order perturbative corrections are then compared with exact results obtained subsequently in section 5.5. In section 5.6, we provide a physical picture of the instability and discuss how compressibility causes the reduction of the KH growth rate. Section 5.7 contains a brief description of the numerical procedure adopted for the simulation of the fluid
equations pertaining to the dusty plasma medium in 2-D. The section also contains the description of results obtained during the linear phase of the instability. These results essentially validate our code. In section 5.8 the salient observations from simulation in the nonlinear phase of the instability have been described and a physical interpretation of the results have been provided. Our results are briefly summarized and discussed in the concluding section 5.9.

5.2 Governing equations

The governing equations for a weakly coupled dusty plasma system comprises of continuity and momentum equations for the dust fluid along with Poisson’s equation as described in Chapter 2. The momentum equation for weakly coupled dusty plasma system could be obtained from Eq. (2.1) in the limit of \( \tau_m d/dt \ll 1 \). The normalized form of the momentum equation for this dust fluid system can be written as:

\[
\left( \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla \right) \mathbf{v}_d + \frac{\alpha}{n_d} \nabla n_d - \nabla \phi = 0
\]  

(5.1)

Here, we have considered inviscid dust fluid. The continuity and Poisson’s equations are referred from Eqs. (2.2, 2.3). We do not consider the effect of the evolution of energy and/or temperature in the present work. It should be noted from Eq. (5.1) that the compressible perturbations can arise both from the \( \nabla n_d \) as well as \( \nabla \phi \) terms in the equation.

5.3 Equations for linear instability analysis

We linearize the Eqs. (2.2,2.3,5.1) around the equilibrium dust density \( n_{d0} \) and a sheared dust flow velocity \( \mathbf{v}_d = \hat{y} v_0(x) \) along \( \hat{y} \) which varies with \( x \). Various specific forms of the flow velocity will be chosen for the analysis later. For the sake of tractability and simplicity we consider here only 2-D perturbations lying in the \( x - y \) plane (of flow and the shear direction) for our analysis. The linearized equations after Fourier analyzing in \( y \) and time coordinates can be written as

\[- i \Omega n_1 + n_{d0}(i k_y v_{1y} + v'_{1z}) = 0, \]  

(5.2)
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\[-i\Omega v_{1x} + \alpha n'_1 - \phi'_1 = 0, \tag{5.3}\]
\[-i\Omega v_{1y} + ik_y(\alpha n_1 - \phi_1) + v_{1x}v'_0 = 0, \tag{5.4}\]
\[\phi''_1 - k_y^2 \phi_1 = n_1 + (\mu_e \sigma_e + \mu_i)\phi_1 \tag{5.5}\]

Here prime (′) as a superscript denotes a derivative with respect to \(x\) (the coordinate along which the equilibrium velocity is sheared). The subscripts 0 and 1 denote the equilibrium and the perturbed quantities respectively and \(\Omega = (\omega - k_y v_0)\).

Eliminating all fields in terms of \(\phi_1\) from the set of Eqs. (5.2, 5.3, 5.4, 5.5) we obtain

\[
\begin{align*}
\left\{ \frac{d^2}{dx^2} - k_y^2 \right\} \left\{ \phi_1 - \alpha \left( \frac{d^2}{dx^2} - k_y^2 - \mu_e \sigma_e - \mu_i \right) \phi_1 \right\} &= \Omega^2 \left( \frac{d^2}{dx^2} - k_y^2 - \mu_e \sigma_e - \mu_i \right) \phi_1 \\
- \frac{2k_y v_0'}{\Omega} \left( \phi_1 - \alpha \left( \frac{d^2}{dx^2} - k_y^2 - \mu_e \sigma_e - \mu_i \right) \phi_1 \right) &= 0,
\end{align*}
\tag{5.6}\]

which represents the linearized final equation for our instability analysis.

A simplified limiting case is when the perturbations are quasineutral. The quasineutral perturbations essentially stand for those perturbations for which the left hand side of the Poisson’s equation [Eq. (5.5)] can be ignored, i.e. \(\nabla^2 \phi_1 \approx 0\).

Thus, for this case there exist a simple relationship

\[\phi_1 = -n_1/(\mu_e \sigma_e + \mu_i) \tag{5.7}\]

in the linear regime between the scalar potential and the density perturbation. In the quasineutral limit when we ignore \(\nabla^2 \phi_1\) it can be shown that the Eq. (5.6) gets simplified to

\[n''_1 - k_y^2 n_1 + \frac{2k_y n'_1 v'_0}{\Omega} + \frac{\Omega^2}{\alpha_1} n_1 = 0 \tag{5.8}\]

Here, \(\alpha_1 = \alpha + 1/(\mu_e \sigma_e + \mu_i)\) and we have used Eq. (5.7) to express \(\phi_1\) in terms of \(n_1\). Thus, \(\alpha_1\) represents the total effect of compressibility arising from finite dust temperature as well as interactions due to ion and electron species of the plasma. It is interesting to note that Eq. (5.8) has two more representations as written below.

\[\psi'' + \left( \frac{\Omega^2}{\alpha_1} - k_y^2 - \frac{k_y n_s'' v_s}{\Omega} - \frac{2k_s^2 v_s'^2}{\Omega^2} \right) \psi = 0 \tag{5.9}\]
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This arises when first derivative of $n_1$ in Eq. (5.8) is eliminated by using the transformation $\psi = n_1/\Omega$. The other form of the equation is in terms of the perturbed velocity field $v_{1x}$

$$k_y \Omega v_{1x} - \left( \frac{k_y v_0' v_{1x} + \Omega v'_{1x}}{k_y - \Omega^2/\alpha_1 k_y} \right)' = 0 \quad (5.10)$$

In fact Eq. (5.10) is the familiar linearized Kelvin - Helmholtz (KH) instability equation for a neutral compressible fluid [130]. This is expected as in the quasineutral limit the equations for dust fluid should reduce to those of the neutral compressible fluid. The perturbed velocity $v_{1x}$ is related to $n_1$ with the relationship $v_{1x} = -in_1'/\Omega$. Any of these Eqs. (5.8, 5.9, 5.10) can be used for the purpose of evaluating the growth rate and the unstable eigenfunction for a compressible fluid without dispersive effects. The choice of a particular form out of these three equations is often guided by whichever simplifies the analysis of any given problem at hand.

For an incompressible fluid we have $1/M_A = \sqrt{\alpha_1/V_0} = \infty$. Here $M_A$ is the Mach number. In this case there exist no density perturbations in the fluid. The variable $n_1$ of Equation (5.8) in this limit essentially represents the pressure perturbations of the fluid. All these three forms, viz., Eq. (5.8), Eq. (5.9) and Eq. (5.10) in the limit of $\alpha_1 \to \infty$ (for finite non zero $V_0$) reduce to the familiar incompressible limit of the linearized KH equation, viz.,

$$\left( \frac{d^2}{dx^2} - k_y^2 \right) v_{1x} + \frac{k_y v_0''}{\Omega} v_{1x} = 0 \quad (5.11)$$

5.4 Perturbative treatment for compressibility

The value of the KH growth rate and the threshold wave vector for instability along the periodic flow direction for the incompressible case (with $\alpha_1 = \infty$) are well known. The growth rate has a typical bell shaped form as a function of $k_y\epsilon$, where $k_y$ is the wavenumber along the flow direction and $\epsilon$ is the shear width of the equilibrium flow. The growth rate is zero at $k_y = 0$, maximizes and then again falls back to zero at a threshold value of wavenumber $k_y = k_y^{th}$. It is observed that $k_y^{th}$ is typically of the order of $1/\epsilon$ and is exactly $1/\epsilon$ for a tangent hyperbolic
form of the shear velocity.

