CHAPTER 4

TRANSIENT ANALYSIS VIA HESSENBERG
DETERMINANTS

4.1 Introduction

Birth and Death processes (BDPs) on a finite state space have a wide range of applications in operations research and biological systems. Queuing models with finite capacity have applications in production and inventory problems, for example, to optimize the size of the storage space, to compute the trade-off between throughput and inventory (or waiting time) and to exhibit the propagation of blockage. The performance of the produce-to-stock manufacturing facility can be determined from the performance of the finite queuing systems. Network of queues with finite buffers occur widely in computer and communications systems. The phenomena of blocking, starvation and server breakdowns can severely restrict throughput and response time.

The transient probability functions of finite BDPs have been discussed in the literature for long time, see, for example Ledermann and Reuter (1954), Rosenlund (1978), Mohanty et al. (1993), Kijima (1997). The standard methods to determine these results usually make use of birth-death polynomials, spectral analysis, numerical analysis and Laplace transforms.

In this chapter, a different approach is discussed to derive the transient probabilities of finite and infinite BDPs. After converting into continued fractions (CFs) from the system of difference-differential equations using laplace transforms, we derive the power
series for this finite continued fraction. We also discuss a power series method to derive the transient probability for the BDPs with infinite state space. For sake convenient, we obtain the empty system size probability only. One can also extend this method to derive the other transient probabilities.

4.2 Finite BDPs and CFs

Let \( \{X(t)\} \) be a birth and death process with state-dependent birth and death rates \( \lambda_n \) and \( \mu_n \) respectively. Then \( P(X(t) = n|X(0) = 0) = P_n(t) \) satisfy the forward Kolmogorov equations

\[
P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)
\]

\[
P_n'(t) = \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t), \quad n = 1, 2, 3, \ldots N - 1, \quad (4.1)
\]

\[
P_N'(t) = \lambda_{N-1} P_{N-1}(t) - \mu_N P_N(t)
\]

Define Laplace transform

\[
f_n(s) = \int_0^\infty e^{-st} P_n(t) \, dt \quad n = 0, 1, 2, \ldots,
\]

for \( \text{Re}(s) > 0 \). Taking Laplace transform of the system of equations given in (4.1)

\[
f_0(s) = \frac{1}{s + \lambda_0 + \frac{f_1(s)}{f_0(s)}}
\]

\[
\frac{f_r(s)}{f_{r-1}(s)} = \frac{-\lambda_{r-1}\mu_r}{s + \lambda_r + \mu_r + \frac{f_{r+1}(s)}{f_r(s)}}, \quad r = 1, 2, 3, \ldots N. \quad (4.2)
\]

These equations are results in

\[
\frac{f_r(s)}{f_{r-1}(s)} = \frac{1}{s + \lambda_0 - s + \lambda_1 + \mu_1 - \frac{\lambda_0\mu_1}{\lambda_0\mu_1}} \frac{\lambda_1\mu_2}{s + \lambda_2 + \mu_2 - \frac{\lambda_1\mu_2}{\lambda_1\mu_2}} \cdots \frac{\lambda_{N-1}\mu_N}{s + \mu_N} \quad (4.3)
\]

\[
f_0(s) = \frac{1}{s + \lambda_0 - s + \lambda_1 + \mu_1 - \frac{\lambda_0\mu_1}{\lambda_0\mu_1}} \frac{\lambda_1\mu_2}{s + \lambda_2 + \mu_2 - \frac{\lambda_1\mu_2}{\lambda_1\mu_2}} \cdots \frac{\lambda_{N-1}\mu_N}{s + \mu_N} \quad (4.4)
\]
This is a J-fraction (or Jacobi fraction) and it can be represented in alternate forms as well.

\[
f_0(s) = \frac{1}{s} \frac{\lambda_0}{1+} \frac{\mu_1}{s} \frac{\lambda_1}{1+} \frac{\mu_2}{s} \frac{\lambda_2}{1+} \cdots \frac{\mu_N}{s}.
\]  
(4.5)

Let

\[a_{2n-1} = \lambda_{n-1}, \quad a_{2n} = \mu_n \quad n = 1, 2, 3 \ldots N \quad \text{and} \quad x = \frac{1}{s}\]

