CHAPTER 3

PO-Γ-FILTERS

PETRICH. M. [51] made a study on filters in general semigroups. LEE. S. K and LEE. S. S. [37], introduced the notion of a left (right) filter in a po-semigroups and gave a characterization of the left(right)-filter of S in term of the right(left prime ideals. N. KEHAYOPULU [48], gave the characterization of the filter of S in terms of the prime ideals in ordered semigroups. Y. B. JUN [27] introduced the notion of a Γ-filter in po-Γ-semigroups and gave a characterization of po-Γ-filter of a po-Γ-semigroup in terms of a prime po-Γ-ideals. LEE. S. K. and KWON. Y. I. [36] introduced the notions of a left (right) Γ-filter in po-Γ-semigroup and gave a characterization of a left (right) Γ-filter of a po-Γ-semigroup in terms of right (left) prime Γ-ideals. In this thesis we made a study on po-Γ-filters and extended many characterizations of semigroups, po-semigroups and Γ-semigroups to po-Γ-semigroups. We introduce the notion of a po-Γ-filter in po-Γ-semigroups and gave a characterization of po-Γ-filters of po-Γ-semigroups in terms of completely prime po-Γ-ideals, c-system, m-system, d-system and n-system.

In this chapter, the terms left po-Γ-filter, right po-Γ-filter, po-Γ-filter, proper po-Γ-filter, left po-Γ-filter of a po-Γ-semigroup S generated by a subset A, right po-Γ-filter of a po-Γ-semigroup S generated by A and po-Γ-filter of a po-Γ-semigroup S generated by A, principal po-Γ-filter are introduced. It is proved that (1) The nonempty intersection of two left po-Γ-filters of a po-Γ-semigroup S is also a left po-Γ-filter, (2) The nonempty intersection of a family of left po-Γ-filters of a po-Γ-semigroup S is also a left po-Γ-filter, (3) The nonempty intersection of two right po-Γ-filters of a po-Γ-semigroup S is also a right po-Γ-filter, (4) The nonempty intersection of a family of right po-Γ-filters of a po-Γ-semigroup S is also a right po-Γ-filter, (5) The nonempty intersection of two po-Γ-filters of a po-Γ-semigroup S is also a po-Γ-filter and (6) The nonempty intersection of a family of po-Γ-filters of a po-Γ-semigroup S is also a po-Γ-filter. Further it is proved that (1) a nonempty subset F of a po-Γ-semigroup S is a left po-Γ-filter if and only if S\F is a completely prime right po-Γ-ideal of S or empty, (2) a nonempty subset F of a po-Γ-semigroup S is a right po-Γ-filter if and only if S\F is a completely prime left po-Γ-ideal of S or empty and (3) a nonempty subset F of a po-Γ-semigroup S is a po-Γ-filter if and only if S\F is a completely prime po-Γ-ideal of S or empty. Further it is
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also proved that (1) if \( F \) is a left po-Γ-filter of a po-Γ-semigroup \( S \), then \( SVF \) is a prime right po-Γ-ideal of \( S \) or empty, (2) a nonempty subset \( F \) of a commutative po-Γ-semigroup \( S \) is a left po-Γ-filter if and only if \( SVF \) is a prime right po-Γ-ideal of \( S \) or empty, (3) if \( F \) is a right po-Γ-filter of a po-Γ-semigroup \( S \), then \( SVF \) is a prime left po-Γ-ideal of \( S \) or empty, (4) a nonempty subset \( F \) of a commutative po-Γ-semigroup \( S \) is a right po-Γ-filter if and only if \( SVF \) is a prime left po-Γ-ideal of \( S \) or empty, (5) if \( F \) is a po-Γ-filter of a po-Γ-semigroup \( S \), then \( SVF \) is (i) a prime po-Γ-ideal, (ii) a completely semiprime po-Γ-ideal, (iii) a semiprime po-Γ-ideal of \( S \) or empty and finally (6) a nonempty subset \( F \) of a commutative po-Γ-semigroup \( S \) is a po-Γ-filter if and only if \( SVF \) is a prime po-Γ-ideal of \( S \) or empty. It is proved that a po-Γ-semigroup \( S \) does not contain proper po-Γ-filters if and only if \( S \) does not contain proper completely prime po-Γ-ideals. Further it is proved that every po-Γ-filter \( F \) of a po-Γ-semigroup \( S \) is (1) a po-c-system of \( S \), (2) a po-d-system of \( S \), (3) a po-m-system of \( S \) and (4) a po-n-system of \( S \). It is proved that (1) the left po-Γ-filter of a po-Γ-semigroup \( S \) generated by a nonempty subset \( A \) of \( S \) is the intersection of all left po-Γ-filters of \( S \) containing \( A \), (2) the right po-Γ-filter of a po-Γ-semigroup \( S \) generated by a nonempty subset \( A \) of \( S \) is the intersection of all right po-Γ-filters of \( S \) containing \( A \) and (3) the po-Γ-filter of a po-Γ-semigroup \( S \) generated by a nonempty subset \( A \) of \( S \) is the intersection of all po-Γ-filters of \( S \) containing \( A \). It is proved that if \( S \) is a po-Γ-semigroup and \( a \in S \), then \( N(a) \) is the least filter of \( S \) containing \( \{a\} \). Further it is proved that if \( N(b) \subseteq N(a) \), then \( N(a) \setminus N(b) \), if it is nonempty, is a completely prime po-Γ-ideal of \( N(a) \). Finally it is proved that (1) if \( a, b \in S \) and \( b \in N(a) \) then \( N(b) \subseteq N(a) \), (2) if \( a, b \in S \) and \( a \leq b \) then \( N(b) \subseteq N(a) \).