It would be interesting to see how the compressibility alters the growth rate and the unstable range of wave numbers. To understand this analytically we employ a perturbative treatment for weakly compressible cases in this section. We consider here the case of large $\alpha_1$ and evaluate the role of compressibility by considering a perturbative expansion in $O(1/\alpha_1)$ around the zeroth order known incompressible result. The problem is thus cast in various orders of $\nu = 1/\alpha_1$. We assume that the eigenvalue and the eigenfunction can be expanded as

$$\omega = \omega^{(0)} + \omega^{(1)} + \ldots,$$

and

$$v_{1x} = v = v^{(0)} + v^{(1)} + \ldots$$

respectively. The superscripts index inside the brackets represent the various orders of $\nu$ the function is dependent upon. We have dropped the suffix 1x from $v_{1x}$ (representing perturbation from equilibrium shear flow). The zeroth order of Eq. (5.10) is

$$v^{(0)''} + \left(-k_y^2 + \frac{k_y v_0''}{\Omega^{(0)}}\right) v^{(0)} = 0 \quad (5.12)$$

The first order expansion of Eq. (5.10) in $1/\alpha_1$ is given by the following equation:

$$v^{(1)''} + \left(-k_y^2 + \frac{k_y v_0''}{\Omega^{(0)}}\right) v^{(1)} - \frac{k_y v_0''}{\Omega^{(0)^2}} \omega^{(1)} v^{(0)}$$

$$+ \frac{1}{\alpha_1 k_y^2 \Omega^{(0)}} \left\{ \Omega^{(0)^2} \left( k_y v_0 v_0' + v^{(0)'} \Omega^{(0)} \right) \right\}' = 0 \quad (5.13)$$

We multiply Eq. (5.13) by $v^{(0)}$ and integrate over $x$. For the first term the $x$ differentiations are transferred to $v^{(0)}$ from $v^{(1)}$. Using Eq.(5.12) the first two terms then vanish. The remaining equation can then be cast as

$$\omega^{(1)} \int \frac{v_0'' v^{(0)^2}}{\Omega^{(0)^2}} dx = \frac{1}{\alpha_1 k_y^2} \int \frac{1}{\Omega^{(0)}} \left[ \Omega^{(0)^2} \left( k_y v_0 v_0' + \Omega^{(0)} v^{(0)'} \right) \right]' v^{(0)} dx \quad (5.14)$$

By evaluating the two integrals for the zeroth order wavefunction for specific shear flow profiles one can get the value for $\omega^{(1)}$. For the step function profile the effect of compressibility can be evaluated exactly analytically (this is shown in the next section). It shows that the growth rate reduces due to compressibility. For other profiles the exact result is obtained numerically which also show that the growth rate reduces due to compressibility. We will show in the section IV when we
consider specific profiles that perturbative expression for \( \omega^{(1)} \) from Eq. (5.14) also shows it to be negative. Furthermore, for weak compressibility the perturbative expressions are in good agreement with the exact results obtained numerically.

We now obtain the expression for the altered threshold wavenumber \( k_y \) for growth, by the first order perturbative treatment. To evaluate the threshold we put \( \omega = 0 \) and look for the change in the value of \( k_y \) from its original incompressible value of \( k_y^{(0)} \). Thus, expanding \( k_y = k_y^{(0)} + k_y^{(1)} + \ldots \) in this case we have the zeroth order equation as

\[
v^{(0)
''} - \left( k_y^{(0)} + \frac{v_y'^2}{v_0^2} \right) v^{(0)} = 0
\]

(5.15)

The first order equation is

\[
v^{(1)
''} - \left( k_y^{(0)} + \frac{v_y'^2}{v_0^2} \right) v^{(1)} - 2k_y^{(0)} k_y^{(1)} v^{(1)}
- \frac{v_0}{\alpha_1} \left( v^{(0)} v_y'' - v^{(0)''} \right) - \frac{2v_0}{\alpha_1} \left( v^{(0)} v_y' - v^{(0)'} v_0 \right) = 0
\]

(5.16)

We again apply the same technique of multiplying Eq. (5.16) by \( v^{(0)} \) and integrating over \( x \). We transfer the two spatial derivatives from \( v^{(1)} \) in the first term of Eq. (5.16) to \( v^{(0)} \) and use Eq. (5.15). The contribution from the first two terms of Eq. (5.16) therefore vanishes from the integration. Again the first order correction for \( k_y \) can be obtained from the following expression:

\[
2k_y^{(0)} k_y^{(1)} \alpha_1 \int v^{(0)2} dx = - \int \left[ v_0 v^{(0)} v_y'' + 2v_0 v^{(0)2} + 2v_0 v_y'^2 - 2v_0 v^{(0)} v_y'' v_0^2 \right] dx
\]

(5.17)

Simplifying Eqn. (5.17) we get a final expression as follows

\[
k^{(1)} = - \frac{1}{2k_y^{(0)} \alpha_1} \frac{\int_{-\infty}^{\infty} \left( v_0 v^{(0)} - v_0 v^{(0)} \right)^2 dx}{\int_{-\infty}^{\infty} v^{(0)2} dx}
\]

(5.18)

Both the integrands of the numerator as well as that of the denominator of Eq. (5.18) being positive definite, it shows that \( k^{(1)} \) would be negative. The shows clearly that the threshold value of the wavenumber decreases in the presence of compressibility. The integrals in Eq. (5.18) has been evaluated numerically for
obtaining the value of $k^{(1)}$ for specific flow profiles. This is discussed in section V.

### 5.5 Instability analysis for specific flows

![Diagram](image)

**Figure 5.1:** The equilibrium dust shear velocity profiles have been shown in the figure. The subplot (a) shows the step function shear flow velocity and (b) represents a piecewise linear flow profile.

We now try solving the linearized equations Eqs. (5.8,5.9,5.10) for specific given profiles of the sheared flow velocity exactly. Often for analytical tractability one considers simple forms of the equilibrium velocity flow profiles. For complicated shear flow structure numerical solutions are obtained. Typically, the simplest case that has often been considered is the case when the flow velocity has a step function form about a point say at $x = 0$. In this case, the velocity is uniform on both sides of $x = 0$ but the values differ by a finite amount. The form of such a flow profile has been illustrated in Fig. 5.1(a). In this case the eigenvalue equation turns out to be homogeneous in the two regions and can be expressed in terms of exponential functions. However, to obtain the final solution the eigenfunctions in the two
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Separate regions have to be matched at the location of discontinuity. There are two matching conditions (the differential equation [Eq. (5.8)] being of second order under the quasineutral assumption) for the fields. The Eq. (5.8) can be expressed in three equivalent forms of differential equations in terms of variables \( v_{1x} \), \( n_1 \) and \( \psi \). The form of two matching conditions in terms of each variables can be written down as follows:

\[
\begin{align*}
  f_1(v_{1x}) &= \Omega v_{1x}' + k_y v_0 v_{1x}; \\
  f_2(v_{1x}) &= \frac{v_{1x}}{\Omega} \\
  f_1(n_1) &= \frac{n_1}{\Omega^2}; \\
  f_2(n_1) &= n_1 \\
  f_1(\psi) &= \frac{\psi'}{\Omega} - \frac{k_y v_0 \psi}{\Omega^2}; \\
  f_2(\psi) &= \Omega \psi
\end{align*}
\]  