Therefore the above CF becomes

\[
f_0 \left( \frac{1}{x} \right) = \frac{x}{1+} \frac{a_1x}{1+} \frac{a_2x}{1+} \frac{a_3x}{1+} \cdots \frac{a_Nx}{1}.
\]  
(4.6)

**Remark:** Finite Continued fraction and its Power Series

Suppose the continued fraction is of the form

\[
\Lambda_n(x) = \frac{1}{1+} \frac{a_1x}{1+} \frac{a_2x}{1+} \frac{a_3x}{1+} \cdots \frac{a_Nx}{1}.
\]  
(4.7)

The above CF can be written in the form

\[
\Lambda_n(x) = \frac{P_n(x)}{Q_n(x)}
\]

where \(P_n(x)\) and \(Q_n(x)\) satisfy the recurrence relation

\[
R_n(x) = R_{n-1}(x) + a_n x R_{n-2}(x)
\]  
(4.8)

with \(P_0(x) = 1, \quad P_1(x) = 1 + a_1x, \quad Q_0(x) = 1, \quad Q_1(x) = 1\)

\[
f_0 \left( \frac{1}{x} \right) = x \frac{Q_{2N}(x)}{P_{2N}(x)}
\]

Using the recurrence relation (4.8), we can express \(P_{2n}(x)\) and \(Q_{2n}(x)\) as a polynomial form.
If we assume the polynomial

\[ P_{2n}(x) = \sum_{k=0}^{n} \phi_k x^k, \quad Q_{2n}(x) = \sum_{k=0}^{n} \theta_k x^k. \]

then

\[
P_{2n}(x) = 1 + x \sum_{i_1=1}^{2n} a_{i_1} + x^2 \sum_{i_1=1}^{2n-2} \sum_{i_2=i_1+1}^{2n} a_{i_1} a_{i_2} + \cdots + x^n \sum_{i_1=1}^{2n-2} \sum_{i_2=i_1+1}^{2n} \cdots \sum_{i_n=i_{n-1}+1}^{2n} a_{i_1} a_{i_2} \cdots a_{i_n}
\]

\[
Q_{2n}(x) = 1 + x \sum_{i_1=2}^{2n} a_{i_1} + x^2 \sum_{i_1=2}^{2n-2} \sum_{i_2=i_1+1}^{2n} a_{i_1} a_{i_2} + \cdots + x^n \sum_{i_1=2}^{2n-2} \sum_{i_2=i_1+1}^{2n} \cdots \sum_{i_n=i_{n-1}+1}^{2n} a_{i_1} a_{i_2} \cdots a_{i_n}
\]

Therefore,

\[ \phi_n = \sum_{i_1=1}^{2} \sum_{i_2=i_1+2}^{4} \cdots \sum_{i_n=i_{n-1}+2}^{2n} a_{i_1} a_{i_2} \cdots a_{i_n}, \quad \phi_0 = 1, \quad (4.9) \]

and

\[ \theta_n = \sum_{i_1=2}^{2} \sum_{i_2=i_1+2}^{4} \cdots \sum_{i_n=i_{n-1}+2}^{2n} a_{i_1} a_{i_2} \cdots a_{i_n}, \quad \theta_0 = 1. \quad (4.10) \]

Using the concept of reciprocal of power series, we can obtain the power series for reciprocal of power series as follows:

\[
\left[ \sum_{k=0}^{n} \phi_k x^k \right]^{-1} = \sum_{k=0}^{\infty} (-1)^k \psi_k x^k
\]

where \( \psi_{n+k} = \psi_{n,k} \)

then

\[
[1 + \phi_1 x + \phi_2 x^2 + \cdots + \phi_n x^n][1 - \psi_1 x + \psi_2 x^2 - \psi_3 x^3 + \cdots] = 1
\]
From the above equation we get

\[ \begin{align*}
\phi_1 &= \phi_0 \psi_1 \\
\phi_2 &= \phi_1 \psi_1 - \phi_0 \psi_2 \\
\phi_3 &= \phi_2 \psi_1 - \phi_1 \psi_2 + \phi_0 \psi_3 \\
\cdots & \cdots \\
\phi_n &= \phi_{n-1} \psi_1 - \phi_{n-2} \psi_2 + \cdots + (-1)^{n+1} \phi_0 \psi_n \\
0 &= \phi_n \psi_1 - \phi_{n-1} \psi_2 + \cdots + (-1)^{n+2} \phi_0 \psi_{n+1} \\
0 &= -\phi_n \psi_2 + \phi_{n-1} \psi_3 + \cdots + (-1)^{n+2} \phi_2 \psi_n + (-1)^{n+3} \phi_1 \psi_{n+1} + (-1)^{n+3} \phi_0 \psi_{n+2} \\
\cdots & \cdots \\
0 &= (-1)^{r+1} \phi_n \psi_r + (-1)^{r+2} \phi_{n-1} \psi_{r+1} + \cdots + (-1)^{n+r+1} \phi_0 \psi_{n+r}.
\end{align*} \]