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3.1. PO-Γ-FILTERS IN PO-Γ-SEMIGROUPS :

**DEFINITION 3.1.1 :** A \( \Gamma \)-subsemigroup \( F \) of a po-Γ-semigroup \( S \) is said to be a *left po-Γ-filter* of \( S \) if

1. \( a, b \in S, \ a \in \Gamma, \ ab \in F \) implies \( a \in F \).
2. \( a \in F, \ c \in S \) and \( a \leq c \) implies \( c \in F \).

**NOTE 3.1.2 :** A \( \Gamma \)-subsemigroup \( F \) of a po-Γ-semigroup \( S \) is a *left po-Γ-filter* of \( S \) iff

1. \( a, b \in S, \ a \Gamma b \subseteq F \) implies \( a \in F \).
2. \( [F] \subseteq F \).
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**Theorem 3.1.3:** The nonempty intersection of two left Po-$\Gamma$-filters of a Po-$\Gamma$-semigroup $S$ is also a left Po-$\Gamma$-filter of $S$.

**Proof:** Let $A$, $B$ be two left Po-$\Gamma$-filters of $S$.

Let $a, b \in S$, $\alpha \in \Gamma$, $aab \in A \cap B$.

$aab \in A \cap B \Rightarrow aab \in A$ and $aab \in B$.

$a, b \in S$, $\alpha \in \Gamma$, $aab \in A$, $A$ is a left Po-$\Gamma$-filter of $S \Rightarrow a \in A$.

$a, b \in S$, $\alpha \in \Gamma$, $aab \in B$, $B$ is a left Po-$\Gamma$-filter of $S \Rightarrow a \in B$.

$a \in A$, $a \in B \Rightarrow a \in A \cap B$.

$a, b \in S$, $\alpha \in \Gamma$, $aab \in A \cap B \Rightarrow a \in A \cap B$.

Let $a \in A \cap B$, and $a \leq c$ for $c \in S$. Now $a \in A \cap B \Rightarrow a \in A$, $a \in B$.

$c \in S$, $a \in A$, $a \leq c$, $A$ is a left Po-$\Gamma$-filter \( \Rightarrow c \in A \).

$c \in S$, $a \in B$, $a \leq c$, $B$ is a left Po-$\Gamma$-filter \( \Rightarrow c \in B \).

$c \in A$, $c \in B \Rightarrow c \in A \cap B$. Thus $a \in A \cap B$, $c \in S$ and $a \leq c \Rightarrow c \in A \cap B$.

Therefore $A \cap B$ is a left Po-$\Gamma$-filter of $S$.

**Theorem 3.1.4:** The nonempty intersection of a family of left Po-$\Gamma$-filters of a Po-$\Gamma$-semigroup $S$ is also a left Po-$\Gamma$-filter.

**Proof:** Let $\{F_{\alpha}\}_{\alpha \in \Delta}$ be a family of left Po-$\Gamma$-filters of $S$ and let $F = \bigcap_{\alpha \in \Delta} F_{\alpha}$.

Let $a, b \in S$, $\gamma \in \Gamma$, $aab \in F$. Now $aab \in F \Rightarrow aab \in \bigcap_{\alpha \in \Delta} F_{\alpha} \Rightarrow aab \in F_{\alpha}$ for each $\alpha \in \Delta$.

$aab \in F_{\alpha}$, $\gamma \in \Gamma$, $F_{\alpha}$ is a left Po-$\Gamma$-filter of $S$.

$\Rightarrow a \in F_{\alpha}$ for each $\alpha \in \Delta \Rightarrow a \in \bigcap_{\alpha \in \Delta} F_{\alpha} \Rightarrow a \in F$.

Let $a \in F$ and $a \leq c$ for $c \in S$. Now $a \in F \Rightarrow a \in \bigcap_{\alpha \in \Delta} F_{\alpha} \Rightarrow a \in F_{\alpha}$ for each $\alpha \in \Delta$.

$a \in F_{\alpha}$, $a \leq c$ for $c \in S \Rightarrow c \in F_{\alpha}$ for all $\alpha \in \Delta$.

$\Rightarrow c \in \bigcap_{\alpha \in \Delta} F_{\alpha} \Rightarrow c \in F$. Therefore $F$ is a left Po-$\Gamma$-filter of $S$.

We now prove a necessary and sufficient condition for a nonempty subset to be a left Po-$\Gamma$-filter in a Po-$\Gamma$-semigroup.

**Theorem 3.1.5:** A nonempty subset $F$ of a Po-$\Gamma$-semigroup $S$ is a left Po-$\Gamma$-filter if and only if $S \setminus F$ is a completely prime right Po-$\Gamma$-ideal of $S$ or empty.

