Chapter 6

Comparison of DTM and MDTM for Solving Time-Dependent Emden-Fowler Type Equations

In this chapter, we present the comparative study of two dimensional differential transform method and the modified differential transform method for solving Time-dependent Emden-Fowler type equations. In fact, the exact solutions can be obtained by known form of the series solutions in both the methods. Several illustrative examples are given to demonstrate the effectiveness of both the methods.

6.1 INTRODUCTION

Many problems in mathematical physics and astrophysics related to the diffusion of perpendicular to the surface of parallel planes can be modeled by the heat equation (Davis, 1962):

\[ u_{xx} + \frac{\alpha}{x} u_x + af(x,t)g(u) + h(x,t) = u_t, \quad 0 < x \leq L, \quad 0 < t < T, \quad r > 0 \]  

\[ u(0, t) = \beta, \quad u_x(0, t) = 0 \]  

where \( \beta \) is a constant, \( u(x, t) \) is the temperature. For steady-state case, and \( \alpha = 2, \) \( h(x, t) = 0, \) (6.1.1) becomes

\[ u_{xx} + \frac{2}{x} u_x + af(x,t)g(u) = u_t, \quad 0 < x \leq L, \quad 0 < t < T, \quad r > 0 \]  

\[ * \]  

* This chapter is based on Ravi Kanth and Aruna (2012)
which is known as Emden-Fowler equation where \( f(x,t) \) and \( g(u) \) are some given function of \( x \) and \( u \) respectively. When \( f(x,t) = 1 \) and \( \alpha = 1 \), (6.1.3) reduces to the Lane-Emden equation with specified \( g(u) \) was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behaviour of a spherical cloud of a gas, isothermal gas sphere and theory of thermionic currents (Richardson, 1921; Chandrasekhar, 1967; Adomian et al., 1995).

In this analysis, we also study the wave type equations with singular behaviour of the form

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\alpha}{x} u_x + af(x,t)g(u) + h(x,t) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x \leq L, \quad 0 < t < T, \quad r > 0 \quad (6.1.4)
\]

where \( u(x,t) \) is the displacement of the wave at the position \( x \) and time \( t \). The solution of the time-dependent Emden-Fowler equation as well as variety of linear, non-linear singular initial value problems in quantum mechanics and astrophysics is numerically challenging because of singularity behavior at the origin. The approximate solutions to the above problems were presented by Shawagfeh (1993), the Adomain decomposition method (Wazwaz, 2001, 2002, 2005), the homotopy analysis method (Sami Bataineh et al., 2007), the variational iteration method (Batiha et al., 2007) and the homotopy perturbation method (Chowdhury and Hashim, 2009).

### 6.2 Description of the Method

In this section, we present the comparison of the differential transform method (DTM) and the modified differential transform method (MDTM) for the Time dependent Emden-Fowler equation (6.1.1) - (6.1.2).
6.2.1 DTM

First, let us consider the two dimensional differential transform for the (6.1.1) with respect to \((x,t)\) according to Theorem 1.3.1 gives

\[
\sum_{m=0}^{k} \delta(m - 1)(k - m + 1)(k - m + 2)U(k - m + 2, h) \\
+ \alpha(k + 1)U(k + 1, h) + a \sum_{m=0}^{k} \sum_{s=0}^{h} \delta(m - 1)F(k - m - 1, h)G(k, h - s) \\
+ \sum_{m=0}^{k} \delta(m - 1)H(k - m - 1, h) = \sum_{m=0}^{k} \delta(m - 1)(h + 1)U(k - m, h + 1), \\
k, h = 0, 1, \cdots \quad (6.2.1)
\]

The transformed version of (6.1.2) is

\[
U(0, h) = \begin{cases} 
\beta, & h = 0 \\
0, & \text{otherwise} 
\end{cases}, \quad U(1, h) = 0, \quad h = 0, 1, \cdots \quad (6.2.2)
\]

where \(\beta\) is known constant. We obtain all values of \(U(k, h)\) by substituting the Eq. (6.2.2) in Eq. (6.2.1) by recursive method. Then by substituting the quantities \(U(k, h)\) in Eq. (1.3.4), we get the following series solution

\[
u(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h)x^{k}t^{h}. \quad (6.2.3)
\]