Solving the above system, we get the coefficient \( \psi_n \)'s in terms of Hessenberg determinants where

\[
\psi_n = \begin{vmatrix}
\phi_1 & \phi_0 & 0 & 0 & 0 & \ldots & 0 \\
\phi_2 & \phi_1 & \phi_0 & 0 & 0 & \ldots & 0 \\
\phi_3 & \phi_2 & \phi_1 & \phi_0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
\phi_{n-1} & \phi_{n-2} & \cdots & \phi_0 \\
\phi_n & \phi_{n-1} & \phi_{n-2} & \cdots & \phi_1
\end{vmatrix} \quad (4.11)
\]
and

\[
\psi_{n,k} = \begin{vmatrix}
\phi_1 & \phi_0 & & \\
\phi_2 & \phi_1 & \phi_0 & \\
\phi_3 & \phi_2 & \phi_1 & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\phi_{n-1} & \phi_{n-2} & \phi_{n-3} & \ddots & \phi_0 \\
\phi_n & \phi_{n-1} & \phi_{n-2} & \ldots & \phi_1 & \phi_0 \\
& \phi_n & \phi_{n-1} & \ldots & \ldots & \phi_1 & \ddots \\
& & & \ddots & \phi_0 \\
& & & \phi_n & \phi_{n-1} & \phi_{n-2} & \ldots & \ldots & \phi_1
\end{vmatrix} \quad (4.12)
\]

Thus,

\[
[P_{2n}(x)]^{-1} = \left[ \sum_{k=0}^{n} \phi_k x^k \right]^{-1} = \sum_{k=0}^{n} (-1)^k \psi_k x^k + \sum_{k=1}^{\infty} (-1)^k \psi_{n,k} x^k
\]

The Hessenberg determinants can be computed using the following recurrence relation

\[
\psi_n = \sum_{m=1}^{n} (-1)^{m-1} \phi_m \psi_{n-m} \quad (4.13)
\]

\[
\psi_{n,k} = \psi_{k+n} = \sum_{m=1}^{n} (-1)^{m-1} \phi_m \psi_{n+k-m} \quad (4.14)
\]

Therefore,

\[
f_0 \left( \frac{1}{x} \right) = x \sum_{k=0}^{N} \theta_k x^k \times \sum_{k=0}^{\infty} (-1)^k \psi_k x^k
\]

\[
f_0(s) = \sum_{m=0}^{N} \left( \frac{1}{s} \right)^{m+1} \sum_{k=0}^{m} (-1)^k \theta_{m-k} \psi_k + \sum_{m=1}^{\infty} \left( \frac{1}{s} \right)^{N+m+1} \sum_{k=0}^{N} (-1)^{k+m} \theta_{N-k} \psi_{k+m}
\]

Inverting, we get

\[
P_0(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{k=0}^{\min(m,N)} (-1)^{k+\max(m-N, 0)} \theta_{\min(m,N)-k} \psi_{k+\max(m-N, 0)}
\]

where \(\theta_n\)’s and \(\psi_n\)’s calculated from (4.9), (4.10), (4.11), and (4.12).
Example 4.1 M/M/1/2 Queue

For M/M/1/2 queue, the arrival rates are $a_1(=\lambda_0)$, $a_3(=\lambda_1)$ and service rates are $a_2(=\mu_1)$, $a_4(=\mu_2)$.