**Proof:** Assume that $S \setminus F \neq \emptyset$. Let $x \in S \setminus F$ and $y \in S$, $\alpha \in \Gamma$.
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Suppose if possible $x\alpha y \notin S\backslash F$. Then $x\alpha y \in F$. Since $F$ is a left Po-$\Gamma$-filter, $x \in F$. It is a contradiction. Thus $x\alpha y \in S\backslash F$, and so $(S\backslash F)\Gamma S \subseteq S\backslash F$.

Let $x \in S\backslash F$, $y \in S$ and $y \leq x$. If $y \notin S\backslash F$, then $y \in F$. Since $F$ is a left Po-$\Gamma$-filter, $x \in F$. It is a contradiction. Thus $y \in S\backslash F$. Therefore $S\backslash F$ is a right Po-$\Gamma$-ideal.

Next we shall prove that $S\backslash F$ is completely prime.

Let $x, y \in S$, $\alpha \in \Gamma$, and $x\alpha y \in S\backslash F$. Suppose if possible $x \notin S\backslash F$ and $y \notin S\backslash F$.

Then $x \in F$ and $y \in F$. Since $F$ is a $\Gamma$-subsemigroup of $S$, $x\alpha y \in F$. It is a contradiction. Thus $x \in S\backslash F$ or $y \in S\backslash F$.

Hence $S\backslash F$ is completely prime.

Therefore $S\backslash F$ is a completely prime right Po-$\Gamma$-ideal of $S$.

Conversely suppose that $S\backslash F$ is a completely prime right Po-$\Gamma$-ideal of $S$ or empty.

If $S\backslash F = \emptyset$, then $F = S$. Thus $F$ is a left Po-$\Gamma$-filter of $S$.

Assume that $S\backslash F$ is a completely prime right Po-$\Gamma$-ideal of $S$.

Let $x, y \in F$, and $\alpha \in \Gamma$. Suppose if possible $x\alpha y \notin F$.

Then $x\alpha y \in S\backslash F$. Since $S\backslash F$ is completely prime, $x \in S\backslash F$ or $y \in S\backslash F$. It is a contradiction. Thus $x\alpha y \in F$ and hence $F$ is a $\Gamma$-subsemigroup of $S$.

Let $x, y \in S$, $\alpha \in \Gamma$, $x\alpha y \in F$. If $x \notin F$, then $x \in S\backslash F$.

Since $S\backslash F$ is a completely prime right Po-$\Gamma$-ideal of $S$, $x\alpha y \in (S\backslash F)\Gamma S \subseteq S\backslash F$.

It is a contradiction. Thus $x \in F$.

Let $x \in F$, $y \in S$ and $x \leq y$. If $y \notin F$, then $y \in S\backslash F$.

Since $S\backslash F$ is a right Po-$\Gamma$-ideal of $S$, $x \in S\backslash F$. It is a contradiction. Thus $y \in F$. Therefore $F$ is a left Po-$\Gamma$-filter of $S$.

**COROLLARY 3.1.6**: Let $S$ be a Po-$\Gamma$-semigroup and $F$ is a left Po-$\Gamma$-filter of $S$. Then $S\backslash F$ is a prime right Po-$\Gamma$-ideal of $S$ or empty.

**Proof**: Since $F$ is a left Po-$\Gamma$-filter, by theorem 3.1.5, $S\backslash F$ is a completely prime right Po-$\Gamma$-ideal of $S$ or empty. By theorem 2.2.15, $S\backslash F$ is a prime right Po-$\Gamma$-ideal of $S$ or empty.

**DEFINITION 3.1.7**: A $\Gamma$-subsemigroup $F$ of a Po-$\Gamma$-semigroup $S$ is said to be *right Po-$\Gamma$-filter* of $S$ if

1. $a, b \in S$, $\alpha \in \Gamma$, $a\alpha b \in F$ implies $b \in F$.
2. $a \in F$, $c \in S$ and $a \leq c$ implies $c \in F$.

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NOTE 3.1.8: A $\Gamma$-subsemigroup $F$ of a po-$\Gamma$-semigroup $S$ is a right po-$\Gamma$-filter of $S$ if

1. $a, b \in S, a\Gamma b \subseteq F$ implies $b \in F$.
2. $|F| \subseteq F$.

THEOREM 3.1.9: The nonempty intersection of two right po-$\Gamma$-filters of a po-$\Gamma$-semigroup $S$ is also a right po-$\Gamma$-filter.

**Proof:** Let $A, B$ be two right po-$\Gamma$-filters of $S$.

Let $a, b \in S, \alpha \in \Gamma, a\alpha b \in A \cap B$.

$a\alpha b \in A \cap B \Rightarrow a\alpha b \in A$ and $a\alpha b \in B$.

$a, b \in S, \alpha \in \Gamma, a\alpha b \in A, A$ is a right po-$\Gamma$-filter of $S \Rightarrow b \in A$.

$a, b \in S, \alpha \in \Gamma, a\alpha b \in B, B$ is a right po-$\Gamma$-filter of $S \Rightarrow b \in B$.

$b \in A, b \in B \Rightarrow b \in A \cap B$.

Let $b \in A \cap B, b \leq c$ for $c \in S$. Now $b \in A \cap B \Rightarrow b \in A, b \in B$.

$b \in A, b \leq c$ for $c \in S, A$ is a right po-$\Gamma$-filter $\Rightarrow c \in A$.