6.2.2 MDTM

The modified differential transform for the (6.1.1) with respect to the variable \(x\) according to Theorem 1.4.1 gives
\[ \sum_{m=0}^{k} \delta(m-1)(k-m-1)(k-m-2)U(k-m-2,t) \]

\[ + \alpha(k+1)U(k+1,t) + a \sum_{r=0}^{k} \sum_{m=0}^{r} \delta(m-1)F(k-r,t)G(r-m,t) \]

\[ + \sum_{m=0}^{k} \delta(m-1)H(k-m-1,t) = \sum_{m=0}^{k} \delta(m-1)(h+1)\frac{\partial U(k-m,t)}{\partial t} \]  

(6.2.4)

The modified transformed version of (6.1.2) with respect to the variable \( x \) is

\[ U(0,t) = \alpha, \quad U(1,t) = 0 \]  

(6.2.5)

From Eq. (6.2.4), we get the following recurrence equation

\[ U(k+1,t) = \frac{1}{\alpha(k+1)} \left[ -a \sum_{r=0}^{k} \sum_{m=0}^{r} \delta(m-1)F(k-r,t)G(r-m,t) \right. \]

\[ - \sum_{m=0}^{k} \delta(m-1)(k-m-1)(k-m-2)U(k-m-2,t) \]

\[ - \sum_{m=0}^{k} \delta(m-1)H(k-m-1,t) + \sum_{m=0}^{k} \delta(m-1)(h+1)\frac{\partial U(k-m,t)}{\partial t} \]  

\[ k = 0, 1, \cdots \]  

(6.2.6)

We can obtain all values of \( U(k,t) \) by substituting the Eq. (6.2.5) in Eq. (6.2.6). Then by substituting the quantities \( U(k,t) \) in Eq. (1.4.3), we get the following series solution

\[ u(x,t) = \sum_{k=0}^{\infty} U(k,t)x^k. \]  

(6.2.7)

6.3 **Numerical Results**

In this section, we implemented the two dimensional differential transform method and the modified differential transform method for solving Time-dependent Emden-
Fowler type equations, which have been widely discussed in the literature.

**Example 6.3.1.** First we consider the following linear non-homogeneous equation (Wazwaz, 2002)

\[ u_{xx} + \frac{2}{x} u_x - (5 + 4x^2)u = u_t + (6 - 5x^2 - 4x^4) \]  

(6.3.1)

subject to

\[ u(0, t) = e^t, \quad u_x(0, t) = 0 \]  

(6.3.2)

**CASE I: DTM**

The transformed version of (6.3.1) is

\[
\begin{align*}
\sum_{m=0}^{k} \sum_{r=0}^{h} & \delta(m-1, h-r)(k-m+1)(k-m+2)U(k-m+2, r) \\
+ 2(k+1)U(k+1, h) & - 5 \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m-1, h-r)U(k-m, r) \\
- 4 \sum_{m=0}^{k} \sum_{r=0}^{h} & \delta(m-3, h-r)U(k-m, r) = 6\delta(k-1, h) - 5\delta(k-3, h) \\
- 4\delta(k-5, h) + \sum_{m=0}^{k} \sum_{r=0}^{h} & \delta(m-1, h-r)(r+1)U(k-m, r+1), \\
k, h = 0, 1, \cdots
\end{align*}
\]

(6.3.3)

The transformed versions of (6.3.2) are

\[
U(0, h) = \frac{1}{h!}, \quad U(1, h) = 0, \quad h = 0, 1, \cdots
\]

(6.3.4)

Using Eq. (6.3.4) in Eq. (6.3.3), we can calculate the values of \(U(k, h)\) by recursive method, some values of \(U(k, h)\) are listed in the Tables 6.1. Then by substituting
the quantities $U(k,h)$ in Eq. (6.2.3), we get the following series solution

\[
    u(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h) x^k t^h 
    \]

\[
    = \left( 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \right) + \left( 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \right) t 
    \]

\[
    + \left( 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \right) \frac{t^2}{2} + \left( 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \right) \frac{t^3}{6} 
    \]

\[
    + \left( 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \right) \frac{t^4}{24} + \cdots
    \]

\hspace{0.5cm} (6.3.5)

**Tab. 6.1:** Some values of $U(k, h)$ for Example 6.3.1

<table>
<thead>
<tr>
<th>$U(k, h)$</th>
</tr>
</thead>
</table>
| $k$ | $h = 0$ | $h = 1$ | $h = 2$ | $h = 3$ | $h = 4$ | $h = 5$
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{24}$ | $\frac{1}{120}$
| 1 | 0 | 0 | 0 | 0 | 0 | 0
| 2 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{24}$ | $\frac{1}{120}$
| 3 | 0 | 0 | 0 | 0 | 0 | 0
| 4 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{12}$ | $\frac{1}{48}$ | $\frac{1}{240}$
| 5 | 0 | 0 | 0 | 0 | 0 | 0