(5.19)

These need to be satisfied at the location of the discontinuity of the equilibrium flow. For the full fourth order differential equation [Eq. (5.6)] for which dispersive effects from Poisson’s equation have been incorporated four matching conditions are required. These matching conditions are as follows:

\[
\begin{align*}
  f_1(\phi_1) &= -\frac{\alpha \phi''}{\Omega^2} + \frac{(1 + \alpha R) \phi'}{\Omega^2}; \\
  f_2(\phi_1) &= -\alpha \phi'' + (1 + \alpha R) \phi_1 \\
  f_3(\phi_1) &= \alpha \phi_1; \\
  f_4(\phi_1) &= \alpha \phi_1
\end{align*}
\]  

(5.20)

where \( R = k_y^2 + \mu_e \sigma_e + \mu_i \).

The step velocity profile is an extreme case of any flow profile with zero shear width. In a realistic situation the flow shear would always have a finite shear width. A piecewise linear velocity flow profile as shown in Fig. 5.1(b) takes account of a finite shear width through a middle region where the equilibrium flow varies linearly with \( x \), thereby connecting the two layers with disparate flows. The matching conditions of Eq. (5.19) being valid for the abrupt step function discontinuity, clearly, also holds for any smoother profile, the piecewise linear case being one.

### 5.5.1 Step profile

The equilibrium velocity profile is chosen to have a step function form as shown in Fig. 5.1(a). For this profile \( v_0 = -V_0 \) for \( -\infty < x < 0 \) (we denote this by Region I) and \( v_0 = V_0 \) for \( 0 < x < \infty \) (Region II). Thus, there is an abrupt jump in the flow
at \( x = 0 \). The simplicity of the profile renders the possibility of exact analytical evaluation of the growth rate as we would now observe. The linearized equation takes a simple homogeneous form in the two regions. We employ Eq. (5.8) for \( n_1 \) for the purpose of analysis here. Denoting the fields in the two regions by suffix \( I \) and \( II \) we have

\[
n''_{1,I,II} - k_y^2 n_{1,I,II} + \frac{\Omega_{I,II}^2}{\alpha_1} n_{1,I,II} = 0 \tag{5.21}
\]

The solutions for \( n_{1,I,II} \) in the two regions can now be obtained easily and have the following form:

\[
n_{1,I} = B \exp \left\{ \sqrt{k_y^2 - \Omega_{I}^2/\alpha_1 x} \right\}; \quad n_{1,II} = A \exp \left\{ -\sqrt{k_y^2 - \Omega_{II}^2/\alpha_1 x} \right\}
\]

The boundary condition for the solution to vanish at \( \pm \infty \) has been used. Now, by employing the matching conditions we seek to obtain the eigenvalue \( \omega \). This is provided by

\[
\omega^2 + (k_y V_0)^2 = \frac{(\omega^2 - (k_y V_0)^2)^2}{2k_y^2 \alpha_1} \tag{5.22}
\]

The above equation in the limit of \( \alpha_1 \to \infty \) gives the correct growth rate \( \gamma = i\omega = k_y V_0 \) for the incompressible step function profile of the flow. The growth rate for the compressible case can be obtained by choosing \( \alpha_1 \) finite. Solving the bi-quadratic equation for \( \omega \), we obtain

\[
\omega^2 = k_y^2 \left[ V_0^2 + \alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_1 V_0^2} \right] \tag{5.23}
\]

Expanding the expression for growth rate in powers of \( 1/\alpha_1 \) we have

\[
\omega^2 = k_y^2 \left[ -V_0^2 + \frac{2V_0^4}{\alpha_1} \right]
\]

showing clearly that compressibility reduces the KH growth rate. There exists a lower threshold on \( \alpha_1 \), which is the square of dust acoustic speed \( c_s^2 \) beyond which the instability cannot be excited. This threshold condition can be obtained by demanding that \( \omega^2 \) remain negative for instability, which yields

\[
| V_0 | < \sqrt{2\alpha_1}
\]
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This shows that for instability, the dust acoustic speed in the medium has to be faster than the flow velocity. In other words the instability is possible only when the flow velocity is subsonic.

In the above derivation we had used quasineutral assumption for simplification. We now consider the full equation including the effect of dispersion arising from the Poisson’s equation. For step flow profile the form of Eq. (5.6) in both regions can be expressed in terms of field variable $\phi_1$ with suffix I and II denoting the respective regions.

$$-\alpha\Phi''_{I,II} + \left(1 + \alpha R + \alpha k^2 - \Omega^2_{I,II}/n_0\right) \Phi''_{I,II} + \left(-k_y^2 + \alpha R k_y^2 + R\Omega^2_{I,II}/n_0\right) \Phi_{I,II} = 0$$

(5.24)

In the above the perturbed field $\phi_1$ has been represented by $\Phi$ to simplify the notation as the suffix due to region I and II are also to be incorporated. Now,

![Graph](image)

Figure 5.2: Figure shows linear growth for $\alpha = 1$ for the quasineutral case (dashed line) and with dispersive correction due to $\nabla^2\phi_1$ (shown by circles). The other parameters are $V_0 = 1, \mu_e = 0.1, \mu_i = 1 + \mu_e$ with step shear flow profile.

Choosing appropriate form of exponential solution for Eq. (5.24) so that they vanish at $\pm\infty$ and utilizing the four matching conditions provided by Eq. (5.20) we obtain a set of four coupled equations relating the coefficients of the four exponential
functions. For non-trivial solutions, the determinant of the coefficient matrix viz., \( \det |M| = 0 \) should vanish. This condition gives the eigenvalue \( \omega \). The matrix \( M \) has the following form:

\[
M = \begin{vmatrix}
1 & -1 & 1 & -1 \\
\alpha p^2 - 1 + \alpha R & \alpha q^2 - 1 - \alpha R & -\alpha r^2 + 1 + \alpha R & (\alpha s^2 - 1 - \alpha R) \\
-\alpha p^2 / \Omega^2_+ & \alpha q^2 / \Omega^2_- & -\alpha r^2 / \Omega^2_+ & (\alpha s^2 / \Omega^2_-) \\
+ (1 + \alpha R)p/ \Omega^2_+ & -(1 + \alpha R)q/ \Omega^2_- & +(1 + \alpha R)r/ \Omega^2_+ & -(1 + \alpha R)s/ \Omega^2_- \\
\end{vmatrix}
\]

Here, the following notations have been used:

\[
\Omega_{\pm} = \omega \pm k_y V_0
\]

\[
p = -(a_+ / 2 + \sqrt{(a_+^2 - 4b_+)/2})^{1/2}
\]

\[
q = -(a_- / 2 + \sqrt{(a_-^2 - 4b_-)/2})^{1/2}
\]

\[
r = -(a_+ / 2 - \sqrt{(a_+^2 - 4b_+)/2})^{1/2}
\]

\[
s = -(a_- / 2 - \sqrt{(a_-^2 - 4b_-)/2})^{1/2}
\]

Also the coefficients \( a_\pm \) and \( b_\pm \) are

\[
a_\pm = -(1/\alpha)(1 + \alpha R + \alpha k_y^2 - \Omega^2_\pm)
\]

\[
b_\pm = -(1/\alpha)(R \Omega^2_\pm - k_y^2 - \alpha R k_y^2)
\]

The roots \( \omega \), for above determinant has been calculated numerically. We shown in Fig. 5.2 the growth rate obtained as a function of \( k_y \) and for \( \alpha = 1 \). A comparison with the non-dispersive compressible growth rate shown in the same figure 5.2 for this range of parameter shows that the dispersive effect reduces the growth rate.