$b \in B, a \leq c$ for $c \in S, B$ is a right po-$\Gamma$-filter $\Rightarrow c \in B$.

$c \in A, c \in B \Rightarrow c \in A \cap B$. Thus $b \in A \cap B, b \leq c$ for $c \in S \Rightarrow c \in A \cap B$.

Therefore $A \cap B$ is a right po-$\Gamma$-filter of $S$.

THEOREM 3.1.10: The nonempty intersection of a family of right po-$\Gamma$-filters of a po-$\Gamma$-semigroup $S$ is also a right po-$\Gamma$-filter.

**Proof:** Let $\{F_\alpha\}_{\alpha \in \Delta}$ be a family of right po-$\Gamma$-filters of $S$ and let $F = \bigcap_{\alpha \in \Delta} F_\alpha$.

Let $a, b \in S, \gamma \in \Gamma, a\alpha b \in F$. Now $a\alpha b \in F \Rightarrow a\alpha b \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow a\alpha b \in F_\alpha$ for each $\alpha \in \Delta$.

$a\alpha b \in F_\alpha, \gamma \in \Gamma, F_\alpha$ is a right po-$\Gamma$-filter of $S \Rightarrow b \in F_\alpha$.

Let $b \in F$ and $b \leq c$ for $c \in S$. Now $b \in F \Rightarrow b \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow b \in F_\alpha$ for each $\alpha \in \Delta$.

$b \in F_\alpha, b \leq c$ for $c \in S \Rightarrow c \in F_\alpha$ for all $\alpha \in \Delta$.

$\Rightarrow b \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow b \in F$ and $b \in F_\alpha \Rightarrow c \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow c \in F$.

Therefore $F$ is a right po-$\Gamma$-filter of $S$.

We now prove a necessary and sufficient condition for a nonempty subset to be a right po-$\Gamma$-filter in a po-$\Gamma$-semigroup.
THEOREM 3.1.11: A nonempty subset $F$ of a po-$\Gamma$-semigroup $S$ is a right po-$\Gamma$-filter if and only if $S \setminus F$ is a completely prime left po-$\Gamma$-ideal of $S$ or empty.

Proof: Assume that $S \setminus F \neq \emptyset$. Let $x \in S \setminus F$ and $y \in S$, $\alpha \in \Gamma$.
Suppose that $y\alpha x \notin S \setminus F$, then $y\alpha x \in F$.
Since $F$ is a right po-$\Gamma$-filter, $x \in F$. It is a contradiction.
Thus $y\alpha x \in S \setminus F$, and so $S\Gamma(S \setminus F) \subseteq S \setminus F$.
Let $x \in S \setminus F$ and $y \leq x$ for $y \in S$.
If $y \notin S \setminus F$, then $y \in S$. Since $F$ is a right po-$\Gamma$-filter,
$x \in F$. It is a contradiction.
Thus $y \in S \setminus F$. Therefore $S \setminus F$ is a left po-$\Gamma$-ideal.
Next we shall prove that $S \setminus F$ is completely prime.
Let $x\alpha y \in S \setminus F$ for $x, y \in S$ and $\alpha \in \Gamma$. Suppose that $x \notin S \setminus F$ and $y \notin S \setminus F$.
Then $x \in F$ and $y \in F$. Since $F$ is a $\Gamma$-subsemigroup of $S$, $x\alpha y \in F$.
It is a contradiction. Thus $x \in S \setminus F$ or $y \in S \setminus F$.
Hence $S \setminus F$ is completely prime and hence $S \setminus F$ is a completely prime left po-$\Gamma$-ideal of $S$.
Conversely suppose that $S \setminus F$ is a completely prime left po-$\Gamma$-ideal of $S$ or empty.
If $S \setminus F = \emptyset$, then $F = S$. Thus $F$ is a right po-$\Gamma$-filter of $S$.
Assume that $S \setminus F$ is a completely prime left po-$\Gamma$-ideal of $S$.
Let $x, y \in F, \alpha \in \Gamma$. Suppose if possible $x\alpha y \notin F$. Then $x\alpha y \in S \setminus F$.
Since $S \setminus F$ is completely prime, $x \in S \setminus F$ or $y \in S \setminus F$. It is a contradiction.
Thus $x\alpha y \in F$ and hence $F$ is a $\Gamma$-subsemigroup of $S$.
Let $x, y \in S, \alpha \in \Gamma, x\alpha y \in F$. If $y \notin F$, then $y \in S \setminus F$.
Since $S \setminus F$ is a completely prime left po-$\Gamma$-ideal of $S$, $x\alpha y \in S\Gamma(S \setminus F) \subseteq S \setminus F$.
It is a contradiction. Thus $y \notin F$.
Let $x \in F$ and $x \leq y$ for $y \in S$. If $y \notin F$, then $y \in S \setminus F$.
Since $S \setminus F$ is a left po-$\Gamma$-ideal of $S, y \in S \setminus F$. It is a contradiction. Thus $y \notin F$.
Therefore $F$ is a right po-$\Gamma$-filter of $S$.

COROLLARY 3.1.12: Let $S$ be a po-$\Gamma$-semigroup and $F$ be a right po-$\Gamma$-filter. Then $S \setminus F$ is a prime left po-$\Gamma$-ideal of $S$ or empty.