First we estimate the values for $\theta$ and $\psi$.

\[
\phi_1 = a_1 + a_2 + a_3 + a_4 = 2(\lambda + \mu);
\]
\[
\phi_2 = a_1(a_3 + a_4) + a_2a_4 = \lambda^2 + \mu^2 + \lambda\mu;
\]
\[
\theta_1 = a_2 + a_3 + a_4 = \lambda + 2\mu;
\]
\[
\theta_2 = a_2a_4 = \mu^2;
\]

Therefore

\[
\psi_1 = 2(\lambda + \mu);
\]
\[
\psi_2 = 3\lambda^2 + 3\mu^2 + 7\lambda\mu;
\]

For $m \geq 3$

\[
\psi_m = \frac{1}{2\lambda\mu}(\lambda + \mu + \sqrt{\lambda\mu}^{m+1}) - (\lambda + \mu - \sqrt{\lambda\mu}^{m+1})
\]

Substitute the values of $\theta$ and $\psi$ in (4.15) and simplifying, we get

\[
P_0(t) = \frac{\mu^2}{\lambda^2 + \mu^2 + \lambda\mu} + \frac{\lambda}{2} \left[ \frac{1}{\lambda + \mu + \sqrt{\lambda\mu}} e^{-(\lambda+\mu+\sqrt{\lambda\mu})t} + \frac{1}{\lambda + \mu - \sqrt{\lambda\mu}} e^{-(\lambda+\mu-\sqrt{\lambda\mu})t} \right]
\]

4.3 Infinite BDQPs and CFs

Let $\{X(t)\}$ be a birth and death process with state-dependent birth and death rates $\lambda_n$ and $\mu_n$ respectively. Then $P[X(t) = n|X(0) = 0] = P_n(t)$ satisfy the forward Kolmogorov equations

\[
P'_0(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)
\]
\[
P'_n(t) = \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t), \quad n = 1, 2, 3, \ldots
\]
Taking Laplace transforms

\[ f_n(s) = \int_0^\infty e^{-st} P_n(t) \, dt, \quad n = 0, 1, 2, \ldots, \]  

(4.17)

for \( \text{Re}(s) > 0 \) of the system of equations given by (4.16) \( f_0(s) \) simplifies as

\[ f_0(s) = \frac{1}{s + \lambda_0 + \frac{f_1(s)}{f_0(s)}}. \]

Similarly,

\[ \frac{f_r(s)}{f_{r-1}(s)} = \frac{-\lambda_{r-1}\mu_r}{s + \lambda_r + \mu_r + \frac{f_{r+1}(s)}{f_r(s)}}, \quad r = 1, 2, 3, \ldots. \]

This results in

\[ f_0(s) = \frac{1}{s + \lambda_0 - \lambda_1\mu_1 + \frac{\lambda_1\mu_2}{s + \lambda_2 + \mu_2 - \cdots}}. \]  

(4.18)

This is a J-fraction (or Jacobi fraction) and it can be represented in alternate forms as well.

\[ f_0(s) = \frac{1}{s + \frac{\lambda_0}{\frac{s}{1+} + \frac{\mu_1}{\frac{s}{1+} + \frac{\lambda_1}{\frac{s}{1+} + \frac{\mu_2}{\frac{s}{1+} + \cdots}}}}}. \]  

(4.19)

If we assume \( x = \frac{1}{s} \), \( a_{2n-1} = \lambda_{n-1} \), \( a_{2n} = \mu_n \) \( n = 1, 2, 3, \ldots \) then the above CF becomes

\[ f_0\left(\frac{1}{x}\right) = \frac{x}{1+ \frac{a_1x}{1+ \frac{a_2x}{1+ \frac{a_3x}{1+ \cdots}}}}. \]  

(4.20)

4.4 Continued fraction to Power Series

Ramanujan (Berndt (1989), Entry 17) has given the power series expansion of S-fraction as follows:

\[ \frac{1}{1+ \frac{a_1x}{1+ \frac{a_2x}{1+ \frac{a_3x}{1+ \cdots}}}} = \sum_{n=0}^{\infty} A_n (-x)^n. \]  

(4.21)
where

\[ A_0 = 1 \]
\[ P_1 = A_1 = a_1 \]
\[ P_2 = A_2 = a_1(a_1 + a_2) \]

and

\[ P_n = a_1 a_2 a_3 \ldots a_{n-1}(a_1 + a_2 + a_3 + \cdots + a_n) \]

