CHAPTER 2

QUEUES WITH CATASTROPHE

2.1 Introduction

Queueing models in which customers leave without getting service either due to systems catastrophic failures or due to customers impatience have been studied in the literature. The loss of customers due to catastrophic failures (also referred to as negative arrivals) in the general context was first introduced by Gelenbe (1989). Such systems are referred to as G-networks or G-systems. Using string transition concept, a different approach to the study of negative arrivals is discussed (Serfozo (1999)). Since the introduction of negative arrivals, a number of authors have been studied different types of queueing models dealing with negative arrivals. For example, Chen and Renshaw (1997) have considered M/M/1 queueing model with disasters.

Queueing models with catastrophes have been studied some years ago, (see for instance Crescenzo et. al(2003, 2008), Giorno et al. (2012) Krinik and Mortensen (2007)). When the queue is not empty, catastrophes may occur and the effect of each catastrophe is to make the queue instantly empty, the system being immediately able to evolve afresh. Simultaneously, the system becomes ready to accept new customers. Queueing models with catastrophes have found applications in manufacturing settings, computer and communications systems. Many authors have been studied certain systems and assuming that they may be subject to catastrophes.
Much of the vast literature on queueing models is confined to results describing steady-state operation only. But time-dependent analysis helps us to understand the behaviour of a system the parameters involved are perturbed. The transient analysis is mainly based on a generating function technique (Sudhesh (2010)). However, an alternative approach is used to derive explicit expressions in terms of power series for the transient state distribution of an infinite systems (see Tarabia and El-Baz (2007), Tarabia et al. (2009)).

In this chapter we discuss the time-dependent system size probabilities of a birth-death process (BDP) with catastrophes \( \{N(t) : t \geq 0\} \) with state space \( S = \{0, 1, 2, \cdots\} \) and transient probabilities are obtained in closed form using continued fractions and their power series. Also we obtain the relations among transition probabilities of BDP’s with in the presence and in the absence of catastrophes and these results agree with (Pakes (1997), Crescenzo et. al(2003)). The stationary probabilities of BDP with catastrophe are also deduced from their Laplace transforms. Examples of transient probability functions are calculated for an M/M/1 queue with catastrophes, a BDP suggested by chain sequence rates and equal birth-death rates with in the presence of catastrophes.

2.2 Model Formulation

The birth-death process with catastrophe is a special case of continuous-time Markov chain where the states represent the current size of a population and where the transitions are limited to births, deaths and catastrophe. Let \( \{N(t), t \geq 0\} \) be a birth and death process with catastrophes defined on a probability space \( (\Omega, \mathcal{F}, \mathcal{P}) \) with state-space \( S = \{t, \infty, \varepsilon, \ldots\} \). The transition diagram of state-dependent birth-death process with catastrophe is given as follows:
Figure 2.1: Transition diagram of Catastrophe Birth-Death Processes

If $P = (p_{ij})_{i,j \geq 0}$ be the transition probability matrix of the Markov chain \{N(t), t \geq 0\} and $N(t)$ represents the size of the population at time $t$. The $(i,j)$-component $p_{ij}$ of $P$ is given by

$$p_{ij} = \begin{cases} 
\alpha_i & \text{if } j = i+1, i \geq 0 \\
\beta_j & \text{if } j = i-1, i \geq 2 \\
\beta_1 + \xi & \text{if } i = 1, j = 0 \\
\xi & \text{if } i \geq 2, j = 0 \\
0 & \text{otherwise}
\end{cases}$$

Hence, the process is specified by birth rates $\{\alpha_i\}_{i=0}^\infty$, death rates $\{\beta_i\}_{i=1}^\infty$ and catastrophes with rate $\xi$, the effect of each catastrophe being the instantaneous transition...
to the reflecting state 0. For all \( k, n \in \mathcal{S} \), and \( t > 0 \) the transition probabilities

\[
P(N(t) = n | N(0) = k) = P_{kn}(t)
\]
satisfy the forward Kolmogorov equations:

\[
P'_{k0}(t) = -(\alpha_0 + \xi)P_{k0}(t) + \beta_1 P_{k1}(t) + \xi, \quad \text{(2.1)}
\]

\[
P'_{kn}(t) = \alpha_{n-1} P_{k,n-1}(t) - (\alpha_n + \beta_n + \xi) P_{kn}(t) + \beta_{n+1} P_{k,n+1}(t), \quad n = 1, 2, 3, \ldots.
\]