**Case II: MDTM**

The modified transformed version of (6.3.1) with respect to the variable $x$ is

\[
    \sum_{m=0}^{k} \delta(m-1)(k-m+1)(k-m+2)U(k-m+2, t) + 2(k+1)U(k+1, t) 
    \]

\[
    - 5 \sum_{m=0}^{k} \delta(m-1)U(k-m, t) - 4 \sum_{m=0}^{k} \delta(m-3)U(k-m, t) 
    \]

\[
    = \sum_{m=0}^{k} \delta(m-1) \frac{\partial U(k-m, t)}{\partial t} + 6\delta(k-1) - 5\delta(k-3) - 4\delta(k-5) 
    \]

\hspace{0.5cm} (6.3.6)
The modified transformed versions of (6.3.2) with respect to the variable $x$ are

$$U(0, t) = e^t, \quad \frac{\partial U(0, t)}{\partial x} = 0 \quad (6.3.7)$$

from Eq. (6.3.6), we get the recurrence form as

$$U(k + 1, t) = \frac{1}{(k + 1)(k + 2)} \left[ 5 \sum_{m=0}^{k} \delta(m - 1)U(k - m, t) + 4 \sum_{m=0}^{k} \delta(m - 3)U(k - m, t) 
+ \sum_{m=0}^{k} \delta(m - 1) \frac{\partial U(k - m, t)}{\partial t} + 6\delta(k - 1) - 5\delta(k - 3) - 4\delta(k - 5) \right],$$

$$k = 0, 1, \ldots \quad (6.3.8)$$

Using the Eq. (6.3.7) in Eq. (6.3.8), we get

$$U(1, t) = 0, \quad U(2, t) = e^t + 1, \quad U(3, t) = 0,$$

$$U(4, t) = \frac{e^t}{2!}, \quad U(5, t) = 0, \quad U(6, t) = \frac{e^t}{3!} \ldots \quad (6.3.9)$$

Substituting the quantities $U(k, t)$ in Eq. (6.2.7), we get the following series solution

$$u(x, t) = \sum_{k=0}^{\infty} U(k, t)x^k = e^t + (e^t + 1)x^2 + e^t \frac{x^4}{2!} + \cdots$$

$$= x^2 + e^t \left( 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \right) \quad (6.3.10)$$

In the limit of infinitely many terms Eqs. (6.3.5) and (6.3.10) yields the exact solution $u(x, t) = x^2 + e^{t + x^2}$ and it is observed that the modified differential transform method minimizes the computational work.

**Example 6.3.2.** We consider the following non-homogeneous singular wave-type equation (Wazwaz, 2002)
\[ u_{xx} + \frac{2}{x} u_x - (5 + 4x^2)u = u_{tt} + (12x - 5x^3 - 4x^5) \quad (6.3.11) \]

subject to

\[ u(0, t) = e^{-t}, \quad u_x(0, t) = 0 \quad (6.3.12) \]

**Case I: DTM**

The transformed version of (6.3.11) is

\[ \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m-1, h-r)(k-m+1)(k-m+2)U(k-m+2, r) \]
\[ + 2(k+1)U(k+1, h) - 5 \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m-1, h-r)U(k-m, r) \]
\[ - 4 \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m-3, h-r)U(k-m, r) \]
\[ = \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m-1, h-r)(r+1)(r+2)U(k-m, r+2) \]
\[ + 12\delta(k-2, h) - 5\delta(k-4, h) - 4\delta(k-6, h) \quad (6.3.13) \]

The transformed versions of (6.3.12) are

\[ U(0, h) = \frac{(-1)^h}{h!}, \quad U(1, h) = 0, \quad h = 0, 1, \cdots \quad (6.3.14) \]

The quantities \( U(k, h) \) are obtained by using the recurrence Eq. (6.3.13) and along with the transformed initial conditions (6.3.14). Then by substituting the quantities \( U(k, h) \) in Eq. (6.2.3), we get the following series solution

\[ u(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h)x^k t^h \]
\[ \begin{align*}
&= 1 + \frac{x^2}{1!} + x^3 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots - \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \right) t \\
&+ \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \right) \frac{t^2}{2} - \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \right) \frac{t^3}{6} \\
&+ \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \right) \frac{t^4}{24} + \cdots
\end{align*} \]

