Chapter 5

DOUBLE DISPERSION EFFECTS ON
CONVECTIVE HEAT AND MASS TRANSFER
FLOW OVER A VERTICAL WAVY SURFACE
IN A POROUS MEDIUM WITH VARIABLE
PROPERTIES
# Chapter 5

**Chapter-5. DOUBLE DISPERSION EFFECTS ON CONVECTIVE HEAT AND MASS TRANSFER FLOW OVER A VERTICAL WAVY SURFACE IN A POROUS MEDIUM WITH VARIABLE PROPERTIES**

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CHAPTER 5

5.1 Introduction

In this chapter, the problem of convective heat and mass flow over a vertical wavy surface in a fluid saturated porous medium with variable properties and double dispersion. To the best of our knowledge, no one has been attempted to investigate the influence of double dispersion effects on convective heat and mass transfer flow over a vertical wavy surface immersed in a porous medium.

The hydrodynamic mixing is called dispersion, which is the secondary effect of a porous medium on the fluid flow takes place in the result of mixing and recirculation of local fluid particles through tortuous paths formed by the porous medium solid particles. There has been renewed interest in studying double diffusive convection due to the effect of thermal and solutal dispersions; these are additional energy and concentration mass transport process. In certain thermal and solutal dispersion applications such as those involving oil reservoir and geothermal engineering applications such as ceramic processing, sensible heat storage beds and petroleum recovery etc.,

In view of the above application, the authors envisage to investigate free and mixed convection along a vertical wavy surface embedded in a fluid saturated porous medium with variable properties and double dispersion effects.
5.3 Free convection

The governing boundary equations for flow mass, momentum, energy and concentration are transformed into non-dimensional nonlinear ordinary differential equations by using appropriate transformation and then solved by using numerical method. The results are reported graphically for various physical parameters for flow velocity, temperature and concentration distributions as well as Nusselt number and Sherwood number. The present results are compared with previously existing results and obtained a very good agreement.

5.2.1 Basic Equations

\[
\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0
\]  

(5.1)

\[
\frac{\partial}{\partial \bar{y}} \left( \frac{\mu}{K} \bar{u} \right) = \frac{\partial}{\partial \bar{x}} \left( \frac{\mu}{K} \bar{v} \right) + \rho g \left( \beta_i \frac{\partial T}{\partial \bar{y}} + \beta_e \frac{\partial C}{\partial \bar{y}} \right)
\]  

(5.2)

\[
\bar{u} \frac{\partial T}{\partial \bar{x}} + \bar{v} \frac{\partial T}{\partial \bar{y}} = \frac{\partial}{\partial \bar{x}} \left( \alpha_x \frac{\partial T}{\partial \bar{x}} \right) + \frac{\partial}{\partial \bar{y}} \left( \alpha_y \frac{\partial T}{\partial \bar{y}} \right)
\]  

(5.3)

\[
\bar{u} \frac{\partial C}{\partial \bar{x}} + \bar{v} \frac{\partial C}{\partial \bar{y}} = \frac{\partial}{\partial \bar{x}} \left( D_x \frac{\partial C}{\partial \bar{x}} \right) + \frac{\partial}{\partial \bar{y}} \left( D_y \frac{\partial C}{\partial \bar{y}} \right)
\]  

(5.4)

The corresponding boundary conditions are

at \( \bar{y} = \bar{\sigma}(\bar{x}) = \bar{\alpha} \sin \left( \frac{\pi \bar{x}}{l} \right) \), \( \bar{u} = 0; \bar{v} = 0; T = T_w; C = C_w \}

(5.5)

as \( \bar{y} \to \infty, T \to T_w; C \to C_w; \bar{u} \to 0 \)

where \( \bar{u} \) and \( \bar{v} \) are velocity components in \( \bar{x} \) and \( \bar{y} \) directions respectively. \( \alpha_x, D_x \) and \( \alpha_y, D_y \) are the effective thermal and solutal diffusivities respectively, have the contribution of both molecular
diffusion and hydrodynamic dispersion, these can be described as (see Telles and Trevisan [86])

\[
\begin{align*}
\alpha_x &= \alpha + \gamma d \bar{v}, \quad \alpha_y = \alpha + \gamma d \bar{u} \\
D_x &= D + \zeta d \bar{v}, \quad D_y = D + \zeta d \bar{u}
\end{align*}
\]  

(5.6)

where \( \alpha \) is the thermal conductivity, \( D \) is the molecular diffusivities, \( \gamma \) is the coefficient of thermal dispersion and \( \zeta \) is the solutal dispersion.

From the eqs. (2.10) – (2.13), (5.6) and non-dimensional variables (2.14), the eqns. (4.1) – (4.5) reduces to

\[
\frac{1}{\theta - \theta_1} \left( \frac{\partial \theta \partial \psi^*}{\partial y \partial y} + \frac{\partial \theta \partial \psi^*}{\partial x \partial x} \right) + \frac{\partial \psi^*}{\partial y^2} + \frac{\partial \psi^*}{\partial x^2} = Ra \left( \frac{1}{\theta} \right) \left( \frac{\partial \theta}{\partial y} + N \frac{\partial \phi}{\partial y} \right)
\]

(5.7)

\[
\frac{\partial \psi^*}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi^*}{\partial x} \frac{\partial \theta}{\partial y} = \beta \left( \frac{\partial \theta}{\partial x} \right)^2 + \left( \frac{\partial \theta}{\partial y} \right)^2 + \left( 1 + \beta \theta \right) \left( \frac{\partial \psi^*}{\partial x} \right)^2 + \left( \frac{\partial \psi^*}{\partial y} \right)^2
\]