Let \( \{\hat{N}(t); t \geq 0\} \) be the time-homogeneous birth and death process which is obtained from \( N(t) \) by removing the possibility of catastrophes, that is, by setting \( \xi = 0 \). The transition probabilities

\[
P(\hat{N}(t) = n | \hat{N}(0) = k) = \hat{P}_{kn}(t)
\]

then satisfy forward Kolmogorov equations of birth and death process. Taking Laplace transforms

\[
f_n^{(k)}(s) = \int_0^\infty e^{-st} P_{kn}(t) \, dt \quad \text{and} \quad \hat{f}_n^{(k)}(s) = \int_0^\infty e^{-st} \hat{P}_{kn}(t) \, dt, \quad n = 0, 1, 2, \ldots, \quad \text{(2.3)}
\]

for \( \text{Re}(s) > 0 \) of the system of equations given by (4.1) and the equation (2.2).

Initially we assume that \( k = 0 \) (i.e., the population size at time \( t = 0 \) is zero), and in that case \( f_0(s) \) simplifies to the expression

\[
f_0(s) = \frac{1 + \frac{\xi}{s}}{s + \xi + \alpha_0 - \beta_1 \frac{f_1(s)}{f_0(s)}},
\]

and

\[
\frac{f_n(s)}{f_{n-1}(s)} = \frac{\alpha_{n-1}}{s + \xi + \alpha_n + \beta_n - \beta_{n+1} \frac{f_{n+1}(s)}{f_n(s)}} = \frac{\alpha_{n-1}}{s + \xi + \alpha_n + \beta_n - \frac{\alpha_n \beta_{n+1}}{s + \xi + \alpha_{n+1} + \beta_{n+1} - \frac{\alpha_{n+1} \beta_{n+2}}{s + \xi + \alpha_{n+2} + \beta_{n+2} - \cdots}}. \quad \text{(2.4)}
\]
By means of the recursion, the continued fraction expression of $f_0(s)$ is given by

$$f_0(s) = \frac{1 + \frac{\xi}{s}}{s + \xi + \alpha_0 - \frac{\alpha_0\beta_1}{s + \xi + \alpha_1 + \beta_1 - \frac{\alpha_1\beta_2}{s + \xi + \alpha_2 + \beta_2 - \cdots}}}.$$  \hspace{1cm} (2.5)

For notational convenience, we use $f_n(s)$ instead of $f_n^{(0)}(s)$ and $P_n(t)$ instead of $P_{\ln}(t)$ throughout this chapter.

In the next section we obtain the power series expression for $P_n(t)$, $n = 0, 1, 2, \ldots$, using CFs.

### 2.3 Time-dependent System size Probabilities

This section presents time-dependent system size probabilities of a state-dependent birth-death process with catastrophe. Pakes (1997) and Crescenzo et. al. (2008) present time-dependent probabilities in terms corresponding birth-death process $\hat{N}(t)$ in the absence of catastrophe. But in their results, the explicit expressions of the birth-death process $\hat{N}(t)$ are not given. We obtained the elegant explicit expressions for the birth-death process with catastrophe process.

The J-fraction given in (2.4) is expressed as a power series leading to the state probabilities of birth-death process with catastrophe. First we prove the following result.

Let us assume that the birth and death rates be $a_{2n}$ and $a_{2n+1}$, $n = 0, 1, 2, \ldots$ respectively (instead of the usual $\alpha_n$ and $\beta_n$). The reason for this notation will become apparent in the sequel.
Theorem 2.1 If
\[
\frac{f_n(s)}{f_{n-1}(s)} = a_{2n-2} \sum_{m=0}^{\infty} (-1)^m B(m, n) \frac{1}{(s + \xi)^{m+1}},
\]  
then \(B(0, n) = 1\) and for \(m = 1, 2, 3, \cdots\),
\[
B(m, n) = \sum_{i_1=2n-1}^{2n} a_{i_1} \sum_{i_2=2n-1}^{i_1+1} a_{i_2} \sum_{i_3=2n-1}^{i_2+1} a_{i_3} \cdots \sum_{i_m=2n-1}^{i_{m-1}+1} a_{i_m} \forall \ n \in \mathbb{N}. \tag{2.7}
\]

Proof. This proof is similar to its counterpart proof in state-dependent BDP (Parthasarathy and Sudhesh (2006a)).

In the following theorem, we expressed the continued fraction \(f_0(s)\) as a power series with an explicit power series coefficients and the power series coefficients are in terms of finite summations.