(6.3.15)

**CASE II: MDTM**

The modified transformed version of (6.3.11) with respect to the variable \(x\) is

\[\sum_{m=0}^{k} \delta(m-1)(k-m+1)(k-m+2)U(k-m+2, t) + 2(k+1)U(k+1, t)\]

\[-5 \sum_{m=0}^{k} \delta(m-1)U(k-m, t) - 4 \sum_{m=0}^{k} \delta(m-3)U(k-m, t)\]

\[= \sum_{m=0}^{k} \delta(m-1) \frac{\partial U(k-m, t)}{\partial t} + 12\delta(k-2) - 5\delta(k-4) - 4\delta(k-6),\]

\(k = 0, 1, \cdots\) \hspace{1cm} (6.3.16)

The modified transformed versions of (6.3.12) with respect to the variable \(x\) are

\[U(0, t) = e^{-t}, \quad \frac{\partial U(0, t)}{\partial x} = 0\] \hspace{1cm} (6.3.17)

Substituting Eq. (6.3.17) in Eq. (6.3.16), we obtain the following \(U(k, t)\) values

\[U(1, t) = 0, \quad U(2, t) = e^{-t} + 1, \quad U(3, t) = 1, \quad U(4, t) = \frac{e^{-t}}{2!},\]

\[U(5, t) = 0, \quad U(6, t) = \frac{e^{-t}}{3!}, \quad U(7, 0) = 0, \quad U(8, t) = \frac{e^{-t}}{4!}, \cdots\] \hspace{1cm} (6.3.18)

Substituting the quantities \(U(k, t)\) in Eq. (6.2.7), we get the following series solution

\[u(x, t) = \sum_{k=0}^{\infty} U(k, t)x^k = x^3 + e^{-t} \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \right)\] \hspace{1cm} (6.3.19)
In the limit of infinitely many terms Eqs. (6.3.15) and (6.3.19) yields the exact solution \( u(x, t) = x^3 + e^{-t+x^2} \) and it is observed that the modified differential transform method minimizes the computational work.

**Example 6.3.3.** We consider the following non-homogeneous wave-type equation (Wazwaz, 2002)

\[
\frac{\partial^2 u}{\partial x^2} + \frac{4}{x} \frac{\partial u}{\partial x} - (18x + 9x^4)u = \frac{\partial^2 u}{\partial t^2} - (18x + 9x^4)t^2 \tag{6.3.20}
\]

subject to

\[
u(x, 0) = e^{x^3}, \quad u_t(x, 0) = 0 \tag{6.3.21}
\]

**CASE I: DTM**

The transformed version of (6.3.20) is

\[
\sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 1, h - r)(k - m + 1)(k - m + 2)U(k - m + 2, r)
\]

\[
+ 4(k + 1)U(k + 1, h) - 18 \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 2, h - r)U(k - m, r)
\]

\[
- 9 \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 5, h - r)U(k - m, r)
\]

\[
= \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 1, h - r)(r + 1)(r + 2)U(k - m, r + 2)
\]

\[
- 2\delta(k - 1, h) - 18\delta(k - 2, h - 2) - 9\delta(k - 5, h - 2) \tag{6.3.22}
\]

The transformed versions of (6.3.21) are

\[
U(k, 0) = \frac{d^k (e^{x^3})}{dx^k} \bigg|_{x=0}, \quad U(k, 1) = 0, \quad k = 0, 1, 2, \cdots \tag{6.3.23}
\]

The quantities \( U(k, h) \) are obtained by using the recurrence Eq. (6.3.22) and along with the transformed initial conditions (6.3.23). We get the following approximate
series solution of the Eq. (6.3.20) by substituting the quantities \( U(k, h) \) in Eq. (6.2.3) is
\[
    u(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h) x^k t^h = t^2 + 1 + \frac{x^3}{1!} + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots \quad (6.3.24)
\]

**CASE II: MDTM**

The modified transformed version of (6.3.20) with respect to the variable \( t \) is
\[
    x \frac{\partial^2 U(x, h)}{\partial x^2} + 4 \frac{\partial U(x, h)}{\partial x} - (18x^2 + 9x^5)U(x, h)
    = x(h + 1)(h + 2)U(x, h + 2) - 2x - (18x^2 + 9x^5)\delta(h - 2) \quad (6.3.25)
\]