Proof: Since $F$ is a right po-$\Gamma$-filter. By theorem 3.1.11, $S \setminus F$ is a completely prime left po-$\Gamma$-ideal of $S$ or empty. By theorem 2.2.16, $S \setminus F$ is a prime left po-$\Gamma$-ideal of $S$ or empty.
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**DEFINITION 3.1.13**: A Γ-subsemigroup $F$ of a po-Γ-semigroup $S$ is said to be a **po-Γ-filter** of $S$ if

1. $a, b \in S$, $\alpha \in \Gamma$, $a\alpha b \in F$ implies $a, b \in F$.
2. $a \in F$, $c \in S$ and $a \leq c$ implies $c \in F$.

**NOTE 3.1.14**: A Γ-subsemigroup $F$ of a po-Γ-semigroup $S$ is a po-Γ-filter of $S$ iff

1. $a, b \in S$, $a\Gamma b \subseteq F$ implies $a, b \in F$.
2. $(F) \subseteq F$

**NOTE 3.1.15**: A Γ-subsemigroup $F$ of a po-Γ-semigroup $S$ is a po-Γ-filter of $S$ iff $F$ is a left po-Γ-filter and a right po-Γ-filter of $S$.

**EXAMPLE 3.1.16**: Let $S = \{a, b, c\}$ and $\Gamma = \{\gamma\}$ with the multiplication defined by

$$xy = \begin{cases} b & \text{if } x = y = b \\ c & \text{if } x = y = c \\ a & \text{otherwise} \end{cases}$$

Define a relation $\leq$ on $S$ as $\leq: 1_S \cup \{(a, b), (a, c)\}$. Then $S$ is a po-Γ-semigroup and \{a, b, c\}, \{b\}, \{c\} are all the po-Γ-filters of $S$.

**DEFINITION 3.1.17**: A po-Γ-filter $F$ of a po-Γ-semigroup $S$ is said to be a **proper po-Γ-filter** if $F \neq S$.

**THEOREM 3.1.18**: The nonempty intersection of two po-Γ-filters of a po-Γ-semigroup $S$ is also a po-Γ-filter of $S$.

**Proof**: Let $A$, $B$ be two po-Γ-filters of $S$.

Let $a, b \in S$, $\alpha \in \Gamma$, $a\alpha b \in A \cap B$.

$a\alpha b \in A \cap B \Rightarrow a\alpha b \in A$ and $a\alpha b \in B$.

$a, b \in S$, $\alpha \in \Gamma$, $a\alpha b \in A$, $A$ is a po-Γ-filter of $S \Rightarrow a, b \in A$.

$a, b \in S$, $\alpha \in \Gamma$, $a\alpha b \in B$, $B$ is a po-Γ-filter of $S \Rightarrow a, b \in B$.

$a, b \in A$, $a, b \in B \Rightarrow a, b \in A \cap B$.

Let $a \in A \cap B$, $c \in S$ and $a \leq c$. Now $a \in A \cap B \Rightarrow a \in A$, $a \in B$.

$a \in A$, $c \in S$, $a \leq c$, $A$ is po-Γ-filter $\Rightarrow c \in A$.

$a \in B$, $c \in S$, $a \leq c$, $B$ is po-Γ-filter $\Rightarrow c \in B$.

$c \in A$, $c \in B \Rightarrow c \in A \cap B$. 

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\[ a \in A \cap B, \ b \in S, \ c \leq d \Rightarrow c \in A \cap B. \]
Therefore $A \cap B$ is a po-$\Gamma$-filter of $S$.

**THEOREM 3.1.19 :** The nonempty intersection of a family of po-$\Gamma$-filters of a po-$\Gamma$-semigroup $S$ is also a po-$\Gamma$-filter of $S$.

**Proof :** Let \( \{ F_a \}_{a \in \Delta} \) be a family of po-$\Gamma$-filters of $S$ and let \( F = \bigcap_{a \in \Delta} F_a \).

Let $a, b \in S, \gamma \in \Gamma$, $aab \in F$. Now $aab \in F \Rightarrow aab \in \bigcap_{a \in \Delta} F_a \Rightarrow aab \in F_a$ for each $a \in \Delta$.

$aab \in F_a, \gamma \in \Gamma$, $F_a$ is a po-$\Gamma$-filter of $S \Rightarrow a, b \in F_a$.

$a \in F, c \in S, a \leq c$. $a \leq c$ for $c \in S \Rightarrow c \in F_a$ for all $a \in \Delta \Rightarrow a, b \in \bigcap_{a \in \Delta} F_a \Rightarrow a, b \in F$

and $a \in F_a, a \leq c$ for $c \in S \Rightarrow c \in \bigcap_{a \in \Delta} F_a \Rightarrow c \in F$. Therefore $F$ is a po-$\Gamma$-filter of $S$.

**NOTE 3.1.20:** In general, the union of two po-$\Gamma$-filters is not a po-$\Gamma$-filter.

**EXAMPLE 3.1.21 :** As in the example 3.1.18, $S$ is a po-$\Gamma$-semigroup and \( \{ b \}, \{ c \} \) are po-$\Gamma$-filters, but \( \{ b \} \cup \{ c \} \) is not a po-$\Gamma$-filter of $S$ because $byc = a$ is not in \( \{ b \} \cup \{ c \} \).