For \( n > 1 \)

\[ P_n = \sum_{0 \leq k < \frac{n-1}{2}} (-1)^k \phi_k(n) A_{n-k} \tag{4.22} \]

where \( \phi_0(n) = 1 \) and \( \phi_r(n) \) is defined recursively as

\[ \phi_r(n + 1) - \phi_r(n) = a_{n-1} \phi_{r-1}(n - 1) \tag{4.23} \]

and

\[ \phi_k(n) = \sum_{1 \leq j_1 \leq j_2 - 2} a_{j_1} a_{j_2} a_{j_3} \ldots a_{j_k} \sum_{1 \leq j_2 \leq j_3 - 2} \cdots \sum_{1 \leq j_k \leq n - 2} a_{j_1} a_{j_2} \ldots a_{j_k} \tag{4.24} \]

Also, \( \phi_r(n) = 0 \) if \( r \geq \left\lceil \frac{n}{2} \right\rceil \)

One can also rewrite the equation (4.24) as

\[ \phi_k(n) = \sum_{i_1=1}^{n-2k} \sum_{i_2=i_1+2}^{n-k-2} \cdots \sum_{i_k=i_{k-1}+2}^{n-2} a_{i_1} a_{i_2} \ldots a_{i_k} \tag{4.25} \]

**Theorem 4.1** If

\[ \frac{1}{1+x} \frac{a_1 x}{1+1} \frac{a_2 x}{1+1} \frac{a_3 x}{1+1} + \cdots = \sum_{n=0}^{\infty} A_n (-x)^n \]
then

\[ A_n = P_n + \sum_{k=1}^{n} P_k \psi_{n-k}(n) \]  

(4.26)

where

\[ \psi_k(n) = \begin{bmatrix} \phi_1(n) & \phi_0(n-1) & 0 & 0 & 0 & \ldots & 0 \\ \phi_2(n) & \phi_1(n-1) & \phi_0(n-2) & 0 & 0 & \ldots & 0 \\ \phi_3(n) & \phi_2(n-1) & \phi_1(n-2) & \phi_0(n-3) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \phi_{k-1}(n) & \phi_{k-2}(n-1) & \phi_{k-3}(n-2) & \ldots & \ddots & \phi_0(n-k+1) \\ \phi_k(n) & \phi_{k-1}(n-1) & \phi_{k-2}(n-2) & \ldots & \phi_1(n-k+1) \end{bmatrix}_{(n \times n)} \]

(4.27)

and \( \psi_0(n) = 1 \) and \( \psi_k(n) = 0 \) if \( n - k < 2 \).

**Proof.** Define

\[ A(z) = \sum_{n=1}^{\infty} A_n z^n \]
\[ \Phi_n(z) = \sum_{k=1}^{\infty} (-1)^k \phi_k(n) z^k \]
\[ P(z) = \sum_{n=1}^{\infty} P_n z^n \]

The equation (4.22) can be expressed as

\[ A_n = P_n - \sum_{1 \leq k < \frac{n}{2}} (-1)^k \phi_k(n) A_{n-k} \]

Multiply both sides by \( z^n \) and taking summation over \( 1 \) to \( \infty \), we get

\[ \sum_{n=1}^{\infty} A_n z^n = \sum_{n=1}^{\infty} P_n z^n - \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^k \phi_k(n) A_{n-k} z^n \]

\[ A(z) = P(z) - \Phi_n(z) A(z) \]
Therefore

\[ A(z) = \frac{P(z)}{1 + \Phi_n(z)} = \frac{P(z)}{\sum_{k=0}^{\infty} (-1)^k \phi_k(n) z^k} = \sum_{n=1}^{\infty} P_n z^n \sum_{k=0}^{\infty} \psi_k(n) z^k \text{ (say)} \quad (4.28) \]

Using the result of Inselberg (1978), we obtain the expression for \( \psi_k(n) \) given in (4.27).

The coefficient of \( z^n \) of (4.28) gives (4.26).

\[ \square \]

**Remark:** The above Hessenberg determinant \( \psi_k(n) \) satisfies the following relation:

\[
\psi_k(n) = \sum_{m=1}^{k} (-1)^{m-1} \phi_m(n) \psi_{k-m}(n-m), \quad (4.29)
\]

\[
\psi_k(n) = \psi_k(n-1) + a_{n-k-1} \psi_{k-1}(n) \quad (4.30)
\]

with \( \psi_0(n) = 1 \).