(5.8)

\[
Le \left( \frac{\partial \psi^*}{\partial y} \frac{\partial \phi}{\partial x} - \frac{\partial \psi^*}{\partial x} \frac{\partial \phi}{\partial y} \right) = \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)
\]

(5.9)

\[
+ \frac{\zeta d}{l} \left[ \frac{\partial^2 \psi^*}{\partial y^2} \frac{\partial \phi}{\partial y} - \frac{\partial^2 \psi^*}{\partial x^2} \frac{\partial \phi}{\partial x} + \frac{\partial \psi^*}{\partial y} \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial \psi^*}{\partial x} \frac{\partial^2 \phi}{\partial y^2} \right]
\]

where \( Ra_d = \frac{g \kappa \beta (T_w - T_m) d}{\alpha_n v} \) is the pore diameter dependent Rayleigh number which describes the relative intensity of the buoyancy force, such that \( d \) is the pore diameter.
The associated boundary conditions are given by
\[
\begin{align*}
&\text{at } y = a \sin(x), \quad \theta = 1, \quad \phi = 1, \quad \psi^* = 0 \\
&\text{as } y \to \infty, \quad \theta \to 0, \quad \phi \to 0, \quad \psi^*_y \to 0
\end{align*}
\] (5.10)

### 5.2.2 Solution Methodology

We incorporate the effect of wavy surface and the usual boundary layer scaling into the governing equations (5.7) – (5.10) for free convection, using the transformations
\[
x = \xi, \quad \hat{\eta} = \frac{y - a \sin(x)}{\xi^{1/2} Ra^{-1/2}}, \quad \psi^* = Ra^{1/2} \psi.
\] (5.11)

along with the substitutions
\[
\eta = \frac{\hat{\eta}}{1 + a^2 \cos^2 \xi}, \quad \psi = \xi^{1/2} f(\eta), \quad \theta = \theta(\eta) \quad \text{and} \quad \phi = \phi(\eta).
\] (5.12)

Substituting eq. (5.11) and (5.12) into eqs. (5.7) – (5.10) and letting Ra→∞, we obtain the following boundary layer equations:
\[
f^* + \frac{1}{\theta - \theta_r} \theta' f' = \left(1 - \frac{\theta}{\theta_r}\right) \left(\theta' + N \varphi'\right)
\] (5.13)

\[
\beta(\theta')^2 + (1 + \beta \theta) \theta'' + \frac{1}{2} f \theta' + Ds \frac{(1 + a^3 \cos^3(\xi))}{(1 + a^2 \cos^2(\xi))^2} (f^* \theta' + \theta^* f') = 0
\] (5.14)

\[
\frac{1}{Le} \varphi^* + \frac{1}{2} f \varphi' + Dc \frac{(1 + a^3 \cos^3(\xi))}{(1 + a^2 \cos^2(\xi))^2} (f^* \varphi' + \varphi^* f') = 0
\] (5.15)

where prime denotes differentiation with respect to $\hat{\eta}$, $Ds = \gamma Ra_d$ is the thermal dispersion parameter $Dc = \zeta Ra_d$ is the solutal dispersion parameter.
From eqns. (5.14) and (5.15), we say that, in natural convection due to vertical wavy surface in a fluid saturated porous medium, the field variable, flow, heat and mass transfer characteristics are not similar because the \( \xi \)-coordinate can’t be eliminated. However, we found the local non-similarity solutions for some convective boundary layer flows dealing with Darcy porous medium, the technique is more complex to extend in this case. Hence, for ease of analysis, it is decided to proceed with evaluating local similarity solutions for the equations, (5.13) – (5.15). For that we take \( \xi = \frac{x}{l} \) and then vary \( \xi \)-location to study the influence of various parameters.

The associated boundary conditions are

\[
\begin{align*}
    f & = 0, \quad \theta = 1, \quad \text{and} \quad \varphi = 1 \quad \text{at} \quad \eta = 0 \\
    f' & \to 0, \quad \theta \to 0 \quad \text{and} \quad \varphi \to 0 \quad \text{as} \quad \eta \to \infty
\end{align*}
\]

The engineering design quantities of physical interest include Nusselt number and Sherwood numbers which are defined as

\[
\begin{align*}
    Nu_\xi & = -\left(1 + D_s \frac{f'(0)}{(1 + a^2 \cos^2(\xi))}\right) \frac{\theta'(0)Ra_\xi^{1/2}}{(1 + a^2 \cos^2(\xi))^{1/2}} \\
    \text{and} \quad Sh_\xi & = -\left(1 + D_s \frac{f'(0)}{(1 + a^2 \cos^2(\xi))}\right) \frac{\varphi'(0)Ra_\xi^{1/2}}{(1 + a^2 \cos^2(\xi))^{1/2}}
\end{align*}
\]

5.2.2.1 Numerical Method

The problem of free convection along a vertical wavy surface embedded in fluid saturated Darcy porous media subject to the variable viscosity, variable thermal conductivity and double dispersion effects has been
investigated. A simple coordinate transformation is employed to reduce
the governing non-linear boundary equations into non-linear ordinary
differential equations and then employed Runge-Kutta method and
Newton-Raphson method with shooting technique. We restrict the
physical parameter values $1 < \theta_r \leq 5$, $1 \leq \beta \leq 5$, $0 < D_s \leq 1$, $0 < D_c \leq 1$, and
$0.5 \leq \xi \leq 2$ with the fixed values $N=1$, $L_e=1$ and $a=0.5$.

