CHAPTER 0
INTRODUCTION AND SUMMARY OF RESULTS

0.1. INTRODUCTION

The concept of a semigroup is very simple and plays a large role in the development of Mathematics. The theory of semigroups is similar to group theory and ring theory. The earliest major contributions to the theory of semigroups are strongly motivated by comparisons with groups and rings. Semigroup theory can be considered as one of the most successful off-springs of ring theory in the sense that the ring theory gives a clue how to develop the ideal theory of semigroups. The algebraic theory of semigroups was widely studied by CLIFFORD and PRESTON [13], [14]; PETRICH [51] and LJAPIN [39].

As a generalization of a semigroup, SEN [60], introduced the notion of \(\Gamma\)-semigroup in 1981 and developed some theory on \(\Gamma\)-semigroups [61], [62], [63] and [64]. JIROJKUL, SRIPAKORN, CHINRAM [26], extended many classical notions of semigroups to \(\Gamma\)-semigroups. DUTTA and CHATTERJEE [16] also studied the properties of Green's relations in \(\Gamma\)-semigroups and generalized the notions; idempotent elements, regular elements and semisimple elements in \(\Gamma\)-semigroups. MADHUSUDHANA RAO, ANJANEYULU and GANGADHARA RAO [17], [18], [19], [20], [40], [41], [42], [43], [44] and [45] developed the algebraic theory of \(\Gamma\)-subsemigroups and \(\Gamma\)-ideals in \(\Gamma\)-semigroups.

Y. I. KWON and S. K. LEE [32], introduced the notion of a po-\(\Gamma\)-semigroup. KOSTAQ HILA and EDMOND PISHA [31] introduced and characterized the left and right simple, completely regular and strongly regular po-\(\Gamma\)-semigroups in terms of bi-ideals and studied their structure by extending the results for ordered semigroups. DHEENA and ELAVARASAN [15] studied about right chain po-\(\Gamma\)-semigroups and obtained some characterizations of po-\(\Gamma\)-semigroups. In this thesis, we make a study on the theory of po-\(\Gamma\)-semigroups and po-\(\Gamma\)-subsemigroups.

The theory of ideals in semigroups was studied by CLIFFORD and PRESTON [13], [14]; PETRICH [51] and LJAPIN [39]. The ideal theory in commutative semigroups was developed by BOURNE [7], HARBANS LAL [23], SATYANARAYANA [55], [56],
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[57], [58]; MANNEPALLI and NAGORE [46]. The ideal theory in semigroups was developed by ANJANEYULU [1], [2], [3], [4], [5], GIRI and WAزالWAR [22], HOEHNKE [24] and SCWARTZ [59]. SATYANARAYANA [55] has developed some literature on prime ideals and prime radicals for commutative semigroups. ANJANEYULU [1], [2], [3]; GIRI and WAزالWAR [22] studied about prime radicals in semigroups. The study of bi-ideals was made by AIYARED IAMPAN [6], BRAJA ISLAM [9] in Γ-semigroups and by KAUSHIK and KHAN MOIN [28] in Γ-semirings. IAMPAN [6] and BRAJA[9] introduced and characterized 0-minimal and 0-maximal bi-ideals in Γ-semigroups. BRAJA ISLAM [8], [9]; JAGATAP and PAWAR [25] studied about quasi Γ-ideals in Γ-semirings. CHINRAM [10] studied about quasi-ideals and obtained some characterizations of regular Γ-semigroups. CHINRAM and SIAMMAI [11] generalized the Green’s relations in semigroups to Γ-semigroups. MADHUSUDHANA RAO, ANJANEYULU and GANGADHARA RAO [17], [18], [19], [20], [40], [41], [42], [43], [44] and [45] studied about prime Γ-ideals, completely prime Γ-ideals, semiprime Γ-ideals, completely semiprime Γ-ideals and prime radicals in Γ-semigroups. In 1997, Y. I. KWON and S. K. LEE [32], introduced the concepts of weakly prime ideals and weakly semiprime ideals in ordered Γ-semigroups and gave some characterizations of weakly prime ideals and weakly semiprime ideals in ordered Γ-semigroups, analogous to the characterizations of weakly prime ideals and weakly semiprime ideals in ordered semigroups. DHEENA and ELAVARASAN [15] made a study on prime ideals, completely prime ideals, semiprime ideals and completely semiprime ideals in partially ordered Γ-semigroups. KOSTAQ HILA and EDMOND PISHA [31] made a study on bi-ideals in po-Γ-semigroups. In this thesis, we make a study about po-Γ-ideals, prime po-Γ-ideals, completely prime po-Γ-ideals, semiprime po-Γ-ideals and completely semiprime po-Γ-ideals in po-Γ-semigroups.

ANJANEYULU [5] studied about m-system and n-system in semigroups. MADHUSUDHANA RAO, ANJANEYULU and GANGADHARA RAO [17] and [18], studied about c-system, m-system, d-system and n-system in Γ-semigroups. These concepts being related to the concepts of prime and semiprime ideals, which play an important role in studying the structure of ordered semigroups. NIOVI KEHAYOPULU [48] introduced the concepts of m-system and n-system in ordered semigroups. In this thesis, we introduce the concepts of po-c-system, po-m-system, po-d-system and
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po-\text{n}-system in po-Γ-semigroups and extend many characteristics from Γ-semigroups to po-Γ-semigroups.

PETRICH. M. [51] made a study on filters in semigroups. LEE. S. K and LEE. S. S. [37], introduced the notion of a left (right) filter in a po-semigroup and gave a characterization of left(right)-filters in terms of the right(left) prime ideals. N. KEHAYOPULU [49], gave a characterization of filters in terms of prime ideals in ordered semigroups. Y. B. JUN [27] introduced the notion of a Γ-filter in po-Γ-semigroups and gave a characterization of po-Γ-filter in terms of a prime po-Γ-ideal. LEE. S. K. and KWON. Y. I. [36] introduced the notions of left (right) po-Γ-filter in po-Γ-semigroup and gave a characterization of left (right) po-Γ-filter of a po-Γ-semigroup in terms of right (left) prime po-Γ-ideal. In this thesis, we make a study on po-Γ-filters and extend many characterizations of semigroups, po-semigroups and Γ-semigroups to po-Γ-semigroups.

PETRICH. M. [51] made a study on \R{}-classes in semigroups. KEHAYOPULU [48] defined the relation \R{} on a po-semigroup and obtained some results. Various kinds of ordered semigroups have been widely studied by many authors by using the notion of filter and the relation \R{}. Y. I. KWON [33] introduced the concept of filter and relation \R{} in po-Γ-semigroups and obtained some results. KOSTAQ HILA [29] also obtained results, dealing with the principal filters on an ordered Γ-semigroup and studied their structure and properties, which are investigated by using the relation \R{}, the smallest complete semilattice congruence. In this thesis, we obtain some new results, dealing with principal filters on po-Γ-semigroups and study their structure and properties.

0.2. PRELIMINARIES:

\textbf{DEFINITION 0.2.1}: A system \( S = (S, \cdot) \), where \( S \) is a nonempty set and \( \cdot \) is an associative binary operation on \( S \), is called a \textit{semigroup}.

\textbf{DEFINITION 0.2.2}: A semigroup \( S \) is said to be \textit{finite} provided the cardinality of \( S \) is finite.

\textbf{DEFINITION 0.2.3}: A semigroup \( S \) is said to be \textit{commutative} provided \( ab = ba \) for all \( a, b \in S \).
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**DEFINITION 0.2.4:** A semigroup $S$ is said to be *quasi commutative* provided for any $a, b \in S$, there exists a natural number $n$ such that $ab = b^n a$.

**DEFINITION 0.2.5:** A semigroup $S$ is said to be *left pseudo commutative* provided $abc = bac$ for all $a, b, c \in S$.

**DEFINITION 0.2.6:** A semigroup $S$ is said to be *right pseudo commutative* provided $abc = acb$ for all $a, b, c \in S$.

**THEOREM 0.2.7:** If $S$ is a commutative semigroup then $S$ is a quasi commutative semigroup.

**THEOREM 0.2.8:** If $S$ is a commutative semigroup then $S$ is a left pseudo commutative semigroup and a right pseudo commutative semigroup.

**NOTE 0.2.9:** If $A, B$ are two subsets of a semigroup $S$ and $x \in S$, then $AB = \{ab : a \in A, b \in B\}$, $xA = \{xa : a \in A\}$ and $Ax = \{ax : a \in A\}$.

**DEFINITION 0.2.10:** A semigroup $S$ is said to be *normal* provided $aS = Sa$ for all $a \in S$.

**THEOREM 0.2.11:** If $S$ is a quasi commutative semigroup then $S$ is a normal semigroup.

**DEFINITION 0.2.12:** An element $a$ of a semigroup $S$ is said to be *left identity* of $S$ provided $as = s$ for all $s \in S$.

**DEFINITION 0.2.13:** An element $a$ of a semigroup $S$ is said to be *right identity* of $S$ provided $sa = s$ for all $s \in S$.

**DEFINITION 0.2.14:** An element $a$ of a semigroup $S$ is said to be a *two sided identity* or an *identity* provided it is both a left identity and a right identity of $S$.

**DEFINITION 0.2.15:** An element $a$ of a semigroup $S$ is said to be a *left zero* of $S$ provided $as = a$ for all $s \in S$.

**DEFINITION 0.2.16:** An element $a$ of a semigroup $S$ is said to be a *right zero* of $S$ provided $sa = a$ for all $s \in S$. 

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DEFINITION 0.2.17: An element $a$ of a semigroup $S$ is said to be a two sided zero or zero of $S$ provided it is both a left zero and a right zero of $S$.

NOTE 0.2.18: A semigroup $S$ has at most one zero element and at most one identity element. If $S$ has a zero element, then it is unique and will be denoted by 0. If $S$ has an identity element, then it is unique and will be denoted by 1.

NOTE 0.2.19: Let $S$ be a semigroup. If $S$ does not have an identity, then $S^1$ be the semigroup $S$ with an identity adjoined, usually denoted by the symbol $1$. If $S$ has an identity, then $S^1 = S$.

DEFINITION 0.2.20: A semigroup in which every element is a left zero is called a left zero semigroup.

DEFINITION 0.2.21: A semigroup in which every element is a right zero is called a right zero semigroup.

DEFINITION 0.2.22: A semigroup with 0 in which the product of any two elements equals to 0 is called a zero semigroup or a null semigroup.

DEFINITION 0.2.23: An element $a$ of a semigroup $S$ is said to be left cancellative provided $x, y \in S; \ ax = ay$ implies $x = y$.

DEFINITION 0.2.24: An element $a$ of a semigroup $S$ is said to be right cancellative provided $x, y \in S; \ xa = ya$ implies $x = y$.

DEFINITION 0.2.25: An element $a$ of a semigroup $S$ is said to be cancellative provided $a$ is both a left cancellative and a right cancellative element.

DEFINITION 0.2.26: A semigroup $S$ is said to be a left cancellative semigroup provided every element of $S$ is a left cancellative element.