The step profile, however, is too simplistic and somewhat unrealistic. It shows that the growth rate increases indefinitely with increasing value of \( k_y \). A realistic flow in general will change over some finite width say \( \epsilon \). When \( k_y^{-1} \) becomes comparable to \( \epsilon \) the effects due to finite width may become important. In fact for the case of incompressible fluid, it has already been shown that the growth
rate vanishes when $k_y \epsilon \geq 0.639$ for a piecewise linear profile and for $k_y \epsilon \geq 1$ for a tangent hyperbolic form of the velocity profile [135]. To discern the effect of compressibility on such a limit, as well as to identify any other role that the compressibility of the fluid may have on the mode, we next carry out analysis for the two cases of piecewise linear and the smooth tangent hyperbolic profiles of the velocity. For these cases it is not possible to obtain the growth rate analytically, we employ the perturbative scheme and numerical eigen value search for our studies.

5.5.2 Piecewise linear profile

The form of the piecewise linear profile is shown in Fig. 5.1(b). We have now Region I and II for $-\infty < x < -\epsilon$ (where $v_0 = V_0$) and $\epsilon < x < \infty$ (where $v_0 = V_0$). The middle region $-\epsilon < x < \epsilon$ is termed as region III for which $v_0 = V_0x/\epsilon$. The eigenvalue for this system in the incompressible limit which is the zeroth order expansion result in the compressibility parameter of $1/\alpha_1$, can be evaluated easily and is given by the following expression:

$$\omega^{(0)2} = \frac{1}{4} \left[ \left( \frac{V_0}{\epsilon} - 2k_yV_0 \right)^2 - \frac{V_0^2}{\epsilon^2} \exp(-4k_y\epsilon) \right] \quad (5.25)$$

This expression (Eq. (5.25)) is same as that obtained by Drazin [126] for $V_0 = \epsilon = 1$. It should be noted that the above expression easily reduces to the result of the step velocity flow profile in the limit of $\epsilon \to 0$. The zeroth order eigenfunctions corresponding to Eq. (5.25) are as follows:

$$v^{(0)}_I = B \exp(k_yx)$$
$$v^{(0)}_II = A \exp(-k_yx)$$
$$v^{(0)}_III = A_0 \exp(-k_yx) + B_0 \exp(k_yx) \quad (5.26)$$

The relationship between the coefficients $A$, $B$, $A_0$ and $B_0$ are obtained from matching conditions and they are given by:

$$B = A_0 f_B = A_0 \left[ 1 + \frac{\epsilon}{V_0} \left( 2 \Omega_+ - \frac{V_0}{\epsilon} \right) \right] \exp(2k_y\epsilon)$$
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\[ B_0 = A_0 f_{B0} = \frac{\epsilon A_0}{V_0} \left[ 2\Omega_+ - \frac{V_0}{\epsilon} \right] \exp(2k_y\epsilon) \]

\[ A = A_0 f_A = A_0 \left[ 1 + \frac{\epsilon}{V_0} \left( 2\Omega_+ - \frac{V_0}{\epsilon} \right) \exp(2k_y\epsilon) \right] \]

This completely determines the eigenfunction in the zeroth order.

To evaluate the correction in the threshold value of the wavevector due to compressibility we substitute \( \omega = 0 \) in the above expressions. The coefficients of the zeroth order wavefunction \( f_B, f_{A0} \) and \( f_{B0} \) in the limit of \( \omega = 0 \) are then related as:

\[ \tilde{f}_B = [1 + (2k_0\epsilon - 1)] \exp(2k_0\epsilon) \]

\[ \tilde{f}_{B0} = (2k_0\epsilon - 1) \exp(2k_0\epsilon) \]

\[ \tilde{f}_B = 1 + (2k_0\epsilon - 1) \exp(4k_0\epsilon) \]

Using these relationships the numerical value for \( k_1 \) from Eq. (5.18) for \( \alpha_1 = 50 \) and \( \epsilon = 0.5 \), turns out to be \( k^{(1)} = -0.5976 \) and \( k^{(1)}\epsilon = -0.2988 \). Thus, the new threshold on the wavenumber for these compressibility parameters is \( k_{\text{yth}} = (0.639 - 0.2988)/\epsilon = 0.3402/0.5 = 0.6804 \). It should be noted that for this value of \( \alpha_1 = 50 \), the second order corrections are of order \( \mathcal{O}(k^{(1)}/k^{(0)})^2 \sim 0.2 \) which is quite high. Since the exact value can not be evaluated for the piecewise linear profile we are in no position to judge the accuracy of the perturbative treatment and to ascertain upto what \( \alpha_1 \) the perturbative treatment would work fine. In the next section we carry out the analysis for a smooth tangent hyperbolic profile. For this particular profile the role of compressibility can be exactly determined by evaluating the eigenvalue and the eigenfunction for the Eqs. (5.2,5.3,5.4,5.5) numerically. We then compare these results with our perturbative analysis.

### 5.5.3 Tangent hyperbolic profile

We now choose a smoothly varying equilibrium profile of the form of a tangent hyperbolic function \( v_0(x) = V_0 \tanh(x/\epsilon) \) and study the linear problem defined by the complete set of Eqs. (5.2,5.3,5.4,5.5) by solving it numerically. This has two objectives, we are able to consider the effects of compressibility non perturbatively, thereby checking the conclusions of a perturbative treatment presented earlier. The second objective is to understand the role of \( \nabla^2 \phi_1 \) on the instability. In the presence
Figure 5.3: The plot of $\gamma/V_0$ (where $\gamma$ is the linear growth rate) as a function of $k_y\varepsilon$ has been shown for a tangent hyperbolic shear flow profile. In subplot (a) the solid line corresponds to the incompressible case $M_A = 0$ ($\alpha = \infty$), the small dots and the stars represent the exact and the perturbative values respectively for $M_A = 0.158$. Subplot (b) solid line again shows the incompressible case and the dots and triangles are the exact and perturbative values for $M_A = 0.7$. Subplot (c) is for $M_A = 0.5$ for which the solid circles and the hollow circles correspond to quasineutral and the dispersive cases. Subplot (d) shows a comparison of the threshold wavenumber evaluated exactly (*) and by perturbative scheme (o) as a function of $1/M_A$. 
of this term the equations are fairly cumbersome to make any analytical progress, hence this effect is investigated numerically with the help of a smooth flow profile. For the numerical scheme, we use the linearized Poisson’s equation Eq. (5.5) to express $n_1$ in terms of $\phi_1$ and its derivatives. The set of Eqs. (5.2,5.3,5.4) are then discretized in $x$ space. Eigenvalue is then obtained by the standard routines of Matrix eigenvalue evaluation.