5.2.2.2 Grid Independence Test

In order to check the effects of step size $(\Delta \eta)$ we found the
Nusselt number and Sherwood number with four different step sizes
as $\Delta \eta = 0.1, \Delta \eta = 0.01, \Delta \eta = 0.001$ and $\Delta \eta = 0.0001$. We observe from
Table-5.1 that the results are independent with the step size $(\Delta \eta)$.
Hence a step size $\Delta \eta = 0.01$ is selected to be satisfactory for a
convergence criterion of $10^{-6}$ in all cases.

Table-5.1: Grid-independence for $\beta=0.5$, $a=0.5$, $D_s=1$, $D_c=1$, $N=2$, $Sc=0.6$, $L_e=1$ and $\theta_r=1.5$.

<table>
<thead>
<tr>
<th>$\Delta \eta$ (Step size)</th>
<th>$Nu_{\xi}Ra_{\xi}^{-1/2}$</th>
<th>$Sh_{\xi}Ra_{\xi}^{-1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.4354625676</td>
<td>2.4841994509</td>
</tr>
<tr>
<td>0.01</td>
<td>2.4354624326</td>
<td>2.4841994212</td>
</tr>
<tr>
<td>0.001</td>
<td>2.4354623455</td>
<td>2.4841993895</td>
</tr>
<tr>
<td>0.0001</td>
<td>2.4354622842</td>
<td>2.4841993104</td>
</tr>
</tbody>
</table>

5.2.2.2 Accuracy of the results

As shown in the Table-5.2 we also checked absolute and relative error
of the present results. It is observed that accuracy of the Nusselt
number and Sherwood number results is obtained when reducing the
effects. In this thesis we restrict the absolute error $10^{-7}$ and relative
error $10^{-7}$ for obtaining the required accuracy ($10^{-6}$) of the flow characteristics.

**Table-5.2:** Absolute and Relative error of the results for $\beta=0.5$, $\alpha=0.5$, $Ds=1$, $Dc=1$, $N=2$, $Sc=0.6$, $Le=1$ and $\theta_r=1.5$.

<table>
<thead>
<tr>
<th>$Nu_{\xi}Ra_{\xi}^{-1/2}$</th>
<th>$Sh_{\xi}Ra_{\xi}^{-1/2}$</th>
<th>Absolute Error</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2949973975</td>
<td>0.3214786387</td>
<td>1e-3</td>
<td>1e-3</td>
</tr>
<tr>
<td>0.2949973264</td>
<td>0.3214785842</td>
<td>1e-5</td>
<td>1e-5</td>
</tr>
<tr>
<td>0.2949972848</td>
<td>0.3214785105</td>
<td>1e-7</td>
<td>1e-7</td>
</tr>
<tr>
<td>0.2949972017</td>
<td>0.3214784729</td>
<td>1e-10</td>
<td>1e-10</td>
</tr>
</tbody>
</table>

**5.2.2.3 Comparison with the earlier published work**

In order to validate the present method the numerical results obtained using the Runge Kutta method with shooting method are compared with Bejan and Khair [103] results. Table-5.3 shows the comparison results in the absence of variable properties and double dispersion effects (i.e. $Ds=0$ and $Dc=0$) over vertical flat surface with Bejan and Khair [103] and the results are found to be in good agreement.

**5.2.3 Results and Discussions**

We have found the numerical solutions for non dimensional velocity, temperature and concentration distributions as well as rate of heat and mass transfer coefficients as shown graphically in Figs. 5.1 – 5.25.

The variation of variable viscosity parameter ($\theta_r$) on non-dimensional velocity, temperature and concentration distributions is presented in figs. 5.1 – 5.3. It is noticed from fig. 5.1 that an increase in variable viscosity parameter $\theta_r$ resulted in depreciation in velocity.
distribution near the plate up to reach certain value and then increase
the velocity profile until approaches a constant value (zero) at outer
boundary layer regime. From figs. 5.2 and 5.3 we conclude that
increasing variable viscosity parameter $\theta_r$, clearly substantially
enhances the temperature and concentration distributions.

**Table-5.3:** Comparison of the rate of heat and mass transfer Bejan
and Khair [103] for $\alpha=0$, $\beta=0$, $\tau=0$ and $\theta_r \to \infty$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$N_{u_T}Ra_{\xi}^{-1/2}$</th>
<th>$S_{h_T}Ra_{\xi}^{-1/2}$</th>
<th>$N_{u_C}Ra_{\xi}^{-1/2}$</th>
<th>$S_{h_C}Ra_{\xi}^{-1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Le N</td>
<td>Bejan and Khair [103]</td>
<td>Bejan and Khair [103]</td>
<td>Present</td>
<td>Present</td>
</tr>
<tr>
<td>2 3</td>
<td>0.810</td>
<td>1.286</td>
<td>0.8102</td>
<td>1.2863</td>
</tr>
<tr>
<td>4 3</td>
<td>0.728</td>
<td>1.852</td>
<td>0.7285</td>
<td>1.8522</td>
</tr>
<tr>
<td>6 1</td>
<td>0.541</td>
<td>1.685</td>
<td>0.5415</td>
<td>1.6850</td>
</tr>
<tr>
<td>6 2</td>
<td>0.618</td>
<td>2.009</td>
<td>0.6188</td>
<td>2.0102</td>
</tr>
</tbody>
</table>