DEFINITION 0.2.27: A semigroup $S$ is said to be a right cancellative semigroup provided every element of $S$ is a right cancellative element.

DEFINITION 0.2.28: A semigroup $S$ is said to be a cancellative semigroup provided every element of $S$ is a cancellative element.
DEFINITION 0.2.29: Let $S$ be a semigroup with identity $1$. An element $a \in S$ is said to be left invertible if there exists an element $b \in S$ such that $ba = 1$. The element $b$ is called left inverse of $a$.

DEFINITION 0.2.30: Let $S$ be a semigroup with identity $1$. An element $a \in S$ is said to be right invertible if there exists an element $c \in S$ such that $ac = 1$. The element $c$ is called right inverse of $a$.

DEFINITION 0.2.31: Let $S$ be a semigroup with identity. An element $a \in S$ is said to be invertible if $a$ is both left invertible and right invertible.

THEOREM 0.2.32: Let $S$ be a semigroup with identity and $a \in S$. If $b$ is a left inverse and $c$ is a right inverse of $a$, then $b = c$.

DEFINITION 0.2.33: Let $S$ be a semigroup with identity and $a \in S$. If $a$ is invertible then the element which is both left inverse and right inverse of $a$ is called inverse of $a$.

NOTE 0.2.34: If $S$ is a group and $a \in S$ then the inverse of $a$ is unique and it is denoted by $a^{-1}$.

DEFINITION 0.2.35: A semigroup $S$ with unity is said to be a group provided every element of $S$ is invertible.

THEOREM 0.2.36: Every group is a cancellative semigroup.

DEFINITION 0.2.37: A nonempty subset $A$ of a semigroup $S$ is said to be a subsemigroup of $S$ provided $a,b \in A$ implies $ab \in A$.

DEFINITION 0.2.38: A nonempty subset $A$ of a semigroup $S$ is said to be an $m$-system provided for any $a,b \in A$, there exists an $x \in S$ such that $axb \in A$.

DEFINITION 0.2.39: A nonempty subset $A$ of a semigroup $S$ is said to be an $n$-system provided for any $a \in A$, there exists an $x \in S$ such that $axa \in A$.

DEFINITION 0.2.40: Let $S$ be a semigroup and $A$ be a nonempty subset of $S$. Then $A$ is said to generate $S$ or $S$ is said to be generated by $A$ provided every element of $S$ is a product of finite number of elements of $A$. 

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DEFINITION 0.2.41: A nonempty subset $A$ of a semigroup $S$ is said to be a \textit{left ideal} of $S$ provided $SA \subseteq A$. 

DEFINITION 0.2.42: A nonempty subset $A$ of a semigroup $S$ is said to be a \textit{right ideal} of $S$ provided $AS \subseteq A$. 

DEFINITION 0.2.43: A nonempty subset $A$ of a semigroup $S$ is said to be a \textit{two sided ideal} or \textit{ideal} of $S$ provided it is both a left ideal and a right ideal of $S$. 

DEFINITION 0.2.44: An ideal $A$ of a semigroup $S$ is said to be a \textit{proper ideal} of $S$ provided $A \neq S$. 

DEFINITION 0.2.45: An ideal $A$ of a semigroup $S$ is said to be a \textit{trivial ideal} of $S$ provided $S \setminus A$ is singleton. 

THEOREM 0.2.46: A semigroup $S$ is a group iff $S$ has neither proper left ideals nor proper right ideals. 

THEOREM 0.2.47: The nonempty intersection of any family of (left or right) ideals of a semigroup $S$ is an (left or right) ideal of $S$. 

THEOREM 0.2.48: The union of any family of (left or right) ideals of a semigroup $S$ is an (left or right) ideal of $S$. 

DEFINITION 0.2.49: Let $S$ be a semigroup. The intersection of all ideals in $S$ is called \textit{kernel} of $S$ and it is denoted by $K$. 

DEFINITION 0.2.50: Let $S$ be a semigroup. The intersection of all left ideals of $S$ containing a nonempty set $A$ is called the \textit{left ideal generated by} $A$. 

DEFINITION 0.2.51: Let $S$ be a semigroup. The intersection of all right ideals of $S$ containing a nonempty set $A$ is called the \textit{right ideal generated by} $A$. 

DEFINITION 0.2.52: Let $S$ be a semigroup. The intersection of all ideals of $S$ containing a nonempty set $A$ is called an \textit{ideal generated by} $A$. It is denoted by $\langle A \rangle$. 

DEFINITION 0.2.53: An ideal $A$ of a semigroup $S$ is said to be a \textit{principal ideal} provided $A$ is an ideal generated by a single element set. If an ideal $A$ is generated by $a$, then $A$ is denoted as $\langle a \rangle$ or $J[a]$. 

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**DEFINITION 0.2.54** : An ideal A of a semigroup S is said to be a *maximal ideal* provided A is a proper ideal of S and is not properly contained in any proper ideal of S.

**DEFINITION 0.2.55** : An ideal A of a semigroup S is said to be a *minimal ideal* provided A does not contain any ideal of S properly.

**THEOREM 0.2.56** : In any semigroup S, the following are equivalent.
1) Principal ideals in S form a chain.
2) Ideals in S form a chain.

**DEFINITION 0.2.57** : A semigroup S is said to be a *noetherian semigroup* provided every ascending chain of ideals becomes stationary.

**DEFINITION 0.2.58** : A semigroup S is said to be an *artinian semigroup* provided every descending chain of ideals becomes stationary.

**DEFINITION 0.2.59** : A semigroup S is said to be a *simple semigroup* provided S has no proper ideals.

**DEFINITION 0.2.60** : Let A be any ideal in a semigroup S. Put $S/A = S \setminus A \cup \{A\}$. Define multiplication in $S/A$ as follows. Let $a, b \in S/A$. Define $a*b = A$, if $a = A$ or $b = A$; $a*b = a.b$ if $a, b \in S\setminus A$; $a*b = A$, if $a, b \in A$. Then $S/A$ is a semigroup. The semigroup $S/A$ is called *Rees quotient (difference) semigroup* of S over the ideal A.

**THEOREM 0.2.61** : An ideal M of a semigroup S is maximal iff $S/M$ is a simple semigroup.

**THEOREM 0.2.62** : Let M be a maximal ideal of a semigroup S. If $a, b \in S\setminus M$, then $<a> = <b>$.

**DEFINITION 0.2.63** : An element $a$ of a semigroup S is said to be an *r-element* provided $as = sa$ for all $s \in S$ and if $x, y \in S$, then $axy = byx$ for some $b \in S$.

**DEFINITION 0.2.64** : A semigroup S is said to be a *generalized commutative semigroup* provided S contains 1 as an r-element.

**DEFINITION 0.2.65** : An ideal A of a semigroup S is said to be a *globally idempotent ideal* provided $A^2 = A$. 

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DEFINITION 0.2.66: A semigroup $S$ is said to be a **globally idempotent semigroup** provided $S^2 = S$.

DEFINITION 0.2.67: An element $a$ of a semigroup $S$ is said to be an **idempotent** provided $a^2 = a$.

DEFINITION 0.2.68: An element $a$ of a semigroup $S$ is said to be a **proper idempotent** provided $a$ is an idempotent which is not the identity of $S$ if identity exists.

DEFINITION 0.2.69: A semigroup $S$ is said to be an **idempotent semigroup** or a **band** provided every element in $S$ is an idempotent.

DEFINITION 0.2.70: An element $a$ of a semigroup $S$ is said to be **regular** provided $a = axa$ for some $x \in S$.

DEFINITION 0.2.71: An element $a$ of a semigroup $S$ is said to be **left regular** provided $a = a^2x$ for some $x \in S$.

DEFINITION 0.2.72: An element $a$ of a semigroup $S$ is said to be **right regular** provided $a = xa^2$ for some $x \in S$.

DEFINITION 0.2.73: An element $a$ of a semigroup $S$ is said to be **intra regular** provided $a = xa^2y$ for some $x, y \in S$.

DEFINITION 0.2.74: A semigroup $S$ is said to be a **regular semigroup** provided every element is regular.

DEFINITION 0.2.75: An element $a$ of semigroup $S$ is said to be **completely regular** provided $a = axa and ax = xa$ for some $x \in S$.

DEFINITION 0.2.76: A semigroup $S$ is said to be a **completely regular semigroup** provided every element in $S$ is completely regular.

THEOREM 0.2.77: If $a$ is an idempotent in a semigroup then $a$ is completely regular.

THEOREM 0.2.78: Let $S$ be a semigroup and $a \in S$. Then $a$ is completely regular iff $a$ is both left regular and right regular.
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**THEOREM 0.2.79**: If a semigroup $S$ is regular then every principal ideal of $S$ is generated by an idempotent.

**DEFINITION 0.2.80**: An element $a$ of a semigroup $S$ is said to be a *midunit* provided $xay = xy$ for any $x, y \in S$.

**DEFINITION 0.2.81**: An element $a$ of a semigroup $S$ is said to be *semisimple* provided $a \in <a>^2$, that is $<a>^2 = <a>$.

**DEFINITION 0.2.82**: A semigroup $S$ is said to be a *semisimple semigroup* provided every element in $S$ is semisimple.

**THEOREM 0.2.83**: Let $S$ be a semigroup and $a \in S$. If $a$ is completely regular then $a$ is regular.

**THEOREM 0.2.84**: Let $S$ be a semigroup and $a \in S$. If $a$ is regular then $a$ is semisimple.

**THEOREM 0.2.85**: Let $a$ be an element of a semigroup $S$. If $a$ is left regular or right regular then $a$ is semisimple.

**THEOREM 0.2.86**: Let $a$ be an element of a semigroup $S$. If $a$ is intra regular then $a$ is semisimple.

**THEOREM 0.2.87**: If $a$ is an element of a duo semigroup $S$, then the following are equivalent.
1. $a$ is completely regular.
2. $a$ is regular.
3. $a$ is left regular.
4. $a$ is right regular.
5. $a$ is intra regular.
6. $a$ is semisimple.

**DEFINITION 0.2.88**: An ideal $A$ of a semigroup $S$ is said to be a *completely prime ideal* provided $x, y \in S, xy \in A$, implies either $x \in A$ or $y \in A$.

**THEOREM 0.2.89**: An ideal $P$ of a semigroup $S$ is completely prime if and only if $S \setminus P$ is either a subsemigroup of $S$ or empty.
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**DEFINITION 0.2.90**: An ideal \( A \) of a semigroup \( S \) is said to be a *prime ideal* provided \( X, Y \) are ideals of \( S \), \( XY \subseteq A \) implies either \( X \subseteq A \) or \( Y \subseteq A \).