In Fig. 5.3(a), we show a comparison between the growth rate as a function of $k_y\epsilon$ for the incompressible $\alpha = \infty$ and the compressible $\alpha = 1000$ cases (solid lines and solid circles respectively). Clearly, the growth rate diminishes in the presence of compressible perturbations. We have also shown the estimates obtained from the first order perturbation treatment by stars for some particular values of the wavenumber in the figure. This has been shown for points around the maximum growth rate where $\omega^{(0)}$ being large the perturbative treatment would hold. For one typical value of $k_y\epsilon$ (say $0.4333$) the incompressible growth rate is $\omega^{(0)} = i1.896$, the exact compressible growth rate $\omega = i1.847$, thereby implying that $\omega^{(1)} = i(1.896 - 1.847) = i0.05$. The ratio, $\omega^{(1)}/\omega^{(0)} \approx 0.02$ is small and the second order corrections are of the order of $10^{-4}$ which can be ignored. Thus, the first order corrections work fine for this high value of $\alpha$. This is the reason that the perturbative treatment provides a very good agreement for this particular case as the figure shows.

In Fig. 5.3(b) we have shown a plot for the case when the value of $\alpha = 50$. In this case the effect of compressibility is not weak to warrant a perturbative analysis. This can be observed by comparing the two exact results obtained numerically, viz., the incompressible (solid line) and the compressible (circles) cases. They differ significantly. Clearly, the second order terms, e.g. $O(\omega_1/\omega_0)^2$ would be significant for this case, which has been ignored in our perturbative treatment. As expected, the estimates obtained from our first order perturbative analysis shown by triangles in the Fig. 5.3(b) are also not close to the exact numerically obtained values for the compressible case denoted by circles. For this case we had also evaluated the exact growth rate with the dispersive corrections through $\nabla^2 \phi$. The dispersion, however, does not seem to play any significant role for this particular value of $\alpha$. The growth rate was found to exactly overlap with the quasineutral plots denoted by circles in Fig. 5.3(b).

At higher compressibility, viz., $\alpha = 1$ the differences between the dispersive and
non-dispersive cases start showing up. In Fig. 5.3(c) the plots show a comparison of growth rates for a quasineutral non dispersive case (solid circles) with that obtained by incorporating dispersive effects for $\alpha = 1$. In this strongly compressible case, the effect of dispersion is clearly evident. The growth rate shows reduction due to dispersion at higher wavenumbers. This effects the threshold wavenumber of the instability which gets considerably reduced in the presence of dispersion. The

![Image](image_url)

**Figure 5.4:** Plot of the eigenvector $v_{1x}$ as a function of $x$ has been shown. The subplot (a) is for $k_y = 0.5334$. the solid, dot-dashed and dashed lines represent $\alpha = 50$, $\alpha = 100$ and $\alpha = \infty$ respectively. Subplot (b) shows the eigenfunctions for the maximally growing mode for $\alpha = 50$, $\alpha = 100$ and $\alpha = \infty$ by solid, dot- dash and dashd line respectively.

exact results clearly show that compressibility reduces the threshold wavenumber for instability and dispersion at higher compressibility further limits the unstable domain. We now provide a comparison with the exact threshold wavenumber with that evaluated from our perturbative analysis presented in section IV. This has
been shown in the plot of Fig. 5.3(d). The symbol asterix(*) and circles(○) denote the exact and perturbative evaluation of \( k_{yth} \). The perturbative result improves with increasing value of \( \alpha \). These results are for \( \epsilon = 0.5 \) and it should be observed that for higher \( \alpha \) values the \( k_{yth} \) slowly asymptotes towards the incompressible limit of \( 1/\epsilon \).

We now study the properties of the eigenfunctions of the unstable KH mode for the various cases. The solid, dot dashed and dashed lines are the unstable eigen modes for \( \alpha = 50, \alpha = 100 \) and the \( \alpha = \infty \) (incompressible) respectively in Fig. 5.4. While the plots in Fig. 5.4(a) correspond for a specific wavenumber value of \( k_y = 0.5333 \), in Fig. 5.4(b) the maximally unstable eigen mode has been plotted. It should be noted that the eigenfunctions have a double humped form. The KH mode is essentially driven by the second derivative of the equilibrium shear flow profile. The second derivative maximizes at two locations in the tangent hyperbolic shear flow. It is at these locations that the unstable KH eigen mode also maximizes. The comparison of various \( \alpha \) values clearly illustrates that compressibility broadens the eigen mode form. This behaviour of the eigenfunction can be readily understood from the analytic expression of the eigenfunction that has been obtained for the step velocity profile in Eq. (5.21). The slow decay of the exponential along the shear direction \( x \) in the presence of finite \( \alpha_1 \) testifies to the broader wave functions for compressible cases.

In Fig. 5.5, we provide a comparison of the eigenfunction for the compressible quasineutral case with the one having contribution from \( \nabla^2 \phi_1 \) and thereby having dispersive contributions. We observe that the eigenfunctions get even more broader when dispersion is taken into account.

### 5.6 Physical interpretation

The process of KH instability can be understood from the schematic cartoon presented in Fig. 5.6. The equilibrium dust flow velocity \( \vec{v}_0 \) has a step profile and is depicted by the arrows pointing along \( \pm V_0 \) in the three subplots. This flow corresponds to a vortex sheet denoted by the thin solid (black) line in the figure. The equilibrium vorticity \( \vec{\Omega}_0 = \nabla \times \vec{v}_0 \) points inside the plane of the paper.
Figure 5.5: Plot of eigenfunction $v_{1x}$ as a function of $x$ for the quasineutral case (solid line) and the one with dispersive corrections arising from $\nabla^2 \phi_1$ (dashed lines) for $\alpha = 1$ and $k_y = 1$.

The curl of the inviscid dust momentum equation (Eq. 5.1) yields the following evolution equation for the vorticity $\overline{\Omega} = \nabla \times \vec{v}$

$$\frac{\partial}{\partial t} \overline{\Omega} + \vec{v} \cdot \nabla \overline{\Omega} - \overline{\Omega} \cdot \nabla \vec{v} = 0. \tag{5.27}$$

Similarly, taking the divergence one obtains

$$\frac{\partial}{\partial t} \nabla \cdot \vec{v} = \overline{\Omega} \cdot \nabla \vec{v} - \vec{v} \cdot \nabla \times \overline{\Omega} + \nabla^2 \left( \phi - \frac{v^2}{2} \right) - \nabla \cdot \overline{\alpha} \nabla n. \tag{5.28}$$

Here, $\overline{\alpha} = (\alpha / n)$. For the 2-D case considered by us one would have $\overline{\Omega} \cdot \nabla \vec{v} = 0$. In addition if the system is incompressible we have $\nabla \cdot \vec{v} = 0$ for all times. Thus, for such a case the right hand side of Eq. (5.28) is balanced for all times. It is clear from Eq. (5.27) that for the 2-D incompressible case the vorticity gets convected by the flow velocity.