We now prove a necessary and sufficient condition for a nonempty subset to be a po-$\Gamma$-filter in a po-$\Gamma$-semigroup.

**THEOREM 3.1.22 :** A nonempty subset $F$ of a po-$\Gamma$-semigroup $S$ is a po-$\Gamma$-filter if and only if $S \setminus F$ is a completely prime po-$\Gamma$-ideal of $S$ or empty.

**Proof :** Assume that $S \setminus F \neq \emptyset$. Let $x, y \in S \setminus F, \alpha \in \Gamma$.

Suppose that $x\alpha y \notin S \setminus F$, then $x\alpha y \in F$. Since $F$ is a po-$\Gamma$-filter and hence $x, y \in F$.

It is a contradiction. Thus $x\alpha y \in S \setminus F$, and so $(S \setminus F)\Gamma S \Gamma (S \setminus F) \subseteq S \setminus F$.

Let $x \in S \setminus F$ and $y \leq x$ for $y \in S$. If $y \notin S \setminus F$, then $y \in F$.

Since $F$ is a po-$\Gamma$-filter, $x \in F$.

It is a contradiction. Thus $y \in S \setminus F$. Therefore $S \setminus F$ is a po-$\Gamma$-ideal.

Next we shall prove that $S \setminus F$ is completely prime.

Let $x\alpha y \in S \setminus F$ for $x, y \in S$ and $\alpha \in \Gamma$. Suppose that $x \notin S \setminus F$ and $y \notin S \setminus F$.

Then $x \in F$ and $y \in F$. Since $F$ is a $\Gamma$-subsemigroup of $S$, $x\alpha y \in F$.

It is a contradiction. Thus $x \in S \setminus F$ or $y \in S \setminus F$.

Hence $S \setminus F$ is completely prime and hence $S \setminus F$ is a completely prime right po-$\Gamma$-ideal of $S$.  

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Conversely suppose that $S \setminus F$ is a completely prime Po-$\Gamma$-ideal of $S$ or empty.

If $S \setminus F = \emptyset$, then $F = S$. Thus $F$ is a Po-$\Gamma$-filter of $S$.

Assume that $S \setminus F$ is a completely prime Po-$\Gamma$-ideal of $S$.

Suppose that for $\alpha \in \Gamma, x, y \in F$, $x \alpha y \notin F$. Then $x \alpha y \in S \setminus F$ for $x, y \in F, \alpha \in \Gamma$.

Since $S \setminus F$ is completely prime, $x \in S \setminus F$ or $y \in S \setminus F$. It is a contradiction.

Thus $x \alpha y \in F$ and hence $F$ is a $\Gamma$-subsemigroup of $S$.

Let $x, y \in S, \alpha \in \Gamma, x \alpha y \in F$. If $x, y \notin F$, then $x, y \in S \setminus F$.

Since $S \setminus F$ is a completely prime Po-$\Gamma$-ideal of $S$, $x \alpha y \in (S \setminus F) \Gamma S \setminus F \subseteq S \setminus F$.

It is a contradiction. Thus $x, y \in F$.

Let $x \in F$ and $x \leq y$ for $y \in S$. If $y \notin F$, then $y \in S \setminus F$.

Since $S \setminus F$ is a Po-$\Gamma$-ideal of $S$, $x \in S \setminus F$. It is a contradiction. Thus $y \in F$.

Therefore $F$ is a Po-$\Gamma$-filter of $S$.

**Corollary 3.1.23**: Let $S$ be a Po-$\Gamma$-semigroup. If $F$ is a Po-$\Gamma$-filter, then $S \setminus F$ is a prime Po-$\Gamma$-ideal of $S$ or empty.

**Proof**: Since $F$ is a Po-$\Gamma$-filter of $S$. By theorem 3.1.24, $S \setminus F$ is a completely prime Po-$\Gamma$-ideal of $S$ or empty. By theorem 2.2.16, $S \setminus F$ is a prime Po-$\Gamma$-ideal of $S$ or empty.

**Corollary 3.1.24**: A nonempty subset $F$ of a commutative Po-$\Gamma$-semigroup $S$ is a Po-$\Gamma$-filter if and only if $S \setminus F$ is a prime Po-$\Gamma$-ideal of $S$ or empty.

**Proof**: Suppose that $S \setminus F$ is a Po-$\Gamma$-filter of commutative Po-$\Gamma$-semigroup $S$. By corollary 3.1.25, $S \setminus F$ is prime Po-$\Gamma$-ideal of $S$ or empty.

Conversely suppose that $S \setminus F$ is a prime Po-$\Gamma$-ideal of $S$ or empty. If $S \setminus F = \emptyset$, then $F = S$. Thus $F$ is a Po-$\Gamma$-filter of $S$. Assume that $S \setminus F$ is a prime Po-$\Gamma$-ideal of $S$. By theorem 2.2.17, $S \setminus F$ is a completely prime Po-$\Gamma$-ideal of $S$ or empty. By theorem 3.1.24, $F$ is a Po-$\Gamma$-filter of $S$.