The modified transformed versions of (6.3.21) with respect to the variable \( t \) are
\[
    U(x, 0) = e^{x^3}, \quad U(x, 1) = 0 \quad (6.3.26)
\]

from Eq. (6.3.25), we get the following recurrence equation
\[
    U(x, h + 2) = \frac{1}{x(h + 1)(h + 2)} \left[ x \frac{\partial^2 U(x, h)}{\partial x^2} + 4 \frac{\partial U(x, h)}{\partial x} - (18x^2 + 9x^5)U(x, h) - 2x - (18x^2 + 9x^5)\delta(h - 2) \right],
    h = 0, 1, \cdots \quad (6.3.27)
\]

The quantities \( U(x, h) \) are obtained by using the recurrence Eq. (6.3.27) and along with the transformed initial condition (6.3.26). We get the following series solution
\[
    u(x, t) = \sum_{h=0}^{\infty} U(x, h) t^h = t^2 + e^{x^3} \quad (6.3.28)
\]

which is the exact solution of (6.3.20) - (6.3.21). Where as in differential transform method, the limit of infinitely many terms equation (6.3.24) yields the exact solution.
Example 6.3.4. Consider the following nonlinear time dependent equation (Wazwaz, 2002)

\[ u_{xx} + \frac{5}{x} u_x + (24t + 16t^2 x^2) e^u - 2x^2 e^{\frac{u}{2}} = u_t \]  \hspace{1cm} (6.3.29)

subject to

\[ u(x, 0) = 0, \quad u_t(x, 0) = -2x^2 \]  \hspace{1cm} (6.3.30)

Case I: DTM

The transformed version of (6.3.29) is

\[
\sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 1, h - r)(k - m + 1)(k - m + 2)U(k - m + 2, r) \\
+ 5(k + 1)U(k + 1, h) + 24 \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 1, h - r - 1)F(k - m, r) \\
+ 16 \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 3, h - r - 2)F(k - m, r) - 2 \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 3, h - r)G(k - m, r) \\
= \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 1, h - r)(r + 1)U(k - m, r + 1), \quad k, h = 0, 1, \cdots \]  \hspace{1cm} (6.3.31)

The transformed versions of (6.3.30) is

\[ U(k, 0) = 0, \quad U(k, 1) = \begin{cases} 
-2, & k = 2 \\
0, & \text{otherwise}
\end{cases} \]  \hspace{1cm} (6.3.32)

The quantities \( U(k, h) \) are obtained by using the recurrence Eq. (6.3.31) and along with the transformed initial conditions (6.3.32). We get the following approximate series solution of the Eq. (6.3.29) by substituting the quantities \( U(k, h) \) in Eq. (6.2.3) is

\[ u(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h)x^k t^h \]
\[-2x^2t + x^4t^2 - \frac{2x^6t^3}{3} + \frac{x^8t^4}{2} - \frac{2x^{10}t^5}{5} + \cdots \quad (6.3.33)\]

**Case II: MDTM**

The modified transformed version of (6.3.29) with respect to the variable \( t \) is

\[ x \frac{\partial^2 U(x, h)}{\partial x^2} + 5 \frac{\partial U(x, h)}{\partial x} + 24x \sum_{m=0}^{h} \delta(m - 1) F(x, h - m) \]

\[ + 16x^3 \sum_{m=0}^{h} \delta(m - 2) F(x, h - m) - 2x^3G(x, h) \]

\[ = x(h + 1)U(x, h + 1) \quad (6.3.34) \]

The modified transformed versions of (6.3.30) with respect to the variable \( t \) are

\[ U(x, 0) = 0, \quad U(x, 1) = -2x^2 \quad (6.3.35) \]

from Eq. (6.3.34), we get the following recurrence equation

\[ U(x, h + 1) = \frac{1}{x(h + 1)} \left[ x \frac{\partial^2 U(x, h)}{\partial x^2} + 5 \frac{\partial U(x, h)}{\partial x} + 24x \sum_{m=0}^{h} \delta(m - 1) F(x, h - m) \right. \]

\[ \left. + 16x^3 \sum_{m=0}^{h} \delta(m - 2) F(x, h - m) - 2x^3G(x, h) \right], \quad h = 1, 2, \cdots \quad (6.3.36) \]

The quantities \( U(x, h) \) are obtained by using the recurrence Eq. (6.3.36) and along with the transformed initial condition (6.3.35). We get the following series solution

\[ u(x, t) = \sum_{h=0}^{\infty} U(x, h)t^h = -2x^2t + x^4t^2 - \frac{2x^6t^3}{3} + \frac{x^8t^4}{2} - \frac{2x^{10}t^5}{5} + \cdots \quad (6.3.37) \]

In the limit of infinitely many terms Eqs. (6.3.33) and (6.3.37) yields the analytical solution \( u(x, t) = -2\ln(1 + tx^2) \) and it is observed that the modified differential transform method minimizes the computational work.