Theorem 2.2 If
\[
P_0(t) = e^{-\xi t} \hat{P}_0(t) + \xi \int_{0}^{t} e^{-\xi y} \hat{P}_0(y) \, dy, \tag{2.8}
\]
then the probabilities \(\hat{P}_0(t)\) are given by
\[
\hat{P}_0(t) = \sum_{m=0}^{\infty} (-1)^m A(m, 0) \frac{t^m}{m!}, \tag{2.9}
\]
where \(A(0, 0) = 1\) and for \(m = 1, 2, 3, \ldots\),
\[
A(m, 0) = a_0 \sum_{i_1=0}^{1} a_{i_1} \sum_{i_2=0}^{i_1+1} a_{i_2} \sum_{i_3=0}^{i_2+1} a_{i_3} \cdots \sum_{i_m=0}^{i_{m-1}+1} a_{i_m}. \tag{2.10}
\]

Further, \(P_0(t)\) is given by
\[
P_0(t) = \sum_{m=0}^{\infty} (-1)^m \frac{A(m, 0)}{\xi^m} - e^{-\xi t} \sum_{k=0}^{\infty} \frac{\xi^k}{k!} \sum_{m=k+1}^{\infty} (-1)^m \frac{A(m, 0)}{\xi^m} \tag{2.11}
\]

Proof. The continued fraction (2.5) can be written as
\[
f_0(s) = (1 + \xi/s)\hat{f}_0(s)
\]
where $\hat{f}_0(s)$ is the Laplace transform of (2.2) when $n = 0$ and $k = 0$.

Using the Theorem 3.2 of Parthasarathy and Sudhesh (2006a), the continued fraction of $f_0(s)$ can be as expressed as power series.

\[ f_0(s) = \left(1 + \frac{\xi}{s}\right) \sum_{m=0}^{\infty} (-1)^m \frac{A(m, 0)}{(s + \xi)^{m+1}} \]  \hspace{1cm} (2.12)

Inversion of (2.12) yields,

\[ P_0(t) = e^{-\xi t} \left(\sum_{m=0}^{\infty} (-1)^m \frac{A(m, 0)}{m!} \right) + \xi \int_0^t e^{-\xi y} \left(\sum_{m=0}^{\infty} (-1)^m \frac{A(m, 0)}{\xi^m} \right) \frac{y^m}{m!} dy \]  \hspace{1cm} (2.13)

Equation (2.13) can be written as in the form of (2.8) (see for instance Theorem 3.2 of Parthasarathy and Sudhesh (2006a)).

After term by term integration of the above equation is simplified as follows:

\[ P_0(t) = e^{-\xi t} \sum_{m=0}^{\infty} (-1)^m \frac{A(m, 0)}{m!} + \sum_{m=0}^{\infty} (-1)^m \frac{A(m, 0)}{\xi^m} - e^{-\xi t} \sum_{k=0}^{\infty} (-1)^k \frac{\xi^k}{k!} \sum_{m=k}^{\infty} (-1)^{m-k} \frac{A(m, 0)}{\xi^{m-k}} \]

and rearranging the terms which results in equation (2.11).

**Theorem 2.3** For $n = 1, 2, 3, \ldots$,

\[ P_n(t) = e^{-\xi t} \hat{P}_n(t) + \xi \int_0^t e^{-\xi y} \hat{P}_n(y) dy, \]  \hspace{1cm} (2.14)

then for $n = 1, 2, 3, \ldots$ the probabilities $\hat{P}_n(t)$ are given by

\[ \hat{P}_n(t) = L_{n-1} \sum_{m=0}^{\infty} (-1)^m A(m, 2n) \frac{t^{m+n}}{(m+n)!}, \]  \hspace{1cm} (2.15)

where

\[ A(m, n) = \sum_{i_1=0}^{n} a_{i_1} \sum_{i_2=0}^{i_1+1} a_{i_2} \sum_{i_3=0}^{i_2+1} a_{i_3} \cdots \sum_{i_m=0}^{i_{m-1}+1} a_{i_m} \quad \forall \ n \in \mathbb{N}, \]  \hspace{1cm} (2.16)

and $L_{n-1} = a_0 a_2 a_4 \cdots a_{2n-2}$, $L_1 = 1$ with $A(0, n) = 1$.