The variation of variable thermal conductivity ($\beta$) on velocity,
temperature and concentration distributions is illustrated in figs. 5.4 –
5.6. The velocity profile results for different values of $\beta$ are given by fig.
(5), these results are having similar behavior as shown in fig. 5.1. From
fig. 5.5 it is evident that temperature profiles are more pronounced with
increasing values of $\beta$ while a strong decrease in concentration
distribution is noticed(Fig. 5.6). The effect of thermal dispersion
parameter (Ds) on the non-dimensional velocity, temperature and
concentration is depicted in figs. 5.7 – 5.9. From fig. 5.7 we conclude
that the results of velocity profile reduce near the surface for larger values of thermal dispersion parameter and opposite results are observed as the radial distance moves far away from the surface with increase in thermal dispersion parameter. The presence of thermal dispersion in the energy equation gives thermal conduction more dominance. It is observed from fig. 5.8 that increasing thermal dispersion parameter tends to enhance the temperature distribution. i.e. thermal dispersion enhances the transport of heat along radial direction to the plate. It is noticed from fig. 5.9 that the solutal boundary layer thickness is reduced with increase in thermal dispersion parameter.

The figs. 5.10 – 5.12 represent the variation of non-dimensional velocity, temperature and concentration distributions across the boundary layer for different values of solutal dispersion parameter (Dc). From fig. 5.10 we noticed that an increase in Dc enhances significantly the momentum boundary thickness. It is observed from fig. 5.11 that temperature profile reduced with increase in Dc. It can be evident from this figure that as Dc increases thermal boundary layer thickness increases. It is observed from fig. 5.12 that increasing the solutal dispersion parameter (Dc), accelerates the concentration of the fluid. Hence the concentration boundary layer thickness increases with an increase in solutal dispersion parameter (Dc).

Figs. 5.13 – 5.15 illustrate the velocity, temperature and concentration distributions for different values of \( \xi \)-location. It can be found from fig. 5.13 that velocity profile is increased with increase in \( \xi \)-
location. Hence the hydrodynamic boundary layer thickness increases with increase in $\xi$-location. We noticed from figs. 5.14 and 5.15 that the similar behaviour of temperature and concentration profile, in comparison with velocity distribution as shown in fig. 5.13. It is important to note that it quickly reaches similarity solutions not far away from the leading edge.

The variation of rate of heat and mass transfer (Nusselt number and Sherwood number) with streamwise coordinate at the wall are shown in figs. 5.16 & 5.17 for different values of variable viscosity parameter ($\theta_r$). It is noticed from these figures that both Nusselt number and Sherwood number decreases with increase in $\theta_r$. Hence, it is clear that increase in $\theta_r$ results in a depreciation in the amplitude of the Nusselt number and Sherwood. Figs 5.18 & 5.19 represent the variation of Nusselt number and Sherwood number for different values of variable thermal conductivity parameter ($\beta$). Figs. 5.18 & 5.19 demonstrates that Nusselt number and Sherwood number reduces considerably for larger values of $\beta$. Figs. 5.20 & 5.21 reveals that enhancement of thermal dispersion parameter results in an enhancement in the amplitude of the Nusselt number and Sherwood number. The variation of Nusselt number and Sherwood number for different values of solutal dispersion parameter (Dc) is given in figs. 5.22 & 5.23. Figs. 5.22 & 5.23 exhibits the similar behavior of Nusselt number and Sherwood with streamwise coordinate, in comparison with what observed in figs. 5.20 & 5.21. Figs. 5.24 & 5.25 illustrate the variation of Nusselt number and Sherwood number for different values
of the amplitude of the wavy surface. For ‘a = 0’ the vertical wavy
surface reduces to a vertical flat surface. It is noticed from figs. 5.24
and 5.25 that increasing the amplitude of the wavy surface,
substantially enhances the amplitude of the Nusselt number and
Sherwood number with streamwise coordinate.
5.2.4 Graphs

Fig. 5.1. Velocity profile \((u)\) for different values of variable viscosity \((\theta_r)\) with transverse coordinate \((\eta)\) for \(N=1, \, Le=1, \, Ds=0.3, \, Dc=0.3, \, a=0.5, \, \beta=0.5, \) and \(\xi=1\).

Fig. 5.2. Temperature profile \((\theta)\) for different values of variable viscosity \((\theta_r)\) with transverse coordinate \((\eta)\) for \(N=1, \, Le=1, \, Ds=0.3, \, Dc=0.3, \, a=0.5, \, \beta=0.5, \) and \(\xi=1\)
Fig. 5.3. Concentration profile ($\phi$) for different values of variable viscosity ($\theta_r$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, $\beta=0.5$, and $\xi=1$.

Fig. 5.4. Velocity profile ($u$) for different values of variable thermal conductivity ($\beta$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, $\theta_r=1.5$, and $\xi=1$. 
Fig. 5.5. Temperature profile ($\theta$) for different values of variable thermal conductivity ($\beta$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, $\theta_t=1.5$, and $\xi=1$.

Fig. 5.6. Concentration profile ($\phi$) for different values of variable thermal conductivity ($\beta$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, $\theta_t=1.5$, and $\xi=1$. 
Fig. 5.7. Velocity profile \((u)\) for different values of thermal dispersion parameter \((D_s)\) with transverse coordinate \((\eta)\) for \(N=1\), \(Le=1\), \(D_c=0.3\), \(a=0.5\), \(\theta_r=1.5\), \(\beta=0.5\), and \(\xi=1\).