Using the recurrence relation (4.30), \( \psi_k(n) \) can be expressed as

\[
\psi_k(n) = a_1 \psi_{k-1}(k+2) + a_2 \psi_{k-1}(k+3) + a_3 \psi_{k-1}(k+4) + \cdots + a_{n-k-1} \psi_{k-1}(n) \quad (4.31)
\]

Repeated applications of the above equation yields,

\[
\psi_k(n + k + 1) = \sum_{i_1=1}^{n} a_{i_1} \sum_{i_2=1}^{i_1+1} a_{i_2} \sum_{i_3=1}^{i_2+1} a_{i_3} \cdots \sum_{i_k=1}^{i_{k-1}+1} a_{i_k} \quad (4.32)
\]

**Theorem 4.2** The generating function of \( \psi_k(k+2) \) leads to a continued fraction

\[
\sum_{k=0}^{\infty} \psi_k(k+2) x^k = \frac{1}{1 - a_1 x - \frac{a_2 x}{1 - \frac{a_3 x}{1 - \cdots}}} \quad (4.33)
\]

and the power series coefficient for \( f_0(s) \), \( A_k = \psi_k(k+2) \)

**Proof.** Define

\[
g_n(x) = \sum_{k=0}^{\infty} \psi_k(k+n) x^k
\]

\[
= \sum_{k=0}^{\infty} [ \psi_k(k + n - 1) + a_{n-1} \psi_{k-1}(k + n) ] x^k \quad [\text{using (4.30)}]
\]

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Therefore, 
\[ g_n(x) = g_{n-1}(x) + x a_{n-1} g_{n+1}(x) \]

The above equation can be rewritten as 
\[ \frac{g_n(x)}{g_{n-1}(x)} = \frac{1}{1 - x a_{n-1} \frac{g_{n+1}(x)}{g_n(x)}} \]

This results in 
\[ \frac{g_n(x)}{g_{n-1}(x)} = \frac{1}{1 - \frac{a_{n-1} x a_n}{1 - \frac{a_{n+1} x}{1 - \frac{a_n x}{1 - \frac{a_{n+1} x}{1 - ...}}}}} \] (4.34)

Since \( g_1(x) = \sum_{k=0}^{\infty} \psi_k(k+1) x^k = \psi_0(1) + 0 = 1 \)

Therefore, 
\[ g_2(x) = \sum_{k=0}^{\infty} \psi_k(k+2) x^k = \frac{1}{1 - \frac{a_1 x}{1 - \frac{a_2 x}{1 - \frac{a_3 x}{1 - ...}}}} \] (4.35)

Replace \( x \) by \( \frac{-1}{s} \), we get 
\[ \sum_{k=0}^{\infty} \psi_k(k+2) \left( \frac{-1}{s} \right)^k = s f_0(s) = \sum_{k=0}^{\infty} (-1)^k A_k \left( \frac{1}{s} \right)^k \] (4.36)

Comparing co-efficient of \( s^{-k} \) on both sides of (4.36), we get \( A_k = \psi_k(k+2) \).

### 4.5 A Single Server Queue

Zajta (1975) has partition approach for the conversion of continued fraction into power series.

\[ \frac{1}{1 - \frac{a_1 x}{1 - \frac{a_2 x}{1 - \frac{a_3 x}{1 - ...}}} = 1 + \sum_{n=1}^{\infty} F_n(a_1, a_2, a_3, \ldots, a_n) x^n } \] (4.37)

The coefficient \( F_n = \sum a_1^{e_1} \prod_{k=2}^{n} \left( \frac{e_k}{e_{k-1}} \right)^{e_k} a_k^{e_k} \)

where the exponents \( e_k(k = 1, 2, 3, \ldots, n) \) are non-negative integers and the summation
is to be extended over all partitions of the positive integer \(n\) such that
\[e_1 + e_2 + e_3 + \cdots + e_j = n, \quad j \leq n.\]

The co-efficient is also be defined as a convolution formula
\[
F_{n+1}(a_1, a_2, a_3, \ldots, a_{n+1}) = a_1 \sum_{k=0}^{n} F_k(a_1, a_2, a_3, \ldots, a_k) F_{n-k}(a_2, a_3, a_4 \ldots, a_{n-k+1})(4.38)
\]
where we put \(F_0 = 1\) and \(F_1 = a_1\)

Define the generating function
\[
G_k(x) = 1 + \sum_{n=1}^{\infty} F_n(a_k, a_{k+1}, a_{k+2}, \ldots, a_{n+k-1}) x^n
\]