Also, $P_n(t)$ is given by

\[ P_n(t) = L_{n-1} \sum_{m=0}^{\infty} (-1)^m \frac{A(m, 2n)}{\xi^{m+n}} - L_{n-1} e^{-\xi t} \sum_{k=0}^{\infty} \frac{(\xi t)^k}{k!} \sum_{m=max(k-n, 0)}^{\infty} (-1)^m \frac{A(m, 2n)}{\xi^{m+n}} \]  \hspace{1cm} (2.17)
Proof. Combining Theorems 1 and 2, we obtain,

\[ f_n(s) = L_{n-1} \left( 1 + \frac{\xi}{s} \right) \sum_{m=0}^{\infty} (-1)^m \frac{A(m, 2n)}{(s + \xi)^{m+n+1}}. \tag{2.18} \]

where \( L_{n-1} = a_0a_2a_4 \cdots a_{2n-2}, \quad L_{-1} = 1. \)

Equation (2.18) can also be represented as

\[ f_n(s) = \hat{f}_n(s + \xi) + \frac{\xi}{s} \hat{f}_n(s + \xi) \tag{2.19} \]

where \( \hat{f}_n(s) \) is the Laplace transform of (2.2) when \( k = 0. \)

Using the Theorem 3.3 given in Parthasarathy and Sudhesh (2006a), we obtain the power series coefficient \( A(m, n) \) given in (2.16).

Inversion of (2.18) yields (2.14) and the simplification of (2.14) leads the equation (2.17).

\[ \square \]

Remark 1 Steady-state Probability Distribution

It is well known that for steady-state

\[ \pi_n = \lim_{t \to \infty} P_n(t) = \lim_{s \to 0} sf_n(s), \quad \text{if } n = 0, 1, 2, \ldots. \tag{2.20} \]

Due to the above equation (2.19), the steady state distribution can be given by

\[ \pi_n = \xi \hat{f}_n(\xi) = L_{n-1} \sum_{m=0}^{\infty} (-1)^m \frac{A(m, 2n)}{\xi^{m+n}}, \quad n = 0, 1, 2, \ldots \]

Remark 2 Catalan numbers and its Generating function

It is known that the J-fraction (or Jacobi fraction)

\[ \phi(z) = \frac{1}{z + a_0 - \frac{a_0a_1}{z + a_1 + a_2 - \frac{a_2a_3}{z + a_3 + a_4 - \cdots}}} \]
can be represented as an S-fraction:

\[
\phi(z) = \frac{1}{z} + \frac{a_0/z}{1 + a_1/z} + \frac{a_2/z}{1 + a_3/s} + \frac{a_3/s}{1 + \cdots}
\]  

(2.21)

Using Theorem 2, the power series expression for \( \phi(z) \) is given by

\[
\phi(z) = \sum_{m=0}^{\infty} (-1)^m A(m, 0) \frac{a_m}{z^{m+1}}
\]

If \( a_0 = a_1 = a_2 = \cdots = a \) then one can easily prove that

\[
A(m, n) = \frac{n + 1}{n + m + 1} \left( \frac{n + 2m}{m} \right) a^m \text{ for every } n.
\]  

(2.22)

The proof of the above result is given in the following Example 1.

Therefore,

\[
\phi(z) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m+1} \frac{2m}{m} \frac{a^m}{z^{m+1}} = \sum_{m=0}^{\infty} (-1)^m C_m \frac{a^m}{z^{m+1}}
\]  

(2.23)

where \( C_m = \frac{1}{m+1} \left( \frac{2m}{m} \right) \) is a Catalan number.

It is known that from the theory of Catalan numbers,

\[
C(z) = \sum_{m=0}^{\infty} C_m z^m = 1 - \sqrt{1 - 4z} \text{ for } z \leq 1/4.
\]

Due to the above result, the equation (2.23) is simplified as

\[
\phi(z) = \frac{\sqrt{1 + (4a/z) - 1}}{2a} \text{ for } 1 + (4a/z) \geq 0.
\]  

(2.24)

**Remark 3** Consider

\[
A(m, n) = \sum_{i_1=0}^{n} a_{i_1} \sum_{i_2=0}^{i_1+1} a_{i_2} \sum_{i_3=0}^{i_2+1} a_{i_3} \cdots \sum_{i_m=0}^{i_{m-1}+1} a_{i_m}
\]

\[
= \sum_{i_1=0}^{n-1} a_{i_1} \sum_{i_2=0}^{i_1+1} a_{i_2} \cdots \sum_{i_m=0}^{i_{m-1}+1} a_{i_m} + a_n \sum_{i_1=0}^{n+1} a_{i_1} \sum_{i_2=0}^{i_2+1} a_{i_2} \cdots \sum_{i_m=0}^{i_{m-1}+1} a_{i_m}
\]

\[
= A(m, n - 1) + a_n A(m - 1, n + 1).
\]  

(2.25)
Repeated application of the recurrence relation (2.25) yields,

\[ A(m, n) = a_0 A(m - 1, 1) + a_1 A(m - 1, 2) + \cdots + a_n A(m - 1, n + 1). \]

This recurrence relation is useful to obtain elegant power series coefficients, as illustrated below.