Fig. 5.8. Temperature profile \((\theta)\) for different values of thermal dispersion parameter \((D_s)\) with transverse coordinate \((\eta)\) for \(N=1\), \(Le=1\), \(D_c=0.3\), \(a=0.5\), \(\theta_r=1.5\), \(\beta=0.5\), and \(\xi=1\).
Fig. 5.9. Concentration profile $\phi$ for different values of thermal dispersion parameter ($Ds$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Dc=0.3$, $a=0.5$, $\theta_r=1.5$, $\beta=0.5$, and $\xi=1$.

Fig. 5.10. Velocity profile ($u$) for different values of solutal dispersion parameter ($Dc$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $a=0.5$, $\theta_r=1.5$, $\beta=0.5$, and $\xi=1$. 
Fig.5.11. Temperature profile ($\theta$) for different values of solutal dispersion parameter ($Dc$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $a=0.5$, $\theta_r=1.5$, $\beta=0.5$, and $\xi=1$.

Fig.5.12. Concentration profile ($\phi$) for different values of solutal dispersion parameter ($Dc$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $a=0.5$, $\theta_r=1.5$, $\beta=0.5$, and $\xi=1$. 
Fig. 5.13. Velocity profile \((u)\) for different values of \(\xi\)-location with transverse coordinate \((\eta)\) for \(N=1\), \(Le=1\), \(Ds=0.3\), \(Dc=0.3\), \(a=0.5\), \(\theta_r=1.5\), \(\beta=0.5\).

Fig. 5.14. Temperature profile \((\theta)\) for different values of \(\xi\)-location with transverse coordinate \((\eta)\) for \(N=1\), \(Le=1\), \(Ds=0.3\), \(Dc=0.3\), \(a=0.5\), \(\theta_r=1.5\), \(\beta=0.5\).
Fig. 5.15. Concentration profile ($\phi$) for different values of $\xi$-location with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, $\theta_r=1.5$, $\beta=0.5$.

Fig. 5.16. Axial distributions of the Nusselt number (Nu) for different values of variable viscosity ($\theta_r$) with stream wise coordinate ($\xi$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, $\xi=1$, $\beta=0.5$. 
Fig. 5.17. Axial distributions of the Sherwood number (Sh) for different values of variable viscosity ($\theta_\tau$) with stream wise coordinate ($\xi$) for $N=1, \ Le=1, \ D_s=0.3, \ D_c=0.3, \ a=0.5, \ \xi=1, \ \beta=0.5$.

Fig. 5.18. Axial distributions of the Nusselt number (Nu) for different values of variable thermal conductivity ($\beta$) with stream wise coordinate ($\xi$) for $N=1, \ Le=1, \ D_s=0.3, \ D_c=0.3, \ a=0.5, \ \xi=1, \ \theta_\tau=1.5$. 
Fig. 5.19. Axial distributions of the Sherwood number (Sh) for different values of variable thermal conductivity ($\beta$) with stream wise coordinate ($\xi$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, $\xi=1$, $\theta_r=1.5$.

Fig. 5.20. Axial distributions of the Nusselt number (Nu) for different values of thermal dispersion parameter ($Ds$) with stream wise coordinate ($\xi$) for $N=1$, $Le=1$, $\beta=0.5$, $Dc=0.3$, $a=0.5$, $\xi=1$, $\theta_r=1.5$. 
Fig. 5.21. Axial distributions of the Sherwood number (Sh) for different values of thermal dispersion parameter (Ds) with stream wise coordinate ($\xi$) for $N=1$, $Le=1$, $\beta=0.5$, $Dc=0.3$, $a=0.5$, $\xi=1$, $\theta_r=1.5$.

Fig. 5.22. Axial distributions of the Nusselt number (Nu) for different values of solutal dispersion parameter (Dc) with stream wise coordinate ($\xi$) for $N=1$, $Le=1$, $\beta=0.5$, $Ds=0.3$, $a=0.5$, $\xi=1$, $\theta_r=1.5$. 
Fig. 5.23. Axial distributions of the Sherwood number (Sh) for different values of solutal dispersion parameter (Dc) with stream wise coordinate (ξ) for N=1, Le=1, β=0.5, Ds=0.3, a=0.5, ξ=1, θ_r=1.5.

Fig. 5.24. Axial distributions of the Nusselt number (Nu) for different values of amplitude of the wavy surface (a) with stream wise coordinate (ξ) for N=1, Le=1, β=0.5, Ds=0.3, Dc=0.3, ξ=1, θ_r=1.5.
Fig. 5.25. Axial distributions of the Sherwood number (Sh) for different values of the amplitude of the wavy surface (a) with stream wise coordinate ($\xi$) for $N=1$, $Le=1$, $\beta=0.5$, $Ds, =0.3$, $Dc=0.3$, $\xi=1$, $\theta_r=1.5$. 
5.4 Mixed convection

In this section, we analyze the combined effects of double dispersion and variable properties on double diffusive mixed convective heat and mass transfer flow over a vertical wavy surface in a fluid saturated porous medium. The set of corresponding governing equations are solved using shooting technique. A representative set of numerical results is shown graphically for different values of variable viscosity, variable thermal conductivity and thermal and solutal dispersion effects.