The perturbed vortex sheet has been shown by the curve depicted in the form of a ribbon with segments $A-B-C-D-E$ identified in the figure 5.6. The perturbed velocity disturbance shown in the subplot (a) of Fig. 5.6 has an associated perturbed
vorticity which enhances and diminishes the equilibrium vorticity at locations $B$ and $D$ respectively. The equilibrium flow velocity has a configuration such that it brings the vorticity along $A$ to $C$ nearer and extends it from $C$ to $E$ as illustrated in the subplot(b) of the Fig. 5.6. When the vorticity patch between $A$ and $C$ are brought closer it further enhances the vorticity around $B$, thereby setting up an instability process. This is the basis of the conventional KH mode instability process. For the compressible case due to the additional $\nabla \cdot \vec{v}$ dependent term the vorticity is not tied to the fluid flow. As the flow tries to bring the vorticity patches along the segment $A - B - C$ nearer, the compressibility effects come into play. The divergence in the flow acquires a finite value in this case (even if it were zero to begin with) and acoustic perturbations get excited as has been schematically shown in subplot(c) of the Figure. This essentially inhibits the process of bringing the segment $A - B - C$ closer, thereby reducing the growth rate of the KH mode. It should also be noted that if the time scale of the acoustic perturbation is similar and/or faster than the growth period the flow would never be able to bring the segments $A - B - C$ closer and/or move the segments $C - D - E$ farther away. In this case then the instability would get totally suppressed. The mathematical analysis essentially conveys this physical mechanism for the stabilization of the KH mode in the presence of compressible perturbations.

We also wish to point out that the KH instability is related to the convection of the vorticity by the fluid flow velocity. Thus, the vorticity merely re-arranges spatially in the 2-D incompressible case. In the presence of compressibility and/or three dimensional perturbations, the vorticity is not carried by the fluid flow but evolves due to additional terms as well. The presence of compressible acoustic perturbations causes the the growth rate to get reduced, as some energy is spent on its excitation. Any 3-D perturbation would have to bend the equilibrium vorticity lines in the third dimension instead of merely convecting the vorticity lines in the 2-D plane. Since the bending of vorticity lines would require additional straining, this would reduce the growth rate of the KH mode. We, therefore, feel that the maximum growth rate is for those modes which have variations only in the 2-D plane of shear and flow. Next, we investigate the process of nonlinear saturation of the KH mode by carrying out nonlinear simulations. The details of the numerical studies are presented in the subsequent sections.
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Figure 5.6: A schematic cartoon illustrating the physical mechanism of the KH instability. The equilibrium flow has a step profile and is represented by the arrows pointing at \( \pm V_0 \). The associated vorticity sheet is shown by the horizontal solid line. This equilibrium vorticity is directed into the plane of paper. The cross and dot indicate the perturbed vorticity directed into and out of plane of paper respectively. Subplot (a) shows the initial sinusoidal perturbation in the flow due to which the equilibrium vorticity gets enhanced over locations B and reduced over locations D. Subplot(b) and (c) show the development of the KH instability schematically for the incompressible and compressible cases. The physical distinction between the two have been described in section VI in detail.
5.7 Numerical scheme and validation of the code

The governing equations for the dust fluid are simulated numerically in two dimensional $x - y$ plane for the purpose of nonlinear studies. The continuity and the momentum equations (2.2, 5.1) have been solved with the help of flux corrected scheme of Boris et al. [107]. A time splitting method is adopted to integrate along the two directions. At each time step the scalar potential $\phi$ is determined from the Poisson equation (2.3). It should be noted that the Poisson equation is a nonlinear equation in $\phi$ here, as the right hand side has an explicit dependence on $\phi$. To obtain $\phi$ from Eq. (2.3) we therefore employ a successive relaxation scheme at each time step. A converged solution is fed at each time step in Eq. (5.1) for the purpose of evolution.

We choose an equilibrium sheared flow configuration for the dust fluid defined by a flow of the form

\[ \vec{v}_0 = V_0 \tanh(x/\epsilon)\hat{y} \]  

along the $\hat{y}$ direction. The equilibrium velocity profile thus has a tangent hyperbolic form and its shear width is defined by the value of $\epsilon$. At any time the flow velocity of the dust fluid in 2-D is then given by

\[ \vec{v}(x, y, t) = \vec{v}_0 + \vec{v}_1(x, y, t) = V_0 \tanh\left(\frac{x}{\epsilon}\right)\hat{y} + \vec{v}_1(x, y, t) \]

which is the sum of the equilibrium flow velocity $\vec{v}_0$ and the perturbed flow velocity $\vec{v}_1(x, y, t)$.

In the numerical simulation the instability will manifest as the growth of the deviation of the velocity field $\vec{v}_1$ from the equilibrium flow profile. In general, for an unstable system $\vec{v}_1$ would automatically emerge in simulation from numerical noise. However, such a process would take a long time. To hasten the development of the fluctuations to a level where the perturbation could be easily distinguished from the equilibrium, we choose an initial finite but small amplitude (compared to the equilibrium amplitude) of $\vec{v}_1$. The two components of $\vec{v}_1$ have been chosen in
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our simulations here to have the following form at $t = 0$

$$v_{x1} = V_1 k_y \cos(k_{yp}y) \exp\left(-\frac{x^2}{\sigma^2}\right)$$

$$v_{y1} = V_1 \left(\frac{2x}{\sigma^2}\right) \sin(k_{yp}y) \exp\left(-\frac{x^2}{\sigma^2}\right)$$

(5.31)

Here, $1/k_{yp}$ and $\sigma$ define the length scales of the chosen initial velocity perturbation. It should be noted that with this choice, the initial velocity profile satisfies the incompressible condition, viz., $\nabla \cdot \vec{v} = 0$. For compressible dusty plasma medium, imposing such an initial condition is, however, not essential. We have, however, chosen such initial condition to have it identical to the case of neutral hydrodynamic case with which comparisons would be made in the present Chapter. Here, $V_1$ is the amplitude of initial perturbation and $\sigma$ is a parameter which defines the extent of initial perturbation around the shear width of the equilibrium flow. To confine it within a shear width one makes a choice of $\sigma \leq \epsilon$.

As time progresses, the perturbed velocity grows exponentially. The perturbed kinetic energy associated with the perturbed velocity field is given by the expression

$$\tilde{E} = \frac{\int (v_{x1}^2 + v_{y1}^2) \, dx \, dy}{\int \, dx \, dy}$$

(5.32)

The tracking of this quantity as a function of time provides a good measure of the growth of the instability and its saturation in the nonlinear regime. We show the evolution of $\tilde{E}$ as a function of time in the semilog plot in Fig. 5.7 for four different cases.

In subplot (a) and (b) we have shown the result of the studies for the case $\alpha = \infty$ and $\alpha = 50$ respectively, $V_0 = 5$ for both the cases. All the other parameters in this case are identical and have been provided in the figure caption (Fig. 5.7). In both cases the perturbed energy initially grows exponentially as illustrated by the linear behaviour of the energy evolution in the semilog plots of the Fig. 5.7. The slope of the curve in this regime (for example, Fig. 5.7 (a)) = 3.45 matches closely with twice the value of the growth rate $= 1.896$ corresponding to the maximally growing mode of the KH instability. The dotted line drawn alongside represents the curve with slope twice the analytical growth rate of the fastest growing KH mode for the
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system Subplot (c) and (d) we have chosen $\alpha = 1$ where effects of dispersion start playing role for the shorter scales permitted by the simulation box size. The results in subplot (c) show the case for which the effect of dispersion was deliberately dropped. Instead of using Poisson’s equation one assumed quasineutrality here. The subplot(d) on the other hand retains the effect of dispersion as the Poisson equation is used. For (c) and (d) we had also chosen the value of $V_0 = 1$. It should be noted that while the linear growth rate scales with $V_0$, the perturbed energy for the smaller choice of $V_0$ saturates at a smaller value. This is reasonable as the nonlinear effects sets in when the perturbed velocity starts becoming comparable to the value of the equilibrium flow velocity.