### 2.4 Examples

In this section, we deduced some interesting examples such as a catastrophic birth-death process with equal rates, M/M/1 queue with catastrophe, catastrophic birth-death process with rates suggested by chain sequence. These results are agree with the results available in the literature.

#### 2.4.1 Example 1: Catastrophic Birth and Death process with equal birth-death rates

Let us assume that the birth and death rates are equal and the catastrophe rate is \( \xi \).

\[ i.e., a_{2n} = a_{2n+1} = \lambda > 0, \quad n = 0, 1, 2, \ldots, \quad (i.e., \alpha = \beta). \]

Then

\[ A(m, n - m) = \frac{n - m + 1}{m} \binom{n + m}{m - 1} \lambda^m. \]  
(2.26)

We use induction to prove the result (2.26).

If \( m = 1 \) then

\[ A(1, n - 1) = a_0 + a_1 + a_2 + \cdots + a_{n-1} = n\lambda = \frac{n}{1} \binom{n + 1}{0} \lambda. \]
If \( m = 2 \) then

\[
A(2, n - 2) = \lambda \sum_{i=0}^{n-2} A(1, i + 1)
= \lambda \sum_{i=0}^{n-2} (i + 2) = \frac{n-1}{2} \binom{n+2}{1} \lambda^2.
\]

We assume that this is true up to and including \( m \)

\[
i.e., \quad A(m, n - m) = \frac{n - m + 1}{m} \binom{n + m}{m - 1} \lambda^m.
\]

Then,

\[
A(m + 1, n - m - 1) = \sum_{i=0}^{n-m-1} a_i A(m, i + 1) = \lambda \sum_{i=0}^{n-m-1} A(m, i + 1)
= \frac{\lambda^m}{m} \sum_{i=0}^{n-m} (i + 1) \binom{2m + i}{m - 1}.
\]

It is well known that,

\[
\sum_{k=1}^{n} \binom{a + k}{m} = \binom{a + n + 1}{m + 1} - \binom{a + 1}{m + 1}.
\] (2.27)

Using the identity (2.27), we prove that

\[
\sum_{k=1}^{n} k \binom{a + k}{m} = n \binom{a + n + 1}{m + 1} - \binom{a + 1}{m + 2} + \binom{a + 1}{m + 2}.
\] (2.28)

Using the identity (2.27) and (2.28),

\[
\sum_{i=1}^{n-m} (i + 1) \binom{2m + i}{m - 1} = \sum_{i=1}^{n-m} i \binom{2m + i}{m - 1} + \sum_{i=1}^{n-m-2} \binom{2m + i}{m - 1}
= (n - m) \binom{n + m + 1}{m + 1} + \binom{2m + 1}{m + 1} + \binom{n + m + 1}{m} - \binom{2m + 1}{m}
= \frac{m(n - m)}{m + 1} \binom{n + m + 1}{m}.
\]

Therefore,

\[
A(m + 1, n - m - 1) = \frac{n - m}{m + 1} \binom{n + m + 1}{m} \lambda^{m+1},
\]
The result is true for $m$.

Using (2.26),

$$A(m, 2n) = \frac{2n + 1}{2n + m + 1} \binom{2n + 2m}{m} \lambda^m.$$  

Thus, for $n = 0, 1, 2, \ldots$,

$$P_{0n}(t) = \sum_{m=0}^{\infty} (-1)^m C_{m+n, n} \frac{(\lambda t)^{m+n}}{(m+n)!}$$

where $C_{m,n} = \frac{2n + 1}{n + m + 1} \binom{2m}{m+n}$, $0 \leq n \leq m$.

### 2.4.2 Example 2: M/M/1 queue with Catastrophe

In this example, we consider a classical M/M/1 queue with infinite capacity and possibility of catastrophe. Here we assume that the interarrival and service times are exponentially distributed with parameters $\lambda$ and $\mu$. Also we assume that the catastrophe is another process which will occur at the service facility as a Poisson process with rate $\xi$. Further, we assume that at time $t = 0$, the system is in the empty state.