**Example 6.3.5.** Finally, we consider the following nonlinear time dependent ho-
mogeneous equation (Wazwaz, 2002)

\[ u_{xx} + \frac{6}{x} u_x + (14t + x^4)u + 4tu \ln u = u_{tt} \quad (6.3.38) \]

subject to

\[ u(x, 0) = 1, \quad u_t(x, 0) = -x^2 \quad (6.3.39) \]

**Case I: DTM**

The transformed version of (6.3.38) is

\[
\sum_{m=0}^{k} \delta(m - 1)(k - m + 1)(k - m + 2)U(k - m + 2, h) \\
+ 6(k + 1)U(k + 1, h) + 14 \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 1)\delta(h - r - 1)U(k - m, r) \\
+ \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 5)U(k - m, r) + \sum_{m=0}^{k} \sum_{r=0}^{h} \delta(m - 1, h - 1)U(m, h - r)F(k - m, r) \\
= \sum_{m=0}^{k} \delta(m - 1)(h + 1)(h + 2)U(k - m, h + 2), \quad k, h = 0, 1, \cdots \quad (6.3.40)
\]

The transformed versions of (6.3.39) are

\[
U(k, 0) = \begin{cases} 
1, & k = 0 \\
0, & \text{otherwise}
\end{cases}, \quad U(k, 1) = \begin{cases} 
-1, & h = 2 \\
0, & \text{otherwise}
\end{cases} \quad (6.3.41)
\]

The quantities \(U(k, h)\) are obtained by using the recurrence Eq. (6.3.40) and along with the transformed initial conditions (6.3.41). We get the following approximate series solution of the Eq. (6.3.38) by substituting the quantities \(U(k, h)\) in (6.2.3) is

\[
u(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h)x^kt^h = 1 - x^2t + \frac{x^4t^2}{2!} - \frac{x^6t^3}{3!} + \frac{x^8t^4}{4!} - \cdots \quad (6.3.42)
\]
Case II: MDTM

The modified transformed version of (6.3.38) with respect to the variable \(t\) is

\[
x \frac{\partial^2 U(x,h)}{\partial x^2} + 6 \frac{\partial U(x,h)}{\partial x} + 14x \sum_{m=0}^{h} \delta(m - 1)U(x, h - m) + x^5U(x, h) \\
+ 4x \sum_{m=0}^{h} \sum_{n=0}^{m} \delta(m - 1)U(x, h - m)G(x, m - n) = x(h + 1)(h + 2)U(x, h + 2)
\]

(6.3.43)

The modified transformed versions of (6.3.39) with respect to the variable \(t\) are

\[
U(x,0) = 1, \quad U(x,1) = -x^2
\]

(6.3.44)

and therefore from Eq. (6.3.43), we get the following recurrence equation

\[
U(x, h + 2) = \frac{1}{x(h + 1)(h + 2)} \left[ x \frac{\partial^2 U(x,h)}{\partial x^2} + 6 \frac{\partial U(x,h)}{\partial x} \\
+ 14x \sum_{m=0}^{h} \delta(m - 1)U(x, h - m) + x^5U(x, h) \\
+ 4x \sum_{m=0}^{h} \sum_{n=0}^{m} \delta(m - 1)U(x, h - m)G(x, m - n) \right], \quad h = 1, 2 \cdots
\]

(6.3.45)

The quantities \(U(x,h)\) are obtained by using the recurrence Eq. (6.3.45) and along with the transformed initial condition (6.3.44). We get the following series solution

\[
u(x,t) = \sum_{h=0}^{\infty} U(x,h)t^h = 1 - x^2t + \frac{x^4t^2}{2!} - \frac{x^6t^3}{3!} + \frac{x^8t^4}{4!} - \cdots
\]

(6.3.46)

In the limit of infinitely many terms Eqs.(6.3.42) and (6.3.46) yields the analytical solution \(u(x,t) = e^{-tx^2}\) and it is observed that the modified differential transform method minimizes the computational work.