5.3.1 Basic Equations

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0
\]  

\[
\frac{\partial}{\partial y} \left( \frac{\mu}{K} \bar{u} \right) = \frac{\partial}{\partial x} \left( \frac{\mu}{K} \bar{v} \right) \pm \rho g \left( \beta_v \frac{\partial T}{\partial y} + \beta_c \frac{\partial C}{\partial y} \right)
\]  

\[
\bar{u} \frac{\partial T}{\partial x} + \bar{v} \frac{\partial T}{\partial y} = \frac{\partial}{\partial x} \left( \alpha_v \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \alpha_c \frac{\partial T}{\partial y} \right)
\]  

\[
\bar{u} \frac{\partial C}{\partial x} + \bar{v} \frac{\partial C}{\partial y} = \frac{\partial}{\partial x} \left( D_v \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( D_c \frac{\partial T}{\partial y} \right)
\]

The corresponding boundary conditions are

\[
\bar{u} = 0, \bar{v} = 0, T = T_w, C = C_w, \quad \text{at} \quad \bar{y} = \sigma(x) = \bar{a} \sin \left( \frac{\pi \bar{x}}{l} \right) \]

\[
\bar{u} \to U_\infty, T \to T_\infty, C \to C_\infty \quad \text{as} \quad \bar{y} \to \infty
\]

where \(\bar{u}\), and \(\bar{v}\) are velocity components in \(\bar{x}\) and \(\bar{y}\) directions respectively. \(U_\infty\) is the uniform free stream velocity.
From the eqs. (2.10) – (2.13), (5.6) and non-dimensional variables (2.14), the eqns. (5.18) – (5.22) reduces to

\[
\frac{1}{\theta - \theta_1} \left( \frac{\partial \theta}{\partial y} \frac{\partial \psi^*}{\partial y} + \frac{\partial \theta}{\partial x} \frac{\partial \psi^*}{\partial x} \right) + \frac{\partial^2 \psi^*}{\partial y^2} + \frac{\partial^2 \psi^*}{\partial x^2} = \pm \frac{Ra}{Pe} \left( 1 - \frac{\theta}{\theta_1} \right) \left( \frac{\partial \theta}{\partial y} + N \frac{\partial \phi}{\partial y} \right) \tag{5.23}
\]

\[
\frac{\partial \psi^*}{\partial y} - \frac{\partial \psi^*}{\partial x} = \beta \left( \frac{\partial \theta}{\partial x} \right)^2 + \left( \frac{\partial \theta}{\partial y} \right)^2 + (1 + \beta \theta) \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) + \frac{\gamma d}{l} \left[ \frac{\partial^2 \psi^*}{\partial y^2} - \frac{\partial^2 \psi^*}{\partial x^2} \frac{\partial \theta}{\partial y} + \frac{\partial \psi^*}{\partial y} \frac{\partial^2 \theta}{\partial y^2} - \frac{\partial \psi^*}{\partial x} \frac{\partial^2 \theta}{\partial x^2} \right] \tag{5.24}
\]

\[
Le \left( \frac{\partial \psi^*}{\partial y} \frac{\partial \phi}{\partial x} - \frac{\partial \psi^*}{\partial x} \frac{\partial \phi}{\partial y} \right) = \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) + \frac{\zeta d}{l} \left[ \frac{\partial^2 \psi^*}{\partial y^2} - \frac{\partial^2 \psi^*}{\partial x^2} \frac{\partial \phi}{\partial y} + \frac{\partial \psi^*}{\partial y} \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial \psi^*}{\partial x} \frac{\partial^2 \phi}{\partial x^2} \right] \tag{5.25}
\]

where \( Ra_d = \frac{gK \beta \left( T_w - T_\infty \right) d}{\alpha_0 \nu} \) is the pore diameter dependent Rayleigh number which describes the relative intensity of the buoyancy force, such that \( d \) is the pore diameter.

The associated boundary conditions are given by

\[
\begin{align*}
\psi^* &= 0, \quad \theta = 1, \quad \phi = 1, \quad \text{on} \quad y = a \sin(x), \\
\psi^*_y &\to \frac{\alpha_0}{l} U_w, \quad \theta \to 0, \quad \phi \to 0 \quad \text{as} \quad y \to \infty.
\end{align*} \tag{5.26}
\]

### 5.3.2 Solution Methodology

We incorporate the effect of wavy surface and the usual boundary layer scaling into the governing equations (5.23) – (5.26) for mixed convection, using the transformations
\[ x = \xi, \quad \psi = Pe^{1/2}\psi. \quad (5.27) \]

along with the substitutions
\[ \eta = \frac{\hat{\eta}}{1 + a^2 \cos^2 \xi}, \quad \psi = \xi^{1/2} f(\eta), \quad \theta = \theta(\eta) \quad \text{and} \quad \phi = \phi(\eta). \quad (5.28) \]

Substituting Eq. (5.27) and (5.28) into Eqs. (5.23) – (5.26) and letting \( Pe \to \infty \), we obtain the following boundary layer equations:

\[ f'' + \frac{1}{\theta - \theta_r} \theta' f' = \Delta \left( 1 - \frac{\theta}{\theta_r} \right) \left( \theta' + N \phi' \right) \quad (5.29) \]

\[ \beta(\theta')^2 + (1 + \beta \theta) \theta'' + \frac{1}{2} f \theta' + D_s \frac{(1 + a^3 \cos^3(\xi))}{(1 + a^2 \cos^2(\xi))^2} (f'' \theta' + \theta'' f') = 0 \quad (5.30) \]

\[ \frac{1}{Le} \phi'' + \frac{1}{2} f \phi' + D_c \frac{(1 + a^3 \cos^3(\xi))}{(1 + a^2 \cos^2(\xi))^2} (f'' \phi' + \phi'' f') = 0 \quad (5.31) \]

where prime denotes differentiation with respect to \( \eta \), \( D_s = \gamma Ra_d \) is the thermal dispersion parameter \( D_c = \zeta Ra_d \) is the solutal dispersion parameter.