In this case, the arrival (birth) and service (death) rates are given by

$$a_{2n} = \alpha_n = \lambda, \quad a_{2n+1} = \beta_{n+1} = \mu, \quad n = 0, 1, 2, \ldots$$

Therefore the continued fraction expansion of $f_n(s)/f_{n-1}(s)$ becomes

$$\frac{f_n(s)}{f_{n-1}(s)} = \frac{\lambda}{s + \xi + \lambda + \mu - \frac{\lambda \mu}{s + \xi + \lambda + \mu - \frac{\lambda \mu}{s + \xi + \lambda + \mu - \cdots}}}$$

$$= \frac{s + \xi + \lambda + \mu - \sqrt{(s + \xi + \lambda + \mu)^2 - 4\lambda \mu}}{2\mu},$$

and

$$f_0(s) = \frac{1 + \xi/s}{s + \xi + \lambda - \frac{\lambda \mu}{s + \xi + \lambda + \mu - \frac{\lambda \mu}{s + \xi + \lambda + \mu - \cdots}}}$$

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Using the equations (2.21) and (2.24), we get
\[ f_n(s) = \left[ \frac{s + \xi + \lambda + \mu - \sqrt{(s + \xi + \lambda + \mu)^2 - 4\lambda\mu}}{2\mu} \right]^n f_0(s), \tag{2.29} \]
and
\[ f_0(s) = \left( 1 + \frac{\xi}{s} \right) \left[ \frac{1}{s + \xi} - \frac{1}{s + \xi} \left( \frac{s + \xi + \lambda + \mu - \sqrt{(s + \xi + \lambda + \mu)^2 - 4\lambda\mu}}{2\mu} \right) \right], \]
\[ = \frac{1}{s} - \frac{1}{s} \left[ \frac{s + \xi + \lambda + \mu - \sqrt{(s + \xi + \lambda + \mu)^2 - 4\lambda\mu}}{2\mu} \right]. \]

Thus, for \( n = 1, 2, 3, \ldots, \)
\[ P_n(t) = \lambda \left( \frac{\sqrt{\lambda}}{\mu} \right)^{n-1} \int_0^t e^{-(\lambda+\mu+\xi)y} \left[ -(\lambda + \mu)y \right] [I_{n-1}(\alpha y) - I_{n+1}(\alpha y)]P_0(t - y) \ dy, \]
and
\[ P_0(t) = 1 - \lambda \int_0^t e^{-(\lambda+\mu+\xi)y} [I_0(2\sqrt{\lambda\mu y}) - I_2(2\sqrt{\lambda\mu y})] \ dy. \]

**Expectation \( E[N(t)] \)**

For an M/M/1 queue with catastrophe, the time-dependent expected system size is computed as follows:

Let \( \hat{m}(s) \) be the Laplace transform of \( m(t) = E[N(t)] \). Then using the equation (2.29) and doing algebraic manipulations, we obtain
\[ \hat{m}(s) = \frac{\lambda}{s(s + \xi)} - \frac{\mu}{s(s + \xi)} \frac{s + \xi + \lambda + \mu - \sqrt{(s + \xi + \lambda + \mu)^2 - 4\lambda\mu}}{2\mu} \quad \tag{2.30} \]

The above equation can be rewritten as:
\[ \hat{m}(s) = \frac{\lambda - \mu}{\xi} \left[ \frac{1}{s} - \frac{1}{s + \xi} \right] + \frac{\mu}{s + \xi} \left[ \frac{1}{s} - \frac{1}{s} \cdot \frac{s + \xi + \lambda + \mu - \sqrt{(s + \xi + \lambda + \mu)^2 - 4\lambda\mu}}{2\mu} \right] \]

Inversion yields,
\[ E[N(t)] = \frac{\lambda - \mu}{\xi} [1 - e^{-\xi t}] + \mu e^{-\xi t} \left[ 1 - \int_0^t e^{-(\lambda+\mu+\xi)y} [I_0(2\sqrt{\lambda\mu y}) - I_2(2\sqrt{\lambda\mu y})] \ dy \right] \]
where ‘*’ denotes convolution.
Steady-state Analysis of M/M/1 Queue with Catastrophe