5.8 Nonlinear phase of the KH instability

After the initial exponential rise, the perturbed energy ultimately saturates. This happens typically when the perturbed velocity fields achieve amplitudes which are comparable to equilibrium values. We now present our observations pertaining to this nonlinear phase of the instability.

The evolution of power in perturbed velocity field for various cases have been shown in Fig. 5.7. The description has been provided in the figure caption. In all these cases a distinct oscillation are observed in the nonlinear phase of the evolution. Furthermore, a comparison of subplot(a) and (b) shows that the amplitudes of the oscillations are more pronounced in the compressible case than that of the incompressible case. Similarly, a comparison of subplot(c) and (d) shows that dispersive effects have more pronounced amplitude of oscillations. In addition to these reversible oscillations, the plots also show that at a later stage an irreversible increase in the power of perturbed velocity occurs. This has been shown by encircling the region in all the subplots of Fig. 5.7. Thus, the salient features during the nonlinear phase are (i) the reversible oscillations in the power of perturbed velocity, the amplitude of which gets pronounced for the compressible and dispersive cases and (ii) the irreversible increase of the saturation level resulting in an observation of second saturation regime at a later time in all these plots. We will provide an interpretation for these observations shortly. The snapshots of vorticity contours have been shown in Fig. 5.8 and Fig. 5.9 for the incompressible
Figure 5.7: The evolution of $\log(E)$ (Eq. (5.32)) with time has been shown. The straight segment represents the linear growth of the instability. The dashed line alongside has been plotted for comparison and has twice the slope of the exact analytical value of maximally growing mode mentioned for each cases below. The curve in all the subplots later saturate corresponding to the nonlinear regime of the instability. In addition reversible oscillations and a second phase of growth and subsequent saturation of $E$ at a higher level (Shown by the encircled region) at later time is also observed in all the cases. Subplot (a) shows the evolution for incompressible case with $M_A = 0.0, (\alpha = \infty), V_0 = 5$, the analytical growth rate for this case is $= 1.896$ and the growth rate evaluated from the simulation is $= 3.45$. Subplot (b) shows the evolution for compressible case with $M_A = 0.7, (\alpha = 50), V_0 = 5$, the analytical growth rate for this case is $= 1.05$ and the growth rate evaluated from the simulation is $= 1.9477$. Subplot (c) and (d) compare the quasineutral and dispersive cases respectively for evolution of $\log(E)$ for which $(\alpha = 1), V_0 = 1$. For quasineutral case (c), the analytical growth rate for this case is $= 0.3063$ and the growth rate evaluated from the simulation is $= 0.5043$. While for dispersive case (d), the analytical growth rate for this case is $= 0.2432$ and the growth rate evaluated from the simulation is $= 0.4165$.  

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Figure 5.8: The various subplots show the vorticity contours at various times obtained from numerical simulation of the incompressible fluid system. At $t = 3.404$ the flow is in the linear growth regime of the KH instability, at $t = 15.77$ the system is in the first saturated nonlinear regime, at $t = 27.47$ the vortices have just started to merge.

case. While in Fig. 5.8, we have shown the snapshots pertaining to the period of linear growth regime and when the power in perturbed flow saturates at the first lower level, the plots in Fig. 5.9 on the other hand correspond to the time when the power in perturbed flow velocity shows irreversible rise and leading subsequently to the second level of saturation. In Fig. 5.10 and Fig. 5.11, the snapshots similarly correspond to the compressible, dispersive case. From these snapshots it is clear that the KH mode initially develops as a perturbation around the shear flow re-
Figure 5.9: The evolution of vorticity contours for the incompressible case for times when the two distinct vorticity patches are about to merge $t = 29.89$, and at other times $t = 34.30, 36.50, 38.71$ the vortex has already merged and the various stages of its rotation has been depicted.

region. The maximally growing mode permissible by the system is observed during the linear phase. For instance in the second subplot of Fig. 5.8 the maximally growing mode permissible for a box size of $L_y = 20$ corresponds to $k_y = 0.63$. This is what develops initially as can be seen from the subplot at $t = 3.404$ of Fig. 5.8. This time corresponds to the linear growth period of the instability as can be observed from the energy evolution shown in Fig. 5.7 for this case. Two modes of this wavenumber get accommodated in the box size, hence two structures develop later
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(subplot at time $t = 15.77$, when the energy shows first saturation regime). We note that for this incompressible case the structure during the saturation regime (e.g. from time $t = 6$ to $t = 26$, the first saturation regime of the nonlinear phase) is essentially isotropic.

In Fig. 5.9, we have plotted the stages where the two isotropic structures begin to coalesce and ultimately merge with each other. The time corresponding to the merging process matches with the irreversible increase of perturbed energy (see Fig. 5.7). The process of merging for the incompressible case can be understood by invoking the existence of the second enstrophy (mean square vorticity) invariant in addition to the energy invariant for the system, which promotes the inverse cascade of flow structures. After the merging process the structure remains somewhat anisotropic. The subplot (a) of Fig. 5.7 for this time regime shows somewhat pronounced oscillations. We have observed that at a later stage as the structure isotropizes the amplitude of the oscillations also become weak. For the compressible dispersive case, the vorticity contours have been shown in the plot of Fig. 5.10 and Fig. 5.11. Here, again in the linear regime the most unstable mode permissible with the simulation box size, namely $k_y = 0.94$ appears. For this case $L_y = 20$, same as before. Thus, three modes develop and ultimately form three distinct vortex structures. In this case, we note, however, that the three vorticity patches that develop are anisotropic. Thus, as they rotate they generate reversible oscillations in the perturbed velocity power. These structures (even though they correspond to compressible dispersive simulations for which the second enstrophy invariant does not exist) also coalesce and show mergers. The merging process in this case also leads to an irreversible increase in $E$. The reversible oscillations of $E$ correspond to the rotation of vorticity patches. The rotation of the asymmetrical vortex patches generates the reversible oscillations in $E$, with minima coinciding with the instant when the longer axis of the patch is aligned along the flow direction and the maxima when it is aligned orthogonal to flow, along the shear direction. Thus, higher the anisotropy of the vortex patch, higher is the amplitude of oscillation. From the snapshots of vorticity contours it is clear that the vorticity patches get distinctly anisotropic as compressibility and/or dispersive effects increase. We would delve into the reason behind this shortly.

We now reflect upon the mechanism of the nonlinear saturation of the KH instability. Once the perturbed velocity reaches a level in which it becomes com-
Figure 5.10: The various subplots show the vorticity contours at different times obtained from numerical simulation of compressible dust fluid. The subplot at $t = 14.85$ shows the linear regime of evolution, while the anisotropy of vortex structures in flow direction ($y$-axis) and direction of flow discontinuity is shown at times $t = 24.999$ and $34.226$.

parable to the order of the original equilibrium flow, it alters the equilibrium shear configuration itself. The new sheared flow profile can be observed by averaging the $\hat{y}$ component of the velocity over $y$ coordinate. Thus, the altered sheared profile is $\bar{v}_y = \int v_y dy/L_y$. The effective shear width of this average flow, viz., $\epsilon_{eff}$ is observed to get broader with time in comparison to $\epsilon$, the shear width of the original flow profile at $t = 0$. When the instability saturates (as evidenced from the evolution of $E$) the effective shear width $\epsilon_{eff}$ also stabilizes at a particular value.
Figure 5.11: The subplots show the evolution of vorticity contours for compressible and dispersive dust fluid with parameters same as in Fig. 5.10. The subplot at \( t = 54.191 \) shows the start of merger of vortices while the vortex coalescing is observed at advance times \( t = 69.393 \) and \( 81.899 \).