The associated boundary conditions are

\[ f = 0, \quad \theta = 1, \quad \text{and} \quad \phi = 1 \quad \text{at} \quad \eta = 0 \]

\[ f' \to 1, \quad \theta \to 0 \quad \text{and} \quad \phi \to 0 \quad \text{as} \quad \eta \to \infty \quad (5.32) \]

The engineering design quantities of physical interest include Nusselt number and Sherwood numbers which are defined as

\[ Nu_\xi = -\left(1 + D_s \frac{f'(0)}{(1 + a^2 \cos^2(\xi))} \right) \frac{\theta'(0) Pe^{1/2}_\xi}{(1 + a^2 \cos^2(\xi))^{1/2}} \quad \text{and} \]

\[ Sh_\xi = -\left(1 + D_s \frac{f'(0)}{(1 + a^2 \cos^2(\xi))} \right) \frac{\phi'(0) Pe^{1/2}_\xi}{(1 + a^2 \cos^2(\xi))^{1/2}} \quad (5.33) \]
5.3.3 Results and Discussions

The governing boundary layer equations (5.29) – (5.32) are solved by using shooting technique. Fig. 5.26 – 5.28 represents the non-dimensional fluid velocity, temperature and concentration profiles for different values of variable viscosity (θ), variable thermal conductivity (β) with transverse coordinate η. We observed from fig. 5.26 that an increase in θ resulted a decrease in velocity profile while an increase in temperature and concentration profiles is observed with in the boundary layer where viscous effects are dominate. The present investigation shows that the flow variable are more pronounced for smaller values of θ. If θ is very large the momentum boundary layer eq. (5.29) reduces to \( f' = \Delta(\theta + N\phi) \). It is also noticed from fig. 5.27 and 5.28 that an increase in β enhances momentum and thermal boundary layer thickness, while depreciation in solutal boundary layer thickness is observed.

Fig. 5.29 – 5.31 shows the non-dimensional flow field variable velocity, temperature and concentration profiles for different values of thermal dispersion (Ds) along with varying values of solutal dispersion parameter. From fig. 5.29 it is clear that the flow variables are influenced by thermal dispersion parameter. For large Ds, in a very small boundary layer region adjacent to the wall, the velocity and temperature profiles are greatly enhanced and as a result momentum and thermal boundary layer thickness is greatly enhanced caused by
increasing thermal dispersion parameter (Ds) as shown in fig. 5.30. But the reverse trend is noted in concentration profile for higher values of Ds, i.e., increase in Ds is seen to reduce solutal boundary layer thickness. We also observed from fig. 5.31 that increase in solutal dispersion results in an enhancement in flow velocity and concentration profile while depreciates in temperature profile.

Figs. 5.32 – 5.34 present the effects of mixed convection parameter (Δ) and ξ-location on velocity, temperature and concentration profiles across boundary layer respectively. From present numerical analysis, we conclude an interesting aspect that when ξ=0, the flow governing boundary layer equations are free from ξ and it quickly approach the similarity solutions. It is noticed that momentum, thermal and solutal boundary layer thickness increases in the downstream direction. We also conclude that as the mixed convective parameter increase, the fluid velocity increases where as temperature and concentration profiles are decrease.

The stream wise variations of Nusselt number and Sherwood number at the walls are shown in figs. 5.35 – 5.36 for different values of variable viscosity (θΓ) and variable thermal conductivity (β) parameters. It is noticed that the non-dimensional Nusselt number and Sherwood number decrease with increasing values of θΓ. It can also be observed from fig. 5.35 and 5.36 that Nusselt number decreases but Sherwood number increases with increasing values of
\(\beta\). Variation of Nusselt number and Sherwood number against the thermal dispersion (Ds) and solutal dispersion (Dc) are shown in figs. 5.37 and 5.38. For higher values of Ds, the temperature and concentration gradient are highly influenced and increased at the wall in the boundary layer regime and hence as a result Nusselt and Sherwood number are greatly increased with increasing values of Ds. The influence of these Nusselt number and Sherwood number against solutal dispersion parameter (Dc) is also seen from figs. 5.37 and 5.38. It is observed from these figs that increase in Dc results in an enhancement in Nusselt number and Sherwood number. The non-dimensional Nusselt number and Sherwood number for different values of amplitude of the wavy surface (a) with fixed values of the other parameters is presented in figs. 5.39 and 5.40. Increasing amplitude of the wavy surface from 0.3 to 0.5 through 0.4, the amplitude of the Nusselt number and Sherwood number increased more significantly. It is an important to note that for \(a=0\) the wavy surface becomes flat plate and the results are true for the model of convective boundary layer flow over vertical flat plate in a porous medium.
5.3.4 Graphs

Fig. 5.26. Velocity profile ($u$) for different values of variable viscosity ($\theta$) and variable thermal conductivity ($\beta$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, and $\xi=1$.