In steady-state, i.e., as \( t \to \infty \), using (2.20) in (2.29) and (2.30), we get

\[
\pi_0 = 1 - \left[ \xi + \lambda + \mu - \sqrt{(\xi + \lambda + \mu)^2 - 4\lambda\mu} \right] = 1 - \rho
\]

\[
\pi_n = (1 - \rho)\rho^n
\]

where \( \rho = \frac{\xi + \lambda + \mu - \sqrt{(\xi + \lambda + \mu)^2 - 4\lambda\mu}}{2\mu} < 1. \)


Also from (2.30), in steady-state, the expected system size of M/M/1 queue with catastrophe is given by,

\[
E(X) = \frac{\lambda - \mu\rho}{\xi}
\]

When \( \xi = 0 \), i.e., in the absence of catastrophe, the above model M/M/1 queue with in the presence of catastrophe reduces to a classical M/M/1 queue and the results are agree with the existing results which are available in the literature.

2.4.3 Example 3: Catastrophic BDP with Chain sequence Rates

In this example, we consider a birth and death process with in the presence of catastrophe whose birth and death rates are suggested by a chain sequence. We deduce the system size probability in terms of modified bessel functions. We assume that initially the system is empty.

We assume that the birth and death rates \( a_{2n} = \lambda_n \) and \( a_{2n+1} = \mu_{n+1} \), \( n = 0, 1, 2, \ldots \), satisfying the conditions,

\[
\lambda_0 = 1, \quad \lambda_n + \mu_n = 1, \quad \lambda_{n-1}\mu_n = \gamma > 0, \quad n = 1, 2, 3, \ldots.
\]

Then the birth and death rates are given by

\[
\lambda_{2n} = \frac{\eta U_{n+1}(\frac{1}{\eta})}{2U_{n}(\frac{1}{\eta})}, \quad n = 0, 1, 2, \ldots, \quad \text{and} \quad \mu_n = \frac{\eta U_{n-1}(\frac{1}{\eta})}{2U_{n}(\frac{1}{\eta})}, \quad n = 1, 2, 3, \ldots,
\]
where $\eta = 2\sqrt{7}$.

The corresponding continued fraction of $f_0(s)$ is given by

$$f_0(s) = \frac{\left(1 + \frac{\xi}{s}\right)^\gamma}{s + \xi + 1 - \frac{\gamma}{s + \xi + 1 - \frac{\gamma}{s + \xi + 1 - \cdots}}},$$

$$= \left(1 + \frac{\xi}{s}\right)\frac{s + \xi + 1 - \sqrt{(s + \xi + 1)^2 - \eta^2}}{2\gamma}.$$

Also for $n = 1, 2, 3, \ldots$,

$$f_n(s) = \frac{\lambda_{n-1} s + \xi + 1 - \sqrt{(s + \xi + 1)^2 - \eta^2}}{2\gamma},$$

and therefore,

$$f_n(s) = \frac{2\eta U_n \left(\frac{1}{\eta}\right) \left(1 + \frac{\xi}{s}\right) \left(\frac{s + \xi + 1 - \sqrt{(s + \xi + 1)^2 - \eta^2}}{\eta}\right)^{n+1}}{\eta}. \quad (2.33)$$

Thus, for $n = 0, 1, 2, \ldots$,

$$P_n(t) = U_n \left(\frac{1}{\eta}\right) \left[e^{-(\xi+1)t}I_n(\eta t) - I_{n+2}(\eta t)\right] + \xi \int_0^t e^{-(\xi+1)y}[I_n(\eta y) - I_{n+2}(\eta y)] \, dy.$$

Also, the stationary probability distribution does not exist for the process $\{\hat{N}(t); t \geq 0\}$ with the birth-death rates are suggested by chain sequence (see Lenin and Parthasarathy (2000) ). But with the suitable choice of birth-death and catastrophe rates, one can easily prove that the stationary probability distribution may exists for the catastrophe process $\{N(t); t \geq 0\}$ with the rates are suggested by chain sequence.

Using (2.20) in the above equation (2.33), we obtain

$$\pi_n = \frac{2\xi}{\eta} U_n \left(\frac{1}{\eta}\right) \left(\frac{\xi + 1 - \sqrt{(\xi + 1)^2 - \eta^2}}{\eta}\right)^{n+1}, n = 0, 1, 2, \ldots. \quad (2.34)$$

2.4.4 Example 4: Linear rates

If $a_{2n} = a_{2n+1} = n + 1, \ n = 0, 1, 2, \ldots$, then

$$A(m, 2n) = m! \left(\frac{n + m}{m}\right)^2.$$
Proof. We use induction to prove this result.