( broader than \( \epsilon \)). In the nonlinear phase the \( \epsilon_{eff} \), acts as the shear width of the modified profile and decides the further course of evolution. In Fig. 5.12 (a) and (b) we have shown the modified average profile of \( \bar{v}_y \) for the cases corresponding to those depicted for \( E \) evolution in subplots (a) and (b) of Fig. 5.7 respectively. These velocity profiles have been shown in Figs. 5.12(a,b) for three different times (i) \( t = 0 \) (original profile depicted by solid lines), (ii) \( t = 9.626 \) (corresponding to first saturated regime of perturbed energy shown by dash lines) and (iii) \( t = 34.303 \)
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(the profile in the second saturated regime shown by dash dot lines). An effective shear width was obtained by fitting these profiles to a tangent hyperbolic form. At $t = 9.626$, the shear width are $\epsilon_{\text{eff}} = \epsilon_{i1} = 1.2$ and $\epsilon_{\text{eff}} = \epsilon_{c1} = 0.9$, here the suffix $i$ and $c$ correspond to incompressible and compressible cases respectively. Similarly, the width at $t = 34.303$ are $\epsilon_{\text{eff}} = \epsilon_{i2} = 2.4$ and $\epsilon_{\text{eff}} = \epsilon_{c2} = 1.8$ for the two cases.

Now, as mentioned before the broad end shear profile decides the future course of action. This profile can be unstable only if there exists a $k < A/\epsilon_{\text{eff}}$ permitted by the simulation box. This relation with $A = 1$ for the incompressible flow and $A < 1$ (for $\alpha = 50$, it can be seen from Fig. 5.3 that $A = 0.72$) for the compressible case decides the threshold wave number for the instability (as the linear growth rate plots of Fig. 5.3(b) for the two cases shows). Clearly, then for similar simulation box sizes and hence similar permissible range of $k$ the $\epsilon_{\text{eff}}$ for the compressible case at saturation would be less compared to the effective shear width for the incompressible simulations. This is indeed what the plots of Fig. 5.12 illustrates. At $t = 9.626$ the perturbation has maximum power in a mode number of 2, i.e. $k_2 = 2 \times \pi / L_y = 0.63$ (two wavelengths in the simulation box). This mode is stable according to the threshold criteria for both the incompressible and compressible cases for their respective shear widths of $\epsilon_{i1}$ and $\epsilon_{c1}$. The perturbed energy, therefore, remains at a stationary level at $k_2$ in the first saturation regime. However, there is another scale $k_1 = 2 \pi / L_y = 0.314$ (longest scale) which is also permitted by the simulation box and is unstable for the shear width $\epsilon_{i,c1}$ of the altered profile of the first stage. For this scale, the calculation shows that $k_1 \epsilon_{i1} = 0.3768 < 1$ for the incompressible case and $k_1 \epsilon_{c1} = 0.2826 < A$ for the compressible case. This longest scale mode which is susceptible to instability then develops from the background and is ultimately responsible for causing the merger of the two vorticities producing an irreversible jump in the energy. After this merger the respective $\epsilon_{\text{eff}}$ increases further as we have already noted and acquires a value such that even the $k_1$ is beyond the threshold of the unstable wavelength domain. Since, no permissible modes of the system are unstable anymore the system relaxes to a final saturated state.

The above description also provides an explanation for the underlying reason for observing more prominent reversible oscillations in energy when compressibility and dispersive effects are added. The vorticity patches, (essentially representing
the perturbation scales) typically have the scale of $k_y^{-1}$ along the flow direction and $\varepsilon_{eff}$ in the shear direction in the nonlinear saturated state. For the incompressible hydrodynamic case the two scales are related by the condition of $k_y\varepsilon_{eff} = 1$, implying that the vortex pattern is symmetrical in this case. For the compressible and dispersive cases, however, the two dimensions of the vortex patch at a saturated state are related by the condition of $k_y\varepsilon_{eff} = A < 1$ and are hence asymmetric. The numerical simulation yields the dimension of the vorticity patches which are very closely related by this theoretical condition. This explains why the reversible oscillations get pronounced with increasing compressibility and dispersion.

Figure 5.12: The subplots (a) and (b) show the evolution of the average profile $\bar{v}_y = \int v_y dy / L_y$ for the incompressible and the compressible cases respectively. The solid, dashed and the dash dot plots show the profile at $t = 0$ (original), $t = 9.626$ (first saturation regime) and $t = 34.303$ (second saturated regime). Their respective fitted shear width also shown by horizontal lines with respective line styles (i.e. solid, dashed and the dash dot lines for $t = 0$, $t = 9.626$ and $t = 34.303$ respectively).
5.9 Summary and conclusion

A weakly coupled dusty plasma system behaves like a fluid which differs from the neutral hydrodynamic fluid in certain ways. The dust fluid can have a very strong compressible nature. The compressibility arises in this case not merely from thermal effects but also due to its interaction with electron and ion charged species. Furthermore, unlike the neutral fluid it can support dispersive compressible perturbations. Keeping these features in view, a prominent fluid instability, namely the Kelvin-Helmholtz mode has been studied for the weakly coupled dusty plasma system in both linear and nonlinear regimes.

A detailed characterization of the instability in the presence of weak and strong compressibility as well as dispersion has been carried out in this Chapter. The behaviour of the KH mode has been investigated both analytically and by the help of numerics for exact eigenvalue evaluation for various shear flow profiles. The studies point out that compressibility has a stabilizing role on the KH mode. The dispersion effect becomes significant only when the fluid is highly compressible. Furthermore, the dispersion is observed to further stabilize the unstable modes, typically at higher wavenumber domain. We have also provided a first order perturbative calculation for weakly compressible cases. The perturbative evaluation of the change in growth rate and the altered threshold wavenumber matches very well with the exact results obtained numerically.

The nonlinear studies have also been carried out by simulating the governing equations numerically. Various distinctive characteristic features in the nonlinear regime associated with compressible and dispersive perturbations have been observed and identified. A physical interpretation of the results have also been provided for its understanding. In short the simulations confirm the characteristics analytical linear growth rate features of the instability. The reduction in growth rate in the presence of compressible and dispersive perturbations have been confirmed through numerical simulations results as well. The presence of compressibility and dispersion also reduces the range of the unstable wave numbers. We have shown that this introduces interesting characteristic features during the nonlinear phase. The effective shear profile in the saturated state of the KH instability shows a weaker broadening for the compressible and dispersive cases as compared to the incompressible fluid. Furthermore, our simulations show that
the vortex merging process, reminiscent of 2-D inverse cascade of energy spectrum for systems preserving energy and enstrophy invariants, is preserved even for the compressible dusty plasma medium. This leads to a coherent nonlinear saturated state in 2-D.