Fig. 5.27. Temperature profile ($\theta$) for different values of variable viscosity ($\theta$) and variable thermal conductivity ($\beta$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, and $\xi=1$. 
Fig. 5.28. Concentration profile ($\phi$) for different values of variable viscosity ($\theta_r$) and variable thermal conductivity ($\beta$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, and $\xi=1$.

Fig. 5.29. Velocity profile ($u$) for different values of thermal dispersion parameter ($Ds$) and solutal dispersion parameter ($Dc$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $a=0.5$, $\theta_r=1.5$, $\beta=0.5$ and $\xi=1$. 
Fig. 5.30. Temperature profile ($\theta$) for different values of thermal dispersion ($D_s$) and solutal dispersion ($D_c$) parameters with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $a=0.5$, $\theta_r=1.5$, $\beta = 0.5$ and $\xi=1$.

Fig. 5.31. Concentration profile ($\phi$) for different values of thermal dispersion ($D_s$) and solutal dispersion ($D_c$) parameters with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $a=0.5$, $\theta_r=1.5$, $\beta = 0.5$ and $\xi=1$. 
Fig. 5.32. Velocity profile \( u \) for different values of \( \xi \)-location and mixed convective parameter \( \Delta \) with transverse coordinate \( \eta \) for \( N=1, \, Le=1, \, Ds=0.3, \, Dc=0.3, \, a=0.5, \, \theta = 1.5, \, \beta = 0.5 \).

Fig. 5.33. Temperature profile \( \theta \) for different values of \( \xi \)-location and mixed convective parameter \( \Delta \) with transverse coordinate \( \eta \) for \( N=1, \, Le=1, \, Ds=0.3, \, Dc=0.3, \, a=0.5, \, \theta = 1.5, \, \beta = 0.5 \).
Fig. 5.34. Concentration profile ($\phi$) for different values of $\xi$-location and mixed convective parameter ($\Delta$) with transverse coordinate ($\eta$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, $\theta_r=1.5$, $\beta=0.5$.

Fig. 5.35. Axial distributions of the Nusselt number (Nu) for different values of variable viscosity ($\theta_r$) and thermal conductivity ($\beta$) with stream wise coordinate ($\xi$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, $\xi=1$. 
Fig. 5.36. Axial distributions of the Sherwood number (Sh) for different values of variable viscosity ($\theta_r$) and thermal conductivity ($\beta$) with stream wise coordinate ($\xi$) for $N=1$, $Le=1$, $Ds=0.3$, $Dc=0.3$, $a=0.5$, $\xi=1$.

Fig. 5.37. Axial distributions of the Nusselt number (Nu) for different values of thermal dispersion ($Ds$) and solutal dispersion ($Dc$) parameters with stream wise coordinate ($\xi$) for $N=1$, $Le=1$, $a=0.5$, $\xi=1$, $\theta_r=1.5$, and $\beta=0.5$. 
Fig. 5.38. Axial distributions of the Sherwood number (Sh) for different values of thermal dispersion (Ds) and solutal dispersion (Dc) parameters with stream wise coordinate ($\xi$) for $N=1$, $Le=1$, $a=0.5$, $\xi=1$, $\theta_r=1.5$, and $\beta = 0.5$.

Fig. 5.39. Axial distributions of the Nusselt number (Nu) for different values of amplitude of the wavy surface (a) with stream wise coordinate ($\xi$) for $N=1$, $Le=1$, $\beta=0.5$, $Ds = 0.3$, $Dc=0.3$, $\xi=1$, $\theta_r=1.5$. 
Fig. 5.40. Axial distributions of the Sherwood number (Sh) for different values of the amplitude of the wavy surface (a) with stream wise coordinate (ξ) for N=1, Le=1, β=0.5, Ds = 0.3, Dc=0.3, ξ=1, θ_r=1.5.
5.5 Conclusions

The effect of thermal and solutal dispersion on free convection along a vertical wavy surface with temperature dependent viscosity and thermal conductivity analyzed and the governing boundary non-dimensional equations are solved by employing shooting technique. The main results in this study are as follows:

Case of Free Convection

1. Increasing variable viscosity parameter leads to decrease velocity distribution, Nusselt number and Sherwood number while temperature and concentration distributions are increases.

2. Increasing thermal dispersion parameter tends to increase velocity and temperature distributions as well as Nusselt number and Sherwood number while opposite results are noticed for concentration profile.

3. Velocity and concentration distributions as well as Nusselt number and Sherwood number are increased with increase in solutal dispersion parameter while we noticed that opposite results are reported for temperature profile.

4. Increase in the amplitude of the wavy surface results an enhancement in Nusselt number and Sherwood number.

Case of Mixed Convection

- Increasing variable viscosity parameter tends to reduce flow velocity, Nusselt number and Sherwood number while enhance temperature and concentration profiles.
• The flow velocity, temperature and Sherwood number are increased greatly for higher values of variable thermal conductivity. But the results of concentration and Nusselt number are reversed for larger values of variable thermal conductivity parameter.

• An increase in Ds is seen to increase the hydrodynamic velocity, temperature, Nusselt number and Sherwood number but decrease in concentration profile.

• For higher values of Dc, The flow velocity, concentration, Nusselt number and Sherwood number results are increased but the actual temperature profiles is decreased. The momentum, thermal and solutal boundary thickness are increased in the downstream direction.

• An increasing mixed convective parameter, the flow velocity is increased but the reverse trend is noticed for temperature and concentration profiles.

• It is also noticed that the amplitude of the both Nusselt number and Sherwood number increased with increasing values of the amplitude of the wavy surface.