It is easily seen that for \( m = 1, 2 \), we obtain

\[
A(1, 2n) = 1! \left( \frac{n + 1}{1} \right)^2; \quad A(1, 2n + 1) = 1! \left( \frac{n + 1}{1} \right) \left( \frac{n + 2}{1} \right)
\]

for every \( n \).

Assume that for every \( n \), this result is true up to \( m - 1 \),

\[
A(m - 1, 2n) = (m - 1)! \left( \frac{n + m - 1}{m - 1} \right)^2 \quad \text{for every } n,
\]

\[
A(m - 1, 2n + 1) = (m - 1)! \left( \frac{n + m - 1}{m - 1} \right) \left( \frac{n + m}{m - 1} \right)
\]

for every \( n \).

Using (2.25),

\[
A(m, 2n) = \sum_{k=0}^{n} a_{2k} A(m - 1, 2k + 1) + \sum_{k=1}^{n} a_{2k-1} A(m - 1, 2k)
\]

\[
= (m - 1)! \sum_{k=0}^{n} (k + 1) \left( \frac{k + m - 1}{m - 1} \right) \left( \frac{k + m}{m - 1} \right) + (m - 1)! \sum_{k=1}^{n} k \left( \frac{k + m - 1}{m - 1} \right)^2
\]

\[
= m! \sum_{k=1}^{n} \left( \frac{k + m - 1}{m} \right) \left[ \left( \frac{k + m - 2}{m - 1} \right) + \left( \frac{k + m - 1}{m - 1} \right) \right] + m! \left( \frac{n + m - 1}{m - 1} \right) \left( \frac{n + m}{m} \right)
\]

\[
= m! \left( \frac{n + m - 1}{m} \right) \left( \frac{n + m}{m} \right) + m! \left( \frac{n + m - 1}{m - 1} \right) \left( \frac{n + m}{m} \right)
\]

\[
= m! \left( \frac{n + m}{m} \right)^2
\]

Thus the result is true for every \( m \).

Therefore,

\[
P_{0n}(t) = e^{-\xi t} \sum_{m=0}^{\infty} (-1)^m \left( \frac{n + m}{m} \right) t^{n+m}
\]

\[
\quad + \xi \int_{0}^{t} e^{-\xi y} \sum_{m=0}^{\infty} (-1)^m \left( \frac{n + m}{m} \right) y^{n+m} dy
\]

\[
= e^{-\xi t} \frac{t^n}{(1 + t)^{n+1}} + \xi \int_{0}^{y} e^{-\xi y} \frac{y^n}{(1 + y)^{n+1}} dy, \quad n = 1, 2, 3, \cdots.
\]
2.4.5 Example 5

Let \(a_n = (n+1)a, \ n = 0, 1, 2, \ldots\), and \(a > 0\), then

\[
A(1, n) = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (k+1)a = \binom{n+2}{2}a,
\]

\[
A(2, n) = \sum_{k=0}^{n} a_k A(1, k+1) = \sum_{k=0}^{n} (k+1) \binom{k+3}{2}a^2 = \frac{4!}{2^2 2!} \binom{n+4}{4}a^2
\]

Assume that, for every \(n\), this result is valid upto \(m-1\).

\[
i.e., \ A(m-1, n) = \frac{(2m-2)!}{2^{m-1} (m-1)!} \binom{n+2m-2}{2m-2} a^{m-1}.
\]

Using (2.25),

\[
A(m, n) = \sum_{k=0}^{n} a_k A(m-1, k+1)
\]

\[
= \frac{(2m-2)!}{2^{m-1} (m-1)!} \sum_{k=0}^{n} (k+1) \binom{k+2m-1}{2m-2} a^m
\]

\[
= \frac{(2m)!}{2^m} \binom{n+2m}{2m} a^m.
\]

Therefore,

\[
L_{n-1}A(m, 2n) = a_0 a_2 a_4 \cdots a_{2n-2} \frac{(2n + 2m)!}{m! (2n)! 2^m} = \frac{(2n + 2m)!}{m! n! 2^{n+m}} a^{n+m}.
\]

Thus,

\[
P_{0n}(t) = e^{-\xi t} \sum_{m=0}^{\infty} (-1)^m \frac{(2n + 2m)!}{m! n! (n + m)!} \left(\frac{at}{2}\right)^{n+m} + \int_{0}^{t} e^{-\xi y} \sum_{m=0}^{\infty} (-1)^m \frac{(2n + 2m)!}{m! n! (n + m)!} \left(\frac{ay}{2}\right)^{n+m} dy.
\]