**THEOREM 0.2.91**: If \( P \) is an ideal in a semigroup \( S \), then the following conditions are equivalent.

1. \( P \) is a prime ideal.
2. If \( a, b \in S \) such that \( aSb \subseteq P \) then either \( a \in P \) or \( b \in P \).
3. If \( S^1aS^1 \) and \( S^1bS^1 \) are principal ideals in \( S \) such that \( (S^1aS^1)(S^1bS^1) \subseteq P \) then either \( a \in P \) or \( b \in P \).
4. If \( U \) and \( V \) are two right ideals in \( S \) such that \( UV \subseteq P \) then either \( U \subseteq P \) or \( V \subseteq P \).
5. If \( U \) and \( V \) are two left ideals in \( S \) such that \( UV \subseteq P \) then either \( U \subseteq P \) or \( V \subseteq P \).

**THEOREM 0.2.92**: Every completely prime ideal of a semigroup is prime.

**THEOREM 0.2.93**: Let \( S \) be a commutative semigroup. An ideal \( P \) of \( S \) is completely prime if and only if \( P \) is prime.

**THEOREM 0.2.94**: An ideal \( P \) of a semigroup \( S \) is a prime ideal of \( S \) if and only if \( S \Delta P \) is an m-system of \( S \) or empty.

**THEOREM 0.2.95**: If \( S \) is a globally idempotent semigroup then every maximal ideal \( M \) of \( S \) is a prime ideal of \( S \).

**THEOREM 0.2.96**: If \( S \) is a globally idempotent semigroup with maximal ideals then \( S \) contains semisimple elements.

**DEFINITION 0.2.97**: An ideal \( A \) of a semigroup \( S \) is said to be completely semiprime provided \( x \in S, \ x^n \in A \) for some natural number \( n \) implies \( x \in A \).

**THEOREM 0.2.98**: An ideal \( A \) of a semigroup \( S \) is completely semiprime iff \( x \in S, \ x^2 \in A \) implies \( x \in A \).

**THEOREM 0.2.99**: If \( A \) is a completely semiprime ideal of a semigroup \( S \), then \( x, y \in S, \ xy \in A \Rightarrow xSy \subseteq A \).
COROLLARY 0.2.100: If \( A \) is completely semiprime ideal of a semigroup \( S \) then \( x, y \in S, xy \in A \Rightarrow <x> <y> \subseteq A \).

THEOREM 0.2.101: If \( A \) is a completely prime ideal of a semigroup \( S \) then \( A \) is a completely semiprime ideal of \( S \).

THEOREM 0.2.102: An ideal \( A \) of a semigroup \( S \) is completely prime iff \( A \) is prime and completely semiprime.

THEOREM 0.2.103: The nonempty intersection of any family of completely prime ideals of a semigroup is completely semiprime.

DEFINITION 0.2.104: An ideal \( A \) of a semigroup \( S \) is said to be semiprime provided \( X \) is an ideal of \( S \), \( X^n \subseteq A \) for some natural number \( n \) implies \( X \subseteq A \).

THEOREM 0.2.105: An ideal \( A \) of a semigroup \( S \) is semiprime iff \( X \) is an ideal of \( S \), \( X^2 \subseteq A \) implies \( X \subseteq A \).

THEOREM 0.2.106: If \( A \) is an ideal of a semigroup \( S \) then the following are equivalent.
1. \( A \) is a semiprime ideal.
2. If \( a \in S \) such that \( aSa \subseteq A \) then \( a \in A \).
3. If \( S^1aS^1 \) is a principal ideal of \( S \) such that \( (S^1aS^1) \subseteq A \) then \( a \in A \).
4. If \( P \) is a right ideal of \( S \) such that \( P^2 \subseteq A \) then \( P \subseteq A \).
5. If \( P \) is a left ideal of \( S \) such that \( P^2 \subseteq A \) then \( P \subseteq A \).

THEOREM 0.2.107: Every prime ideal of a semigroup is semiprime.

THEOREM 0.2.108: Every completely semiprime ideal of a semigroup is semiprime.

THEOREM 0.2.109: An ideal \( A \) of a commutative semigroup \( S \) is completely semiprime iff semiprime.

THEOREM 0.2.110: The nonempty intersection of prime ideals of a semigroup \( S \) is a semiprime ideal of \( S \).
Theorem 0.2.111: An ideal $Q$ of a semigroup $S$ is a semiprime ideal iff $S \setminus Q$ is an $n$-system or empty.

Theorem 0.2.112: If $N$ is an $n$-system in a semigroup $S$ and $a \in N$, then there exists an $m$-system $M$ in $S$ such that $a \in M$ and $M \subseteq N$.

Corollary 0.2.113: An ideal $Q$ of a semigroup $S$ is a semiprime ideal iff $Q$ is the intersection of all prime ideals of $S$ containing $Q$.

Theorem 0.2.114: Let $I$ be a semiprime ideal and $M$ be an $m$-system of a semigroup $S$. If $M^*$ is any $m$-system of $S$ maximal relative to the properties: $M \subseteq M^*$, $I \cap M^* = \emptyset$, then $S \setminus M^*$ is a minimal prime ideal of $S$ containing $I$.

Theorem 0.2.115: Every prime ideal $P$ minimal relative to containing a completely semiprime ideal $A$ of a semigroup $S$ is completely prime.

Definition 0.2.116: A subsemigroup $F$ of a semigroup $S$ is a filter of $S$ if for all $x, y \in S$, $xy \in F$ implies $x, y \in F$.

Notation 0.2.117: For any element $x$ of a semigroup $S$, let $N(x)$ denote the least filter of $S$ containing $x$, and let $N_x = \{ y \in S / N(x) = N(y) \}$.

Definition 0.2.118: A relation $\rho$ on a set $S$ is said to be reflexive on $S$ if $x \rho x$ for all $x \in S$.

Definition 0.2.119: A relation $\rho$ on a set $S$ is said to be symmetric on $S$ if $x, y \in S$ and $x \rho y$ implies $y \rho x$.

Definition 0.2.120: A relation $\rho$ on a set $S$ is said to be antisymmetric on $S$ if $x, y \in S$ and $x \rho y$, $y \rho x$ implies $x = y$.

Definition 0.2.121: A relation $\rho$ on a set $S$ is said to be transitive on $S$ if $x, y, z \in S$, $x \rho y$, $y \rho z$ implies $x \rho z$.

Definition 0.2.122: A relation $\rho$ on a set $S$ is said to be a partial ordering on $S$ if (i) $x \rho x$ for all $x \in S$, (ii) $x, y \in S$ and $x \rho y$, $y \rho x$ implies $x = y$, (iii) $x, y, z \in S$, $x \rho y$, $y \rho z$ implies $x \rho z$ and the set $S$ is called partially ordered set or simply poset.
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**DEFINITION 0.2.123**: A relation ρ on a set S is said to be an *equivalence relation* on S if (i) \(x \rho x\) for all \(x \in S\), (ii) \(x, y \in S, x \rho y\) implies \(y \rho x\) (iii) \(x, y, z \in S, x \rho y, y \rho z\) implies \(x \rho z\).

**DEFINITION 0.2.124**: A partial order \(\leq\) on a set S is said to be *linear* if for any \(a, b \in S\), either \(a \leq b\) or \(b \leq a\).

**DEFINITION 0.2.125**: Let \(\leq\) be a partial order on a set S and is linear, then S is called a *chain*.

**DEFINITION 0.2.126**: An equivalence relation ρ on a semigroup S is a *left congruence* if for all \(a, b, c \in S\), \(apb\) implies \(ca \rho cb\).

**DEFINITION 0.2.127**: An equivalence relation ρ on a semigroup S is a *right congruence* if for all \(a, b, c \in S\), \(apb\) implies \(ac \rho bc\).

**DEFINITION 0.2.128**: An equivalence relation ρ on a semigroup S is a *congruence* if it is both *left* and *right* congruence.

**DEFINITION 0.2.129**: A proper (left or right) congruence is a (left or right) congruence which is proper as an equivalence relation.

**THEOREM 0.2.130**: An equivalence relation \(\rho\) on a semigroup S is congruence if and only if for all \(a, b, c, d \in S\), \(a \rho b\) and \(c \rho d\) implies \(ac \rho bd\).

**DEFINITION 0.2.131**: Let \(\rho\) be a congruence on a semigroup S. Then the set \(S/\rho\) of all \(\rho\)-classes with the multiplication \((a_\rho)(b_\rho) = (ab)_\rho\) for all \(a, b \in S\) is called the *quotient semigroup* relative to the congruence \(\rho\).

**THEOREM 0.2.132**: Let S be a semigroup. If \(\rho_1\) and \(\rho_2\) are two left congruences (resp. right congruences, congruences) on S, then \(\rho_1 \circ \rho_2\) is a left congruence (resp. right congruence, congruence) on S.

**COROLLARY 0.2.133**: Let S be a semigroup. If \(\rho_1, \rho_2, \ldots, \rho_n\) are left congruences (resp. right congruences, congruences) on S, then \(\rho_1 \circ \rho_2 \circ \ldots \circ \rho_n\) is a left congruence (resp. right congruence, congruence) on S.
THEOREM 0.2.134: The intersection of any set of congruences on a semigroup \( S \) is again a congruence on \( S \).

THEOREM 0.2.135: The union of a non-empty family of congruences on a semigroup \( S \) is a congruence on \( S \).

NOTE 0.2.136: The set of congruences on a semigroup \( S \) is denoted by \( C(S) \).

DEFINITION 0.2.137: The intersection of all congruences on a semigroup \( S \) containing a binary relation \( \rho \) on \( S \) is called the congruence generated by \( \rho \).

DEFINITION 0.2.138: A semigroup \( S \) is said to be a band if every element of \( S \) is an idempotent.

DEFINITION 0.2.139: A semigroup \( S \) is said to be a semilattice if \( S \) is a commutative band.

DEFINITION 0.2.140: A congruence \( \rho \) on a semigroup \( S \) is said to be semilattice congruence if for all \( a, b \in S \), \( a^2 \rho a \) and \( ab \rho ba \).

DEFINITION 0.2.141: A semilattice congruence \( \rho \) on a semigroup \( S \) is said to be complete if for any \( a, b \in S \), \( a \leq b \) implies \( a \rho ab \).

NOTATION 0.2.142: We denote \( \mathcal{R} \) the equivalence relation on a semigroup \( S \) defined by \( \mathcal{R} = \{(a, b) \in S \times S / N(a) = N(b)\} \). \((a)_{\mathcal{R}}\) denote the \( \mathcal{R} \)-class of \( S \) containing \( a \) for \( a \in S \). Let \( S/\mathcal{R} = \{(a)_{\mathcal{R}} / a \in S\} \).

THEOREM 0.2.143: If \( S \) is a semigroup and \( a \in S \), then the \( \mathcal{R} \)-class \((a)_{\mathcal{R}}\) is a subsemigroup of \( S \).

LEMMA 0.2.144: If \( S \) is a semigroup, then the set \( S/\mathcal{R} = \{(a)_{\mathcal{R}} / a \in S\} \) is a semigroup with respect to the operation defined by \((a)_{\mathcal{R}}(b)_{\mathcal{R}} = (ab)_{\mathcal{R}}\) for all \((a)_{\mathcal{R}}, (b)_{\mathcal{R}} \in S/\mathcal{R}\).

THEOREM 0.2.145: Let \( S \) be a semigroup, \( z \in S \). If \( A \) is an ideal of an \( \mathcal{R} \)-class \((z)_{\mathcal{R}}\), then \( A \) has no proper completely prime ideals.

COROLLARY 0.2.146: Let \( S \) be a semigroup and \( A \) be a completely prime ideal of \( S \). Then \( A = \cup \{(a)_{\mathcal{R}} / a \in A\} \).
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**DEFINITION 0.2.147**: A semigroup is said to be \(\mathcal{R}\)-simple if \(A\) has no proper completely prime ideals.

**COROLLARY 0.2.148**: Every semigroup is a semilattice of \(\mathcal{R}\)-simple semigroups.

**THEOREM 0.2.149**: If \(A\) is a completely prime ideal of a semigroup \(S\), then \(J = \{ (x)_{\mathcal{R}} \in S/\mathcal{R} : x \in A \}\) is a completely prime ideal of \(S/\mathcal{R}\). Conversely, if \(J\) is a completely prime ideal of \(S/\mathcal{R}\), then \(A = \{ x \in S : (x)_{\mathcal{R}} \in J \}\) is a completely prime ideal of \(S\). This establishes a one-to-one, order preserving (relative to inclusion) correspondence between the partially ordered set of all completely prime ideals of \(S\) and the partially ordered set of all completely prime ideals of \(S/\mathcal{R}\).

**THEOREM 0.2.150**: The following conditions on an ideal \(A\) of a semigroup \(S\) are equivalent

(i) \(A\) is the intersection of completely prime ideals of \(S\) containing \(A\)

(ii) \(A\) is the intersection of minimal completely prime ideals of \(S\).

(iii) \(A\) is the union of \(\mathcal{R}\)-classes.

(iv) \(A\) is a completely semiprime ideal of \(S\).

**COROLLARY 0.2.151**: A semigroup \(S\) is \(\mathcal{R}\)-simple if and only if it contains no proper completely semiprime ideals.

**COROLLARY 0.2.152**: If \(A\) is a completely semiprime ideal of a semigroup \(S\), then \(J = \{ (x)_{\mathcal{R}} \in S/\mathcal{R} : x \in A \}\) is an ideal of \(S/\mathcal{R}\). Conversely, if \(J\) is an ideal of \(S/\mathcal{R}\), then \(A = \{ x \in S : (x)_{\mathcal{R}} \in J \}\) is a completely semiprime ideal of \(S\). This establishes a one-to-one, order preserving (relative to inclusion) correspondence between the partially ordered set of all completely semiprime ideals of \(S\) and the partially ordered set of all ideals of \(S/\mathcal{R}\).

**DEFINITION 0.2.153**: A nonempty subset \(C\) of a semigroup \(S\) is said to be an \(\mathcal{R}\)-subset of \(S\), if \(C\) is completely semiprime and satisfies the condition: for any \(x, y \in S\) and \(z \in S^1\), \(x, yz \in C\) implies \(xy, zx \in C\).

**THEOREM 0.2.154**: Let \(C\) be an \(\mathcal{R}\)-subset of a semigroup \(S\). Then for any \(a, b \in S\) and \(x \in S^1\), \(xab \in C\) implies \(xba \in C\).
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THEOREM 0.2.155: A nonempty subset \( \mathcal{C} \) of a semigroup \( S \) is an \( \mathcal{R} \)-subset, then \( \mathcal{C} \) is a class of semilattice congruences on \( S \).

THEOREM 0.2.156: A semigroup \( S \) is separative if and only if \( S \) is a semilattice of cancellative semigroup. If so, the relation \( \sigma \) defined on \( S \) by \( x \sigma y \) if for any \( a, b \in S \), \( xa = xb \) if and only if \( ya = yb \) is the greatest band congruence on \( S \) all whose classes are cancellative.

COROLLARY 0.2.157: A semigroup \( S \) is separative if and only if every \( \mathcal{R} \)-class of \( S \) is cancellative.

THEOREM 0.2.158: If \( S \) is a commutative and separative semigroup, then \( S \) is a semilattice of commutative cancellative semigroup.

COROLLARY 0.2.159: If a semigroup \( S \) is commutative and separative, then every \( \mathcal{R} \)-class of \( S \) is commutative and cancellative.

DEFINITION 0.2.160: Let \( S \) and \( \Gamma \) be two non-empty sets. Then \( S \) is called a \( \Gamma \)-semigroup if there exists a mapping from \( S \times \Gamma \times S \) to \( S \) which maps \( (a, \alpha, b) \rightarrow a \alpha b \) satisfying the condition : \( (a\gamma b)\mu c = a\gamma(b\mu c) \) for all \( a, b, c \in S \) and \( \gamma, \mu \in \Gamma \).

DEFINITION 0.2.161: A \( \Gamma \)-semigroup \( S \) is said to be finite provided the cardinality of \( S \) is finite.

DEFINITION 0.2.162: A \( \Gamma \)-semigroup \( S \) is said to be commutative provided \( a\gamma b = b\gamma a \) for all \( a, b \in S \) and \( \gamma \in \Gamma \).

NOTE 0.2.163: If \( S \) is a commutative \( \Gamma \)-semigroup then \( a \Gamma b = b \Gamma a \) for all \( a, b \in S \).

NOTE 0.2.164: Let \( S \) be a \( \Gamma \)-semigroup and \( a, b \in S, \alpha \in \Gamma \). Then \( aaaaab \) is denoted by \( (aa)^3b \) and consequently \( a \alpha a a a a ... (n \text{ terms}) b \) is denoted by \( (a\alpha)^n b \).

DEFINITION 0.2.165: A \( \Gamma \)-semigroup \( S \) is said to be quasi commutative provided for each \( a, b \in S \), there exists a natural number \( n \) such that \( a\gamma b = (b\gamma)^n a \ \forall \gamma \in \Gamma \).

NOTE 0.2.166: If a \( \Gamma \)-semigroup \( S \) is quasi commutative then for each \( a, b \in S \), there exists a natural number \( n \) such that, \( a\Gamma b = (b \Gamma)^n a \).
THEOREM 0.2.167: If $S$ is a commutative $\Gamma$-semigroup then $S$ is a quasi commutative $\Gamma$-semigroup.

DEFINITION 0.2.168: A $\Gamma$-semigroup $S$ is said to be normal provided $aaS = Saa \quad \forall a \in \Gamma$ and $\forall a \in S$.

NOTE 0.2.169: If a $\Gamma$-semigroup $S$ is normal then $aS = Sa$ for all $a \in S$.

THEOREM 0.2.170: If $S$ is a quasi commutative $\Gamma$-semigroup then $S$ is a normal $\Gamma$-semigroup.

COROLLARY 0.2.171: Every commutative $\Gamma$-semigroup is a normal $\Gamma$-semigroup.

DEFINITION 0.2.172: An element $a$ of a $\Gamma$-semigroup $S$ is said to be a left identity of $S$ provided $aas = s$ for all $s \in S$ and $a \in \Gamma$.

DEFINITION 0.2.173: An element ‘$a$’ of a $\Gamma$-semigroup $S$ is said to be a right identity of $S$ provided $sa = s$ for all $s \in S$ and $a \in \Gamma$.

DEFINITION 0.2.174: An element ‘$a$’ of a $\Gamma$-semigroup $S$ is said to be a two sided identity or an identity provided it is both a left identity and a right identity of $S$.

THEOREM 0.2.175: If $a$ is a left identity and $b$ is a right identity of a $\Gamma$-semigroup $S$, then $a = b$.

THEOREM 0.2.176: Any $\Gamma$-semigroup $S$ has at most one identity.

NOTE 0.2.177: The identity (if exists) of a $\Gamma$-semigroup is usually denoted by $e$ or 1.

NOTATION 0.2.178: Let $S$ be a $\Gamma$-semigroup. If $S$ has an identity, let $S^1 = S$ and if $S$ does not have an identity, let $S^1$ be the $\Gamma$-semigroup $S$ with an identity adjoined usually denoted by the symbol 1.

DEFINITION 0.2.179: A $\Gamma$-semigroup $S$ with identity is called a $\Gamma$-monoid.

DEFINITION 0.2.180: An element $a$ of a $\Gamma$-semigroup $S$ is said to be a left zero of $S$ provided $aas = a$ for all $s \in S$ and $a \in \Gamma$. 

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**DEFINITION 0.2.181**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be a **right zero** of $S$ provided $saa = a$ for all $s \in S$ and $a \in \Gamma$.

**DEFINITION 0.2.182**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be a **two sided zero** or **zero** provided it is both a left zero and a right zero of $S$.

**DEFINITION 0.2.183**: A $\Gamma$-semigroup in which every element is a left zero is called a **left zero $\Gamma$-semigroup**.

**DEFINITION 0.2.184**: A $\Gamma$-semigroup in which every element is a right zero is called a **right zero $\Gamma$-semigroup**.

**DEFINITION 0.2.185**: A $\Gamma$-semigroup with 0 in which the product of any two elements equals to 0 is called a **zero $\Gamma$-semigroup** or a **null $\Gamma$-semigroup**.

**THEOREM 0.2.186**: If $a$ is a left zero and $b$ is a right zero of a $\Gamma$-semigroup $S$, then $a = b$.

**THEOREM 0.2.187**: Any $\Gamma$-semigroup $S$ has at most one zero element.

**NOTE 0.2.188**: The zero (if exists) of a $\Gamma$-semigroup is usually denoted by 0.

**NOTATION 0.2.189**: Let $S$ be a $\Gamma$-semigroup. If $S$ has a zero, let $S^0 = S$ and if $S$ does not have zero, let $S^0$ be the $\Gamma$-semigroup $S$ with zero adjoined usually denoted by the symbol 0.

**DEFINITION 0.2.190**: Let $S$ be a $\Gamma$-semigroup with identity and $a \in S$, $\alpha \in \Gamma$. An element $b$ of $S$ said to be a **left $\alpha$-inverse** of $a$ in $S$ provided, $baa = e$.

**DEFINITION 0.2.191**: An element $b$ of a $\Gamma$-semigroup $S$ with identity $e$ is said to be a **right $\alpha$-inverse** of $a$ in a $\Gamma$-semigroup $S$ provided, $aab = e$ for $\alpha \in \Gamma$.

**THEOREM 0.2.192**: If $b$ is a left $\alpha$-inverse and $c$ is a right $\alpha$-inverse of an element $a$ in a $\Gamma$-semigroup $S$ with identity $e$, then $b = c$.

**DEFINITION 0.2.193**: An element $b$ of a $\Gamma$-semigroup $S$ with identity $e$ is said to be a **$\alpha$-inverse** of $a$ in a $\Gamma$-semigroup $S$ provided, $aab = baa = e$ for $\alpha \in \Gamma$.

**THEOREM 0.2.194**: The $\alpha$-inverse of an element $a$ in a $\Gamma$-semigroup $S$ with identity $e$ (if exists) is unique.
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**DEFINITION 0.2.195** : An element $b$ of a $\Gamma$-semigroup $S$ with identity $e$ is said to be a *left $\Gamma$-inverse* of $a$ in a $\Gamma$-semigroup $S$ provided, $b \alpha a = e$ for all $\alpha \in \Gamma$.

**NOTE 0.2.196** : An element $b$ of a $\Gamma$-semigroup $S$ with identity $e$ is the *left $\Gamma$-inverse* of $a$ in a $\Gamma$-semigroup $S$ provided $b \Gamma a = e$.

**DEFINITION 0.2.197** : An element $b$ of a $\Gamma$-semigroup $S$ with identity $e$ is said to be a *right $\Gamma$-inverse* of $a$ in a $\Gamma$-semigroup $S$ provided, $a \alpha b = e$ for all $\alpha \in \Gamma$.

**NOTE 0.2.198** : An element $b$ of a $\Gamma$-semigroup $S$ with identity $e$ is the *right $\Gamma$-inverse* of $a$ in a $\Gamma$-semigroup $S$ provided $a \Gamma b = e$.

**THEOREM 0.2.199** : If $b$ is a left $\Gamma$-inverse and $c$ is a right $\Gamma$-inverse of an element $a$ of a $\Gamma$-semigroup $S$ with identity $e$, then $b = c$.

**DEFINITION 0.2.200** : An element $b$ of a $\Gamma$-semigroup $S$ with identity $e$ is said to be a $\Gamma$-inverse of $a$ of a $\Gamma$-semigroup $S$ provided it is both a left $\Gamma$-inverse of $a$ and a right $\Gamma$-inverse of $a$.

**NOTE 0.2.201** : An element $b$ of a $\Gamma$-semigroup $S$ with identity $e$ is the $\Gamma$-inverse of $a$ of a $\Gamma$-semigroup $S$ provided $a \Gamma b = b \Gamma a = e$.

**THEOREM 0.2.202** : The $\Gamma$-inverse of an element $a$ in a $\Gamma$-semigroup $S$ with identity $e$ (if exists) is unique.

**DEFINITION 0.2.203** : An element $a$ of a $\Gamma$-semigroup $S$ is said to be a *unit* if it has $\Gamma$-inverse.

**DEFINITION 0.2.204** : A $\Gamma$-semigroup $S$ is said to be a *$\Gamma$-group* if

1. $\exists \ e \in S \exists \ a \Gamma e = e \Gamma a = a$ for all $a \in S$.
2. every element $a \in S$ has a $\alpha$-inverse in $S$ for some $\alpha \in \Gamma$.

**DEFINITION 0.2.205** : Let $S$ be a $\Gamma$-semigroup. A nonempty subset $T$ of $S$ is said to be a $\Gamma$-subsemigroup of $S$ if $a \gamma b \in T$, for all $a, b \in T$ and $\gamma \in \Gamma$.

**NOTE 0.2.206** : A nonempty subset $T$ of a $\Gamma$-semigroup $S$ is a $\Gamma$-subsemigroup of $S$ iff $T \Gamma T \subseteq T$. 

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**THEOREM 0.2.207**: The nonempty intersection of two $\Gamma$-subsemigroups of a $\Gamma$-semigroup $S$ is a $\Gamma$-subsemigroup of $S$.

**THEOREM 0.2.208**: The nonempty intersection of any family of $\Gamma$-subsemigroups of a $\Gamma$-semigroup $S$ is a $\Gamma$-subsemigroup of $S$.

**DEFINITION 0.2.209**: Let $S$ be a $\Gamma$-semigroup and $A$ be a nonempty subset of $S$. The smallest $\Gamma$-subsemigroup of $S$ containing $A$ is called a $\Gamma$-subsemigroup of $S$ generated by $A$. It is denoted by $(A)$.

**THEOREM 0.2.210**: Let $S$ be a $\Gamma$-semigroup and $A$ be a nonempty subset of $S$. Then $(A) = \{ a_1 \alpha_1 a_2, \alpha_2, \ldots, a_{n-1} \alpha_{n-1} a_n : n \in \mathbb{N}, a_1, a_2, \ldots, a_n \in A, \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \Gamma \}$. 

**THEOREM 0.2.211**: Let $S$ be a $\Gamma$-semigroup and $A$ be a nonempty subset of $S$. Then $(A) = \bigcap \{ T \mid T$ is a $\Gamma$-subsemigroup of $S$ containing $A \}$.

**DEFINITION 0.2.212**: Let $S$ be a $\Gamma$-semigroup. A $\Gamma$-subsemigroup $T$ of $S$ is said to be cyclic $\Gamma$-subsemigroup of $S$ if $T$ is generated by a single element subset of $S$.

**NOTE 0.2.213**: Let $T$ be a $\Gamma$-subsemigroup of $\Gamma$-semigroup $S$. Then $T$ is cyclic iff $T = \bigcup_{n \in \mathbb{N}} (a \Gamma)^{n-1} a$ for some $a \in S$.

**DEFINITION 0.2.214**: A $\Gamma$-semigroup $S$ is said to be a cyclic $\Gamma$-semigroup if $S$ is a cyclic $\Gamma$-subsemigroup of $S$ itself.

**DEFINITION 0.2.215**: An element $a$ of $\Gamma$-semigroup $S$ is said to be an $\alpha$-idempotent provided $a \alpha a = a$.

**NOTE 0.2.216**: The set of all $\alpha$-idempotent elements in a $\Gamma$-semigroup $S$ is denoted by $E_\alpha$.

**DEFINITION 0.2.217**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be an idempotent or $\Gamma$-idempotent if $aaa = a$ for all $\alpha \in \Gamma$.

**NOTE 0.2.218**: In a $\Gamma$-semigroup $S$, $a$ is an idempotent iff $a$ is an $\alpha$-idempotent for all $\alpha \in \Gamma$.

**NOTE 0.2.219**: If an element $a$ of $\Gamma$-semigroup $S$ is an idempotent, then $a \Gamma a = a$. 

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**DEFINITION 0.2.220** : A $\Gamma$-semigroup $S$ is said to be an *idempotent* $\Gamma$-$\text{semigroup}$ provided every element of $S$ is $\alpha$–idempotent for some $\alpha \in \Gamma$.

**DEFINITION 0.2.221** : A $\Gamma$-semigroup $S$ is said to be a *strongly idempotent* $\Gamma$-$\text{semigroup}$ provided every element in $S$ is an idempotent.

**DEFINITION 0.2.222** : An element $a$ of $\Gamma$-$\text{semigroup}$ $S$ is said to be a *midunit* provided $x\Gamma a \Gamma y = x \Gamma y$ for all $x, y \in S$.

**NOTE 0.2.223** : Identity of a $\Gamma$-semigroup $S$ is a midunit.

**DEFINITION 0.2.224** : An element ‘$a$’ of $\Gamma$-semigroup $S$ is said to be an *$r$-element* provided $a \Gamma s = s \Gamma a$ for all $s \in S$ and if $x, y \in S$, then $a \Gamma x \Gamma y = b \Gamma y \Gamma x$ for some $b \in S$.

**DEFINITION 0.2.225** : A $\Gamma$-semigroup $S$ with identity $1$ is said to be a *generalized commutative* $\Gamma$-$\text{semigroup}$ provided $1$ is an $r$-element in $S$.

**DEFINITION 0.2.226** : An element $a$ of a $\Gamma$-semigroup $S$ is said to be *regular* provided $a = a \alpha \beta a$, for some $x \in S$ and $\alpha, \beta \in \Gamma$. i.e, $a \in a \Gamma S \Gamma a$.

**DEFINITION 0.2.227** : A $\Gamma$-semigroup $S$ is said to be a *regular* $\Gamma$-$\text{semigroup}$ provided every element is regular.

**THEOREM 0.2.228** : Every $\alpha$-idempotent element in a $\Gamma$-semigroup is regular.

**DEFINITION 0.2.229** : A nonempty subset $A$ of a $\Gamma$-semigroup $S$ is said to be a *left $\Gamma$-ideal* of $S$ if $s \in S, a \in A, \alpha \in \Gamma$ implies $s \alpha a \in A$.

**NOTE 0.2.230** : A nonempty subset $A$ of a $\Gamma$-semigroup $S$ is a left $\Gamma$- ideal of $S$ iff $S \Gamma A \subseteq A$.

**DEFINITION 0.2.231** : A nonempty subset $A$ of a $\Gamma$-semigroup $S$ is said to be a *right $\Gamma$-ideal* of $S$ if $s \in S, a \in A, \alpha \in \Gamma$ implies $a \alpha s \in A$.

**NOTE 0.2.232** : A nonempty subset $A$ of a $\Gamma$-semigroup $S$ is a right $\Gamma$- ideal of $S$ iff $A \Gamma S \subseteq A$.

**DEFINITION 0.2.233** : A nonempty subset $A$ of a $\Gamma$-semigroup $S$ is said to be a *two sided $\Gamma$- ideal* or simply a $\Gamma$-$\text{ideal}$ of $S$ if $s \in S, a \in A, \alpha \in \Gamma$ imply $s \alpha a \in A, a \alpha s \in A$. 


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**NOTE 0.2.234** : A nonempty subset $A$ of a $\Gamma$-semigroup $S$ is a two sided $\Gamma$-ideal iff it is both a left $\Gamma$-ideal and a right $\Gamma$-ideal of $S$.

**DEFINITION 0.2.235** : A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a *proper $\Gamma$-ideal* of $S$ if $A$ is different from $S$.

**DEFINITION 0.2.236** : A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a *trivial $\Gamma$-ideal* provided $S \setminus A$ is singleton.

**DEFINITION 0.2.237** : A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a *maximal left $\Gamma$-ideal* provided $A$ is a proper left $\Gamma$-ideal of $S$ and is not properly contained in any proper left $\Gamma$-ideal of $S$.

**DEFINITION 0.2.238** : A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a *maximal right $\Gamma$-ideal* provided $A$ is a proper right $\Gamma$-ideal of $S$ and is not properly contained in any proper right $\Gamma$-ideal of $S$.

**DEFINITION 0.2.239** : A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a *maximal $\Gamma$-ideal* provided $A$ is a proper $\Gamma$-ideal of $S$ and is not properly contained in any proper $\Gamma$-ideal of $S$.

**DEFINITION 0.2.240** : A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be *globally idempotent* if $A \Gamma A = A$.

**THEOREM 0.2.241** : If $A$ is a $\Gamma$-ideal of a $\Gamma$-semigroup $S$ with unity $1$ and $1 \in A$ then $A = S$.

**DEFINITION 0.2.242** : Let $S$ be a $\Gamma$-semigroup and $A$ be a nonempty subset of $S$. The smallest left $\Gamma$-ideal of $S$ containing $A$ is called *left $\Gamma$-ideal of $S$ generated by $A$*.

**THEOREM 0.2.243** : The left $\Gamma$-ideal of a $\Gamma$-semigroup $S$ generated by a nonempty subset $A$ is the intersection of all left $\Gamma$-ideals of $S$ containing $A$.

**DEFINITION 0.2.244** : Let $S$ be a $\Gamma$-semigroup and $A$ be a nonempty subset of $S$. The smallest right $\Gamma$-ideal of $S$ containing $A$ is called *right $\Gamma$-ideal of $S$ generated by $A$*.

**THEOREM 0.2.245** : The right $\Gamma$-ideal of a $\Gamma$-semigroup $S$ generated by a nonempty subset $A$ is the intersection of all right $\Gamma$-ideals of $S$ containing $A$. 
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**DEFINITION 0.2.246**: Let $S$ be a Γ-semigroup and $A$ be a nonempty subset of $S$. The smallest Γ-ideal of $S$ containing $A$ is called \textit{Γ-ideal of $S$ generated by $A$}.

**THEOREM 0.2.247**: The Γ-ideal of a Γ-semigroup $S$ generated by a nonempty subset $A$ is the intersection of all Γ-ideals of $S$ containing $A$.

**DEFINITION 0.2.248**: A left Γ-ideal $A$ of a Γ-semigroup $S$ is said to be the \textit{principal left Γ-ideal generated by $a$} if $A$ is a left Γ-ideal generated by \{a\} for some $a \in S$. It is denoted by $L(a)$.

**THEOREM 0.2.249**: If $S$ is a Γ-semigroup and $a \in S$ then $L(a) = a \cup S\Gamma a$.

**DEFINITION 0.2.250**: A right Γ-ideal $A$ of a Γ-semigroup $S$ is said to be the \textit{principal right Γ-ideal generated by $a$} if $A$ is a right Γ-ideal generated by \{a\} for some $a \in S$. It is denoted by $R(a)$.

**NOTE 0.2.251**: The principal right Γ-ideal generated by $a$ is denoted by $R(a)$.

**THEOREM 0.2.252**: If $S$ is a Γ-semigroup and $a \in S$ then $R(a) = a \cup a\Gamma S$.

**DEFINITION 0.2.253**: A Γ-ideal $A$ of a Γ-semigroup $S$ is said to be a \textit{principal Γ-ideal} provided $A$ is a Γ-ideal generated by \{a\} for some $a \in S$. It is denoted by $J[a]$ or <$a>$.

**THEOREM 0.2.254**: If $S$ is a Γ-semigroup and $a \in S$ then $J(a) = a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S$.

**THEOREM 0.2.255**: In any Γ-semigroup $S$, the following are equivalent.

1. Principal Γ-ideals of $S$ form a chain.
2. Γ-ideals of $S$ form a chain.

**DEFINITION 0.2.256**: A Γ-semigroup $S$ is said to be a \textit{left duo Γ-semigroup} provided every left Γ-ideal of $S$ is a two sided Γ-ideal of $S$.

**DEFINITION 0.2.257**: A Γ-semigroup $S$ is said to be a \textit{right duo Γ-semigroup} provided every right Γ-ideal of $S$ is a two sided Γ-ideal of $S$.

**DEFINITION 0.2.258**: A Γ-semigroup $S$ is said to be a \textit{duo Γ-semigroup} provided it is both a left duo Γ-semigroup and a right duo Γ-semigroup.
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**DEFINITION 0.2.259**: A $\Gamma$-semigroup $S$ is said to be a *left simple $\Gamma$-semigroup* if $S$ is its only left $\Gamma$-ideal.

**THEOREM 0.2.260**: A $\Gamma$-semigroup $S$ is a left simple $\Gamma$-semigroup if and only if $S\Gamma a = S$ for all $a \in S$.

**DEFINITION 0.2.261**: A $\Gamma$-semigroup $S$ is said to be a *right simple $\Gamma$-semigroup* if $S$ is its only right $\Gamma$-ideal.

**THEOREM 0.2.262**: A $\Gamma$-semigroup $S$ is a right simple $\Gamma$-semigroup if and only if $a\Gamma S = S$ for all $a \in S$.

**DEFINITION 0.2.263**: A $\Gamma$-semigroup $S$ is said to be *simple $\Gamma$-semigroup* if $S$ is its only two-sided $\Gamma$-ideal.

**THEOREM 0.2.264**: If $S$ is a left simple $\Gamma$-semigroup or a right simple $\Gamma$-semigroup then $S$ is a simple $\Gamma$-semigroup.

**THEOREM 0.2.265**: A $\Gamma$-semigroup $S$ is simple $\Gamma$-semigroup if and only if $S\Gamma a \Gamma S = S$ for all $a \in S$.

**DEFINITION 0.2.266**: A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be *regular* if every element of $A$ is regular in $S$.

**THEOREM 0.2.267**: Every $\Gamma$-ideal of a regular $\Gamma$-semigroup $S$ is a regular $\Gamma$-ideal of $S$.

**THEOREM 0.2.268**: If a $\Gamma$-semigroup $S$ is a regular $\Gamma$-semigroup then every principal $\Gamma$-ideal is generated by a $\beta$-idempotent for some $\beta \in \Gamma$.

**DEFINITION 0.2.269**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be *left regular* provided $a = a\alpha a\beta x$, for some $x \in S$ and $\alpha, \beta \in \Gamma$. i.e., $a \in a\Gamma a \Gamma S$.

**DEFINITION 0.2.270**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be *right regular* provided $a = x\alpha a\beta a$, for some $x \in S$ and $\alpha, \beta \in \Gamma$. i.e., $a \in S \Gamma a \Gamma a$.

**DEFINITION 0.2.271**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be *completely regular* provided, there exists an element $x \in S$ such that $a = a\alpha x \beta a$ for some $\alpha, \beta \in \Gamma$ and $a\alpha x = x\beta a$ i.e., $a \in a\Gamma x \Gamma a$ and $a\Gamma x = x\Gamma a$.  

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**DEFINITION 0.2.272**: A $\Gamma$-semigroup $S$ is said to be *completely regular $\Gamma$-semigroup* provided every element of $S$ is completely regular.

**DEFINITION 0.2.273**: Let $S$ be a $\Gamma$-semigroup, $a \in S$ and $\alpha, \beta \in \Gamma$. An element $b \in S$ is said to be an $(\alpha, \beta)$-*inverse* of $a$ if $a = aab\beta a$ and $b = b\beta aab$.

**THEOREM 0.2.274**: Let $S$ be a $\Gamma$-semigroup and $a \in S$. Then $a$ is a regular element if and only if $a$ has an $(\alpha, \beta)$-inverse.

**DEFINITION 0.2.275**: An element $a$ of $\Gamma$-semigroup $S$ is said to be *semisimple* provided $a \in < a > \Gamma < a >$, that is, $< a > \Gamma < a > = < a >$.

**DEFINITION 0.2.276**: A $\Gamma$-semigroup $S$ is said to be *semisimple $\Gamma$-semigroup* provided every element of $S$ is a semisimple element.

**DEFINITION 0.2.277**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be *intra regular* provided $a = x\alpha a\beta a y y$ for some $x, y \in S$ and $\alpha, \beta, y \in \Gamma$.

**THEOREM 0.2.278**: If ‘$a$’ is a completely regular element of a $\Gamma$-semigroup $S$, then $a$ is regular and semisimple.

**THEOREM 0.2.279**: If ‘$a$’ is a completely regular element of a $\Gamma$-semigroup $S$, then $a$ is both a left regular element and a right regular element.

**THEOREM 0.2.280**: If ‘$a$’ is a left regular element of a $\Gamma$-semigroup $S$, then $a$ is semisimple.

**THEOREM 0.2.281**: If ‘$a$’ is a right regular element of a $\Gamma$-semigroup $S$, then $a$ is semisimple.

**THEOREM 0.2.282**: If ‘$a$’ is a regular element of a $\Gamma$-semigroup $S$, then $a$ is semisimple.

**THEOREM 0.2.283**: If ‘$a$’ is a intra regular element of a $\Gamma$-semigroup $S$, then $a$ is semisimple.

**DEFINITION 0.2.284**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be *left $\alpha$-cancellative* provided for $\alpha \in \Gamma$, $aab = aac$ implies $b = c$.

**DEFINITION 0.2.285**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be *right $\alpha$-cancellative* provided for $\alpha \in \Gamma$, $baa = c\alpha a$ implies $b = c$. 


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**DEFINITION 0.2.286**: An element $a$ of a Γ-semigroup $S$ is said to be $α$-cancellative provided $a$ is both a left $α$-cancellative element and a right $α$-cancellative element.

**DEFINITION 0.2.287**: An element $a$ of a Γ-semigroup $S$ is said to be *left* Γ-cancellative provided $a$ is left $α$-cancellative for all $α ∈ Γ$.

**DEFINITION 0.2.288**: An element $a$ of a Γ-semigroup $S$ is said to be *right* Γ-cancellative provided $a$ is right $α$-cancellative for all $α ∈ Γ$.

**DEFINITION 0.2.289**: An element $a$ of a Γ-semigroup $S$ is said to be Γ-cancellative provided $a$ is a both left Γ-cancellative and Γ-cancellative.

**DEFINITION 0.2.290**: An element $a$ of a Γ-semigroup $S$ is said to be strongly left Γ-cancellative provided $aΓb = aΓc$ implies $b = c$.

**NOTE 0.2.291**: An element $a$ of a Γ-semigroup $S$ is strongly left Γ-cancellative provided $aαb = aβc$ , $α, β ∈ Γ$ ⇒ $b = c$.

**DEFINITION 0.2.292**: An element $a$ of a Γ-semigroup $S$ is said to be strongly right Γ-cancellative provided $bΓa = cΓa$ implies $b = c$.

**NOTE 0.2.293**: An element $a$ of a Γ-semigroup $S$ is strongly right Γ-cancellative provided $bαa = cβa$ , $α, β ∈ Γ$ ⇒ $b = c$.

**DEFINITION 0.2.294**: An element $a$ of a Γ-semigroup $S$ is said to be strongly Γ-cancellative provided $a$ is a both strongly left Γ-cancellative and strongly right Γ-cancellative.

**THEOREM 0.2.295**: Every Γ-group is a strongly Γ-cancellative Γ-semigroup.

### 0.3. **RESULTS OF THE THESIS**

The thesis is divided into 4 chapters.

Chapter 1 is divided into 3 sections. In section 1, the notion of a po-Γ-semigroup is introduced and some examples are given. Further the terms; commutative po-Γ-semigroup, quasi commutative po-Γ-semigroup, normal po-Γ-semigroup and (left,right) pseudo commutative po-Γ-semigroup are introduced. It is proved that (1) if $S$ is a commutative po-Γ-semigroup then $S$ is a quasi commutative po-Γ-semigroup, (2) if $S$ is
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A quasi commutative po-Γ-semigroup then S is a normal po-Γ-semigroup. It is proved that, if S is a commutative po-Γ-semigroup, then S is both left pseudo commutative po-Γ-semigroup and right pseudo commutative po-Γ-semigroup. Further the terms; left identity, right identity, identity, left zero, right zero, zero, left α-inverse of an element, right α-inverse of an element, α-inverse of an element, left Γ-inverse of an element, right Γ-inverse of an element, Γ-inverse of an element, unit element of a po-Γ-semigroup are introduced. It is proved that, if a is a left identity and b is a right identity of a po-Γ-semigroup, then a = b. It is also proved that any po-Γ-semigroup has at most one identity. It is proved that if a is a left zero and b is a right zero of a po-Γ-semigroup, then a = b and it is also proved that any po-Γ-semigroup has at most one zero element.

In section 2, the terms; po-Γ-subsemigroup, po-Γ-subsemigroup generated by a subset, cyclic po-Γ-subsemigroup of a po-Γ-semigroup and cyclic po-Γ-semigroup are introduced. It is proved that (1) if S is a po-Γ-semigroup and A ⊆ S, B ⊆ S, then (i) A ⊆ (A], (ii) ((A] = (A], (iii) (A]Γ(B] ⊆ (AΓB] and (iv) A ⊆ B ⇒ A ⊆ (B], (v) A ⊆ B ⇒ (A] ⊆ (B]. It is also proved that, (1) the nonempty intersection of any two po-Γ-subsemigroups of a po-Γ-semigroup S is a po-Γ-subsemigroup of S, (2) the nonempty intersection of any family of po-Γ-subsemigroups of a po-Γ-semigroup S is a po-Γ-subsemigroup of S. It is also proved that if A is a nonempty subset of a po-Γ-semigroup S, then the po-Γ-subsemigroup of S generated by A is the intersection of all po-Γ-subsemigroups of S containing A.

In section 3, the terms; α-idempotent, Γ-idempotent, strongly Γ-idempotent, midunit, r-element, regular element, left regular element, right regular element, completely regular element, (α, β)-inverse of an element are introduced. It is proved that, every α-idempotent element of a po-Γ-semigroup is regular. It is also proved that, in a po-Γ-semigroup, if a has an (α, β)-inverse then a is a regular element. Further the terms; left α-cancellative element, right α-cancellative element, α-cancellative element, left Γ-cancellative element, right Γ-cancellative element, Γ-cancellative element, strongly left Γ-cancellative element, strongly right Γ-cancellative element and strongly Γ-cancellative element in a po-Γ-semigroup are introduced. Also the terms; idempotent po-Γ-semigroup, generalized commutative po-Γ-semigroup and separative po-Γ-semigroup are introduced. It is also proved that, in a separative po-Γ-semigroup S, for any x, y, a, b ∈ S, the following statements hold.

(i) xΓa ≤ xΓb if and only if aΓx ≤ bΓx, (ii) x Γ x Γ a ≤ x Γ x Γ b implies x Γ a ≤ x Γ b,
The contents of chapter 1 are published in “International Journal of Engineering Research and Technology” under the title ‘Partially Ordered $\Gamma$-Semigroups’ [66].

Chapter 2 is divided into 3 sections. In section 1, the terms; left po-$\Gamma$-ideal, right po-$\Gamma$-ideal, po-$\Gamma$-ideal are introduced. It is proved that (1) the nonempty intersection of two left po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a left po-$\Gamma$-ideal of $S$, (2) the nonempty intersection of any family of left po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a left po-$\Gamma$-ideal of $S$, (3) the union of two left po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a left po-$\Gamma$-ideal of $S$ and (4) the union of any family of left po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a left po-$\Gamma$-ideal of $S$. It is also proved that (1) the nonempty intersection of two right po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a right po-$\Gamma$-ideal of $S$, (2) the nonempty intersection of any family of right po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a right po-$\Gamma$-ideal of $S$, (3) the union of two right po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a right po-$\Gamma$-ideal of $S$ and (4) the union of any family of right po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a right po-$\Gamma$-ideal of $S$. Further it is proved that (1) the nonempty intersection of two po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a po-$\Gamma$-ideal of $S$, (2) the nonempty intersection of any family of po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a po-$\Gamma$-ideal of $S$, (3) the union of two po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a po-$\Gamma$-ideal of $S$ and (4) the union of any family of po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$ is a po-$\Gamma$-ideal of $S$. Further the terms; proper po-$\Gamma$-ideal, trivial po-$\Gamma$-ideal, maximal left po-$\Gamma$-ideal, maximal right po-$\Gamma$-ideal, maximal po-$\Gamma$-ideal, left po-$\Gamma$-ideal generated by a subset, right po-$\Gamma$-ideal generated by a subset, po-$\Gamma$-ideal generated by a subset are introduced. It is proved that, if $S$ is a po-$\Gamma$-semigroup with unity 1, then the union of all proper po-$\Gamma$-ideals of $S$ is the unique maximal po-$\Gamma$-ideal of $S$. It is also proved that, (1) if $S$ is a po-$\Gamma$-semigroup and $A$ is a nonempty subset of $S$, then $L(A) = (A \cup S\Gamma A)$ (2) the left po-$\Gamma$-ideal of a po-$\Gamma$-semigroup $S$ generated by a nonempty subset $A$ is the intersection of all left po-$\Gamma$-ideals of $S$ containing $A$ (3) if $S$ is a po-$\Gamma$-semigroup and $A$ is a nonempty subset of $S$, then $R(A) = (A \cup A\Gamma S)$ (4) the right po-$\Gamma$-ideal of a po-$\Gamma$-semigroup $S$ generated by a nonempty subset $A$ is the intersection of all right po-$\Gamma$-ideals of $S$ containing $A$. Also it is proved that, if $S$ is a po-$\Gamma$-semigroup and $A \subseteq S$ then $J(A) = (A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma a \Gamma S)$. Further the terms; principal left po-$\Gamma$-ideal, principal right po-$\Gamma$-ideal, principal po-$\Gamma$-ideal of a po-$\Gamma$-semigroup are introduced. It is proved that, if $S$ is a po-$\Gamma$-semigroup and $a \in S$ then (i) $L(a) = (a \cup S\Gamma a)$ (ii) $R(a) = (a \cup a \Gamma S)$, (iii) $J(a) = a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S$. 

(iii) $x \Gamma y \Gamma a \leq x \Gamma y \Gamma b$ implies $y \Gamma x \Gamma a \leq y \Gamma x \Gamma b$. 

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The terms left simple, right simple, simple po-\(\Gamma\)-semigroups are introduced. It is proved that, (1) a po-\(\Gamma\)-semigroup \(S\) is a left simple po-\(\Gamma\)-semigroup if and only if \((S \Gamma a) = S\) for all \(a \in S\) (2) a po-\(\Gamma\)-semigroup \(S\) is a right simple po-\(\Gamma\)-semigroup if and only if \((a \Gamma S) = S\) for all \(a \in S\) (3) a po-\(\Gamma\)-semigroup \(S\) is simple po-\(\Gamma\)-semigroup if and only if \((S \Gamma a \Gamma S) = S\) for all \(a \in S\). (4) a po-\(\Gamma\)-semigroup \(S\) is left and right simple, then it is regular. Further the terms; regular po-\(\Gamma\)-ideal, semisimple element, intra regular element are introduced. It is proved that, every po-\(\Gamma\)-ideal of a regular po-\(\Gamma\)-semigroup \(S\) is a regular po-\(\Gamma\)-ideal of \(S\). Further it is also proved that, (1) if ‘\(a\)’ is a completely regular element of a po-\(\Gamma\)-semigroup \(S\), then \(a\) is regular and semisimple, (2) if \(a\) is a completely regular element of po-\(\Gamma\)-semigroup, then \(a\) is both left regular and right regular, (3) if ‘\(a\)’ is a left regular element of a po-\(\Gamma\)-semigroup \(S\), then \(a\) is semisimple, (4) if ‘\(a\)’ is a right regular element of a po-\(\Gamma\)-semigroup \(S\), then \(a\) is semisimple, (5) if ‘\(a\)’ is a regular element of a po-\(\Gamma\)-semigroup \(S\), then \(a\) is semisimple and (6) if ‘\(a\)’ is an intra regular element of a po-\(\Gamma\)-semigroup \(S\), then \(a\) is semisimple.

In section 2, the terms; completely prime po-\(\Gamma\)-ideal, po-\(c\)-system, prime po-\(\Gamma\)-ideal, po-\(m\)-system are introduced. It is proved that every po-\(\Gamma\)-subsemigroup of a po-\(\Gamma\)-semigroup is a po-\(c\)-system. It is also proved that a po-\(\Gamma\)-ideal \(P\) of a po-\(\Gamma\)-semigroup \(S\) is completely prime if and only if \(S \cap P\) is either a po-\(c\)-system or empty. It is proved that if \(P\) is a po-\(\Gamma\)-ideal of a po-\(\Gamma\)-semigroup \(S\), then the conditions (1) if \(A, B\) are \(\Gamma\)-ideals of \(S\) and \(A \Gamma B \subseteq P\) then either \(A \subseteq P\) or \(B \subseteq P\), (2) if \(a, b \in S\) such that \(a \Gamma S \Gamma b \subseteq P\), then either \(a \in P\) or \(b \in P\) are equivalent. It is also proved that, every completely prime po-\(\Gamma\)-ideal of a po-\(\Gamma\)-semigroup \(S\) is a prime po-\(\Gamma\)-ideal of \(S\). Further it is proved that, a po-\(\Gamma\)-ideal \(P\) of a po-\(\Gamma\)-semigroup \(S\) is a prime po-\(\Gamma\)-ideal of \(S\) if and only if \(S \cap P\) is an po-\(m\)-system or empty. The term globally idempotent po-\(\Gamma\)-semigroup is introduced. It is proved that, in a globally idempotent po-\(\Gamma\)-semigroup, every maximal po-\(\Gamma\)-ideal is a prime po-\(\Gamma\)-ideal. It is also proved that, a globally idempotent po-\(\Gamma\)-semigroup having a maximal po-\(\Gamma\)-ideal contains semisimple elements.

In section 3, the terms; completely semiprime po-\(\Gamma\)-ideal, po-\(d\)-system, semiprime po-\(\Gamma\)-ideal, po-\(n\)-system are introduced. It is proved that (1) every completely prime po-\(\Gamma\)-ideal of a po-\(\Gamma\)-semigroup is completely semiprime (2) every completely semiprime po-\(\Gamma\)-ideal of a po-\(\Gamma\)-semigroup is semiprime, (3) every prime po-\(\Gamma\)-ideal of a po-\(\Gamma\)-semigroup is a semiprime po-\(\Gamma\)-ideal. It is also proved that the nonempty
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intersection of a family of (1) completely prime po-Γ-ideals of a po-Γ-semigroup is completely semiprime, (2) prime po-Γ-ideals of a po-Γ-semigroup is semiprime. It is proved that a po-Γ-ideal P of a po-Γ-semigroup S is completely semiprime iff S\{P is a po-d-system of S or empty. It is also proved that an ideal Q of a po-Γ-semigroup S is semiprime iff S\{Q is either an po-n-system or empty. Further it is proved that if N is a po-n-system in a po-Γ-semigroup S and a ∈ N, then there exists a po-m-system M of S such that a ∈ M and M ⊆ N.