**Theorem 3.1.25**: Every Po-$\Gamma$-filter $F$ of a Po-$\Gamma$-semigroup $S$ is a po-$c$-system of $S$.

**Proof**: Suppose that $F$ is a Po-$\Gamma$-filter. By theorem 3.1.24, $S \setminus F$ is completely prime Po-$\Gamma$-ideal of $S$. By theorem 2.2.9, $F$ is a po-$c$-system of $S$.

**Theorem 3.1.26**: A Po-$\Gamma$-semigroup $S$ does not contain proper Po-$\Gamma$-filters if and only if $S$ does not contain proper completely prime Po-$\Gamma$-ideals.
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**Proof**: Suppose that a po-$\Gamma$-semigroup $S$ does not contain proper po-$\Gamma$-filters. Let $A$ be a completely prime po-$\Gamma$-ideal of $S$ and $A \subseteq S$. Then $\emptyset \neq S \setminus A \subseteq S$ and $S \setminus (S \setminus A)$ (= $A$) is a completely prime po-$\Gamma$-ideal of $S$. Since $S \setminus A$ is the complement of $A$ to $S$, by theorem 3.1.24, $S \setminus A$ is a po-$\Gamma$-filter of $S$. Then $S \setminus A = S$ and hence $A = \emptyset$. It is a contradiction. Therefore $S$ does not contain proper completely prime po-$\Gamma$-ideals.

Conversely suppose that $S$ does not contain proper completely prime po-$\Gamma$-ideals. Let $F$ be a po-$\Gamma$-filter of $S$ and $F \subseteq S$. Since $S \setminus F \neq \emptyset$, by theorem 3.1.24, $S \setminus F$ is a completely prime po-$\Gamma$-ideal of $S$. Then $S \setminus F = S$ and hence $F = \emptyset$. It is a contradiction. Therefore $S$ does not contain proper po-$\Gamma$-filters.

**Theorem 3.1.27**: Every po-$\Gamma$-filter $F$ of a po-$\Gamma$-semigroup $S$ is a po-$m$-system of $S$.

**Proof**: Suppose that $F$ is a po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$. By corollary 3.1.25, $S \setminus F$ is a prime po-$\Gamma$-ideal of $S$. By theorem 2.2.20, $S \setminus (S \setminus F) = F$ is a po-$m$-system of $S$ or empty.

**Theorem 3.1.28**: Let $S$ be a po-$\Gamma$-semigroup. If $F$ is a po-$\Gamma$-filter, then $S \setminus F$ is a completely semiprime po-$\Gamma$-ideal of $S$.

**Proof**: Since $F$ is a po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$, by theorem 3.1.24, $S \setminus F$ is a completely prime po-$\Gamma$-ideal of $S$. By theorem 2.3.2, $S \setminus F$ is a completely semiprime po-$\Gamma$-ideal of $S$.

**Theorem 3.1.29**: Every po-$\Gamma$-filter $F$ of a po-$\Gamma$-semigroup $S$ is a po-$d$-system of $S$.

**Proof**: Since $F$ is a po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$, by theorem 3.1.30, $S \setminus F$ is a completely semiprime po-$\Gamma$-ideal of $S$. By theorem 2.3.6, $S \setminus (S \setminus F) = F$ is a po-$d$-system of $S$ or empty.

**Theorem 3.1.30**: Let $S$ be a po-$\Gamma$-semigroup. If $F$ is a po-$\Gamma$-filter, then $S \setminus F$ is a semiprime po-$\Gamma$-ideal of $S$.

**Proof**: Since $F$ is a po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$. By theorem 3.1.24, $S \setminus F$ is a completely prime po-$\Gamma$-ideal of $S$. By theorem 2.3.2, $S \setminus F$ is a completely semiprime po-$\Gamma$-ideal of $S$. By theorem 2.3.9, $S \setminus F$ is a semiprime po-$\Gamma$-ideal of $S$.  

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THEOREM 3.1.31: Every po-$\Gamma$-filter $F$ of a po-$\Gamma$-semigroup $S$ is a po-$n$-system of $S$.

Proof: Since $F$ is a po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$. By theorem 3.1.30, $S\setminus F$ is a semiprime po-$\Gamma$-ideal of $S$. By theorem 2.2.17, $S\setminus(S\setminus F) = F$ is a po-$n$-system of $S$.

DEFINITION 3.1.32: Let $S$ be a po-$\Gamma$-semigroup and $A$ be a nonempty subset of $S$. The smallest left po-$\Gamma$-filter of $S$ containing $A$ is called left po-$\Gamma$-filter of $S$ generated by $A$ and it is denoted by $F_l(A)$.

THEOREM 3.1.33: The left po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$ generated by a nonempty subset $A$ of $S$ is the intersection of all left po-$\Gamma$-filters of $S$ containing $A$.

Proof: Let $\Delta$ be the set of all left po-$\Gamma$-filters of $S$ containing $A$.

Since $S$ itself is a left po-$\Gamma$-filter of $S$ containing $A$, $S \in \Delta$. So $\Delta \neq \emptyset$.

Let $F^* = \bigcap_{F \in \Delta} F$. Since $A \subseteq F$ for all $F \in \Delta$, $A \subseteq F^*$. So $F^* \neq \emptyset$.

By theorem 3.1.4, $F^*$ is a left po-$\Gamma$-filter of $S$.