Using the convolution formula (4.38), the cauchy product of \(G_k(x)\) and \(G_{k+1}(x)\) gives
\[
G_k(x) = \frac{1}{1 - a_k x G_{k+1}(x)} (4.39)
\]
If \(a_1 = a_2 = a_3 = \cdots = \lambda\) then \(G_k(x) = G_{k+1}(x)\) and
\[
G_1(x) = \frac{1}{1 - \lambda x G_1(x)}
\]
which leads to the quadratic equation
\[
\lambda x [G_1(x)]^2 - G_1(x) + 1 = 0
\]
\[
G_1(x) = \frac{1 - \sqrt{1 - 4\lambda x}}{2\lambda x}
\]
\[
= \frac{1}{2\lambda x} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n+1} (-4x)^{n+1}
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \lambda^n
\]
Therefore,
\[
F_n(\lambda, \lambda, \lambda, \ldots, \lambda) = \frac{1}{n+1} \binom{2n}{n} \lambda^n (4.40)
\]
For the busy period of M/M/1 queue, the continued fraction expansion for \( f_1(s) \) is given by

\[
f_1(s) = \frac{1}{p - \frac{\lambda \mu}{p^2} - \frac{\lambda \mu}{p^2} - \frac{\lambda \mu}{p^2} - \cdots}
\]

\[
= \frac{1}{p} + \sum_{n=1}^{\infty} F_n(\lambda \mu, \lambda \mu, \lambda \mu, \ldots, \lambda \mu) \cdot \frac{1}{p^{2n+1}}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n + 1} \binom{2n}{n} \frac{(\lambda \mu)^n}{p^{2n+1}}
\]

where \( p = s + \lambda + \mu \)

\[
p_1(t) = e^{-(\lambda+\mu)t} \sum_{n=0}^{\infty} \frac{1}{n + 1} \binom{2n}{n} (\lambda \mu)^n \frac{t^{2n}}{(2n)!}
\]

One can easily verify that the above series converges to \( e^{-(\lambda+\mu)t} \frac{I_1[2t\sqrt{\lambda\mu}]}{t\sqrt{\lambda\mu}} \), \( t > 0 \)

where \( I_n(.) \) is the modified Bessel function of the first kind of order \( n \).

For an M/M/1 queue, the arrival and service rates are assumed to be \( \lambda \) and \( \mu \) respectively.

That is, if we assume \( a_{2n-1} = \lambda, \ a_{2n} = \mu \) then (4.37) is a Continued fraction expansion of \( sf_0(s) \) where \( s = 1/x \).

Also for the above assumption, we have

\[
G_1(x) = G_3(x) = G_5(x) = \ldots
\]

\[
G_2(x) = G_4(x) = G_6(x) = \ldots
\]

\[
G_1(x) = \frac{1}{1 - \lambda x G_2(x)} \quad \text{(4.41)}
\]

and

\[
G_2(x) = \frac{1}{1 - \lambda t G_3(x)} = \frac{1}{1 - \lambda x G_1(x)}
\]

Substitute in (4.41) and simplify, we get a quadratic equation

\[
\mu x \left[ G_1(x) \right]^2 + [(\lambda - \mu)x - 1] G_1(x) + 1 = 0
\]
and this leads the expression as follows:

\[ G_1(x) = 1 + \frac{1}{\mu} \sum_{n=1}^{\infty} \sum_{i=0}^{[n+1]} (-1)^{-i} \left( \frac{\lambda + \mu}{2} \right)^{n+1} \frac{1}{2n - 2i + 1} \binom{2n - 2i + 1}{n - i + 1} \binom{n - i + 1}{i} \left( \frac{\lambda - \mu}{\lambda + \mu} \right)^{2i} x^n \]

From the above equation, one can obtain the empty system size probability of an M/M/1 queue which is given by

\[ P_0(t) = 1 + \frac{1}{\mu} \sum_{n=1}^{\infty} \left( \frac{\lambda + \mu}{2} \right)^{n+1} \frac{t^n}{n!} \sum_{i=0}^{[n+1]} (-1)^{n-i} \binom{2n - 2i + 1}{n - i + 1} \binom{n - i + 1}{i} \left( \frac{\lambda - \mu}{\lambda + \mu} \right)^{2i} \]

This result agrees with Krinik (1992).