The contents of chapter 2 are published in “International Organization of Scientific Research Journal of Mathematics(IOSRJM)” under the title ‘Po-Γ-Ideals in Po-Γ-Semigroups’ [67].

In chapter 3, the terms left po-Γ-filter, right po-Γ-filter, proper po-Γ-filter, left po-Γ-filter of a po-Γ-semigroup S generated by a subset A, right po-Γ-filter of a po-Γ-semigroup S generated by A, principal po-Γ-filter are introduced. It is proved that (1) the nonempty intersection of two left po-Γ-filters of a po-Γ-semigroup S is also a left po-Γ-filter, (2) the nonempty intersection of a family of left po-Γ-filters of a po-Γ-semigroup S is also a left po-Γ-filter, (3) the nonempty intersection of two right po-Γ-filters of a po-Γ-semigroup S is also a right po-Γ-filter, (4) the nonempty intersection of a family of right po-Γ-filters of a po-Γ-semigroup S is also a right po-Γ-filter, (5) the nonempty intersection of two po-Γ-filters of a po-Γ-semigroup S is also a po-Γ-filter and (6) the nonempty intersection of a family of po-Γ-filters of a po-Γ-semigroup S is also a po-Γ-filter. Further it is proved that (1) a nonempty subset F of a po-Γ-semigroup S is a left po-Γ-filter if and only if S\{F is a completely prime right po-Γ-ideal of S or empty, (2) a nonempty subset F of a po-Γ-semigroup S is a right po-Γ-filter if and only if S\{F is a completely prime left po-Γ-ideal of S or empty and (3) a nonempty subset F of a po-Γ-semigroup S is a po-Γ-filter if and only if S\{F is a completely prime po-Γ-ideal of S or empty. Further it is also proved that (1) if F is a left po-Γ-filter of a po-Γ-semigroup S, then S\{F is a prime right po-Γ-ideal of S or empty, (2) if F is a right po-Γ-filter of a po-Γ-semigroup S, then S\{F is a prime left po-Γ-ideal of S or empty, (3) if F is a po-Γ-filter of a po-Γ-semigroup S, then S\{F is (i) a prime po-Γ-ideal, (ii) a completely semiprime po-Γ-ideal, (iii) a semiprime po-Γ-ideal of S or empty and finally (4) a nonempty subset F of a commutative po-Γ-semigroup S is a po-Γ-filter if and only if S\{F is a prime po-Γ-ideal of S or empty. It is proved that a po-Γ-semigroup S does not contain proper po-Γ-filters if and only if S does not contain proper completely prime po-Γ-ideals. Further it is proved that every po-Γ-filter F of a po-Γ-semigroup S is (1) a po-c-system of S, (2) a po-d-system
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of $S$, (3) a po-$m$-system of $S$ and (4) a po-$n$-system of $S$. It is proved that, (1) the left po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$ generated by a nonempty subset $A$ of $S$ is the intersection of all left po-$\Gamma$-filters of $S$ containing $A$, (2) the right po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$ generated by a nonempty subset $A$ of $S$ is the intersection of all right po-$\Gamma$-filters of $S$ containing $A$ and (3) the po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$ generated by a nonempty subset $A$ of $S$ is the intersection of all po-$\Gamma$-filters of $S$ containing $\{a\}$. Further it is proved that if $N(b) \subseteq N(a)$, then $N(a) \setminus N(b)$, if it is nonempty, is a completely prime po-$\Gamma$-ideal of $N(a)$. Finally it is proved that (1) if $a, b \in S$ and $b \in N(a)$ then $N(b) \subseteq N(a)$, (2) if $a, b \in S$ and $a \leq b$ then $N(b) \subseteq N(a)$.

The contents of chapter 3 are published in “International Journal of Mathematical Sciences, Technology and Humanities” under the title ‘Po-$\Gamma$-filters in Po-$\Gamma$-Semigroups’ [68].

Chapter 4 is divided into 2 sections. In section 1, the terms; equivalence relation, left $\Gamma$-congruence, right $\Gamma$-congruence are introduced. It is proved that an equivalence relation $\rho$ on a po-$\Gamma$-semigroup $S$ is $\Gamma$-congruence if and only if for all $a, b, c, d \in S$, $a \in \Gamma$, $a \rho b$ and $c \rho d$ implies $(a\Gamma c)\rho(b\Gamma d)$. It is proved that if $\rho_1$ and $\rho_2$ are two left $\Gamma$-congruences (resp. right $\Gamma$-congruences, $\Gamma$-congruences) of a po-$\Gamma$-semigroup $S$, then $(\rho_1 \circ \rho_2)$ is a left $\Gamma$-congruence (resp. right $\Gamma$-congruence, $\Gamma$-congruence) of $S$. It is proved that the intersection of family of $\Gamma$-congruences on a po-$\Gamma$-semigroup $S$ is again a $\Gamma$-congruence on $S$. Further it is proved that, the union of a non-empty family of $\Gamma$-congruences on a po-$\Gamma$-semigroup $S$ is a $\Gamma$-congruence on $S$. The terms; $\Gamma$-congruence generated by $\rho$, $\Gamma$-band, $\Gamma$-semilattice, $\Gamma$-semilattice $\Gamma$-congruence and complete are introduced.

In section 2, the terms, $\mathcal{R}$-class, $\mathcal{R}$-simple, $\mathcal{R}$-subset are introduced. It is proved that the $\mathcal{R}$-class $(a)_{\mathcal{R}}$ is a po-$\Gamma$-subsemigroup of a po-$\Gamma$-semigroup $S$. It is proved that if $a \leq b$ for $a, b$ in a po-$\Gamma$-semigroup $S$, then $(a, a\mathcal{R}b) \in \mathcal{R}$ for every $a \in \Gamma$. Also it is proved that in a po-$\Gamma$-semigroup $S$, $(a)_{\mathcal{R}} = (b)_{\mathcal{R}}$ if and only if $(a)_{\mathcal{R}} = (a\mathcal{R}b)_{\mathcal{R}}$ for all $a \in \Gamma$. It is proved that if $F$ is a po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$, $a \in F \cap (z)_{\mathcal{R}}$ for $z \in S$, then $(z)_{\mathcal{R}} \subseteq F$. It is also proved that in a po-$\Gamma$-semigroup $S$ (1) if $z \in S$ and $A$ is a po-$\Gamma$-ideal of an $\mathcal{R}$-class $(z)_{\mathcal{R}}$, then $A$ has no proper completely prime po-$\Gamma$-ideals, (2) if $A$ is a completely prime po-$\Gamma$-ideal of $S$, then $A = \cup\{(a)_{\mathcal{R}} / a \in A\}$, (3) every po-$\Gamma$-semigroup $S$ is a semilattice of $\mathcal{R}$-simple po-$\Gamma$-semigroups. It is proved that in a po-$\Gamma$-semigroup $S$, if $A$ is a completely prime po-$\Gamma$-ideal of $S$, then $J = \{ (x)_{\mathcal{R}} \in S/\mathcal{R} : x \in A\}$ is a completely
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prime po-$\Gamma$-ideal of $S/\mathcal{R}$ and conversely, if $J$ is a completely prime po-$\Gamma$-ideal of $S/\mathcal{R}$, then $A = \{ x \in S : (x)_{\mathcal{R}} \in J \}$ is a completely prime po-$\Gamma$-ideal of $S$. It is also proved that, if $A$ is a completely semiprime po-$\Gamma$-ideals of a po-$\Gamma$-semigroup $S$, then $J = \{ (x)_{\mathcal{R}} \in S/\mathcal{R} : x \in A \}$ is a po-$\Gamma$-ideal of $S/\mathcal{R}$ and conversely, if $J$ is a po-$\Gamma$-ideal of $S/\mathcal{R}$, then $A = \{ x \in S : (x)_{\mathcal{R}} \in J \}$ is a completely semiprime po-$\Gamma$-ideal of $S$. It is proved that every $\mathcal{R}$-simple po-$\Gamma$-subsemigroup of a po-$\Gamma$-semigroup $S$ is contained in an $\mathcal{R}$-class of $S$. Also it is proved that (1) a po-$\Gamma$-semigroup $S$ is $\mathcal{R}$-simple if and only if $S$ has a single $\mathcal{R}$-class (2) the $\mathcal{R}$-classes of $S$ are precisely all maximal $\mathcal{R}$-simple po-$\Gamma$-subsemigroups of $S$ (3) The set $(a)_{\mathcal{R}}$ can be characterized either as union of all $\mathcal{R}$-simple po-$\Gamma$-subsemigroup of $S$ containing $a$ or as the greatest such po-$\Gamma$-subsemigroup. It is proved that on a po-$\Gamma$-ideal $A$ of a po-$\Gamma$-semigroup $S$ the conditions (i) $A$ is the intersection of completely prime po-$\Gamma$-ideals of $S$ containing $A$, (ii) $A$ is the intersection of all minimal completely prime po-$\Gamma$-ideals of $S$ containing $A$, (iii) $A$ is the union of $\mathcal{R}$-classes and (iv) $A$ is a completely semiprime po-$\Gamma$-ideal of $S$, are equivalent. Further it is proved that, a po-$\Gamma$-semigroup $S$ is $\mathcal{R}$-simple if and only if it contains no proper completely semiprime po-$\Gamma$-ideals. It is proved that if $\mathcal{C}$ is an $\mathcal{R}$-subset of a po-$\Gamma$-semigroup $S$, then (1) for any $a, b \in S$ and $x \in S^1$, $x_{\Gamma}a_{\Gamma}b \subseteq \mathcal{C}$ implies $x_{\Gamma}b_{\Gamma}a \subseteq \mathcal{C}$, (2) $\mathcal{C}$ is a class of semilattice $\Gamma$-congruences. It is proved that (1) a po-$\Gamma$-semigroup $S$ is separative if and only if $S$ is a semilattice of strongly $\Gamma$-cancellative po-$\Gamma$-semigroups (2) the relation $\sigma$ defined on $S$ by $x \sigma y$ if for any $a, b \in S$, $x_{\Gamma}a = x_{\Gamma}b$ if and only if $y_{\Gamma}a = y_{\Gamma}b$ is the greatest band $\Gamma$-congruence on $S$ all whose classes are strongly $\Gamma$-cancellative. Further it is also proved that, a po-$\Gamma$-semigroup $S$ is separative if and only if every $\mathcal{R}$-class of $S$ is strongly $\Gamma$-cancellative. It is proved that if $S$ is a commutative and separative po-$\Gamma$-semigroup, then $S$ is a semilattice of commutative strongly $\Gamma$-cancellative po-$\Gamma$-semigroups.

The contents of chapter 4 are published in “International eJournal of Mathematics and Engineering” under the title ‘$\Gamma$-congruence and $\mathcal{R}$-classes in po-$\Gamma$-semigroups’ [69].