Let $K$ be a left po-$\Gamma$-filter of $S$ containing $A$.

Clearly $A \subseteq K$ and $K$ is a left po-$\Gamma$-filter of $S$.

Therefore $K \in \Delta \Rightarrow F^* \subseteq K$. Therefore $F^*$ is the smallest left po-$\Gamma$-filter of $S$ containing $A$ and hence $F^*$ is the left po-$\Gamma$-filter of $S$ generated by $A$.

DEFINITION 3.1.34: Let $S$ be a po-$\Gamma$-semigroup and $A$ be a nonempty subset of $S$. The smallest right po-$\Gamma$-filter of $S$ containing $A$ is called right po-$\Gamma$-ideal of $S$ generated by $A$ and it is denoted by $F_r(A)$.

THEOREM 3.1.35: The right po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$ generated by a nonempty subset $A$ of $S$ is the intersection of all right po-$\Gamma$-filters of $S$ containing $A$.

Proof: Let $\Delta$ be the set of all right po-$\Gamma$-filters of $S$ containing $A$.

Since $S$ itself is a right po-$\Gamma$-filter of $S$ containing $A$, $S \in \Delta$. So $\Delta \neq \emptyset$.

Let $F^* = \bigcap_{F \in \Delta} F$. Since $A \subseteq F$ for all $F \in \Delta$, $A \subseteq F^*$. So $F^* \neq \emptyset$.

By theorem 3.1.11, $T^*$ is a right po-$\Gamma$-filter of $S$.

Let $K$ be a right po-$\Gamma$-filter of $S$ containing $A$.

Clearly $A \subseteq K$ and $K$ is a right po-$\Gamma$-filter of $S$.

Therefore $K \in \Delta \Rightarrow F^* \subseteq K$. Therefore $F^*$ is the smallest right po-$\Gamma$-filter of $S$ containing $A$ and hence $F^*$ is the right po-$\Gamma$-filter of $S$ generated by $A$. 

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DEFINITION 3.1.36: Let $S$ be a po-$\Gamma$-semigroup and $A$ be a nonempty subset of $S$. The smallest po-$\Gamma$-filter of $S$ containing $A$ is called \textit{po-$\Gamma$-filter of $S$ generated by $A$} and it is denoted by $N(A)$.

THEOREM 3.1.37: The po-$\Gamma$-filter of a po-$\Gamma$-semigroup $S$ generated by a nonempty subset $A$ is the intersection of all po-$\Gamma$-filters of $S$ containing $A$.

\textbf{Proof}: Let $\Delta$ be the set of all po-$\Gamma$-filters of $S$ containing $A$. Since $S$ itself is a po-$\Gamma$-filter of $S$ containing $A$, $S \in \Delta$. So $\Delta \neq \emptyset$.

Let $F^* = \bigcap_{F \in \Delta} F$. Since $A \subseteq F$ for all $F \in \Delta$, $A \subseteq F^*$. So $F^* \neq \emptyset$.

By theorem 3.1.21, $F^*$ is a po-$\Gamma$-filter of $S$.

Let $K$ be a po-$\Gamma$-filter of $S$ containing $A$.

Clearly $A \subseteq K$ and $K$ is a po-$\Gamma$-filter of $S$.

Therefore $K \in \Delta \Rightarrow F^* \subseteq K$.

Therefore $F^*$ is the po-$\Gamma$-filter of $S$ generated by $A$.

DEFINITION 3.1.38: A po-$\Gamma$-filter $F$ of a po-$\Gamma$-semigroup $S$ is said to be a \textit{principal po-$\Gamma$-filter} provided $F$ is a po-$\Gamma$-filter generated by \{a\} for some $a \in S$. It is denoted by $N(a)$.

EXAMPLE 3.1.39: As in the example 3.1.18, $S$ is a po-$\Gamma$-semigroup and $N(a) = \{a, b, c\}$, $N(b) = \{b\}$ and $N(c) = \{c\}$ are all the principal po-$\Gamma$-filters of the po-$\Gamma$-semigroup $S$.

COROLLARY 3.1.40: Let $S$ is a po-$\Gamma$-semigroup and $a \in S$. Then $N(a)$ is the least filter of $S$ containing \{a\}.

NOTE 3.1.41: For every $a \in S$, the intersection of all po-$\Gamma$-filters containing \{a\} is again a po-$\Gamma$-filter and thus the least po-$\Gamma$-filter containing \{a\}.

THEOREM 3.1.42: If $N(b) \subseteq N(a)$, then $N(a) \n N(b)$, if it is nonempty, is a completely prime po-$\Gamma$-ideal of $N(a)$.

\textbf{Proof}: By theorem 3.1.24, $N(a) \n N(b)$ is a completely prime po-$\Gamma$-ideal of $N(a)$.

LEMMA 3.1.43: Let $a, b \in S$ and $b \in N(a)$, then $N(b) \subseteq N(a)$.
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**Proof**: From the definition of the principal po-Γ-filter, it is clear.

**COROLLARY 3.1.44**: Let \( a, b \in S \) and \( a \leq b \) then \( N(b) \subseteq N(a) \).

**Proof**: Since \( a \leq b \) then it is clear that \( b \in N(a) \). By lemma 3.1.45, we have \( N(b) \subseteq